Theoretical Physics, Vol. 3, No. 3, September 2018
https://dx.doi.org/10.22606/tp.2018.33002

Aharonov-Bohm Effect, Dirac Monopole, and Bundle Theory

Miguel Socolovsky

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria, 04510, México D. F., México
Email: socolovs@nucleares.unam.mx

Abstract We discuss the Aharonov-Bohm ($A-B$) effect and the Dirac (D) monopole of magnetic charge $g=\frac{1}{2}$ in the context of bundle theory, which allows to exhibit a deep geometric relation between them. If ξ_{A-B} and ξ_D are the respective $U(1)$-bundles, we show that ξ_{A-B} is isomorphic to the pull-back of ξ_D induced by the inclusion of the corresponding base spaces. The fact that the $A-B$ effect disappears when the magnetic flux in the solenoid equals an integer number of times the quantum of flux associated with the unit of electric charge, reflects here as a consequence of the pull-back of the Dirac connection in ξ_D to ξ_{A-B}, and the Dirac quantization condition.

Keywords: Aharonov-Bohm effect, magnetic monopole, fiber bundles

PACS numbers: 02.40.-k, 02.40.Re, 03.65.Vf, 03.65.-w

1 Introduction

As is well known, the Aharonov-Bohm ($A-B$) effect [1] and the Dirac (D) magnetic monopole [2],[3] proposal have had a profound influence on the development of the gauge theories of fundamental interactions. The first one of these phenomena was immediately verified experimentally [4] and by many others later on [5], while even if Dirac monopoles have not yet being seen in Nature, both grand unified theories [6] and string theories [7] predict their existence.

The description of both the $A-B$ effect and the D monopole are deeply rooted in the concept of gauge potential and therefore in the concept of connection in fiber bundles. The first one provides an explicit evidence of the non-local character of quantum mechanics describing the motion of electrically charged particles in a non-simply connected space [8],[9], while the second one makes unavoidable the use of at least two charts on manifolds to define the gauge potential, leading to the necessity of a description in terms of a non-trivial bundle [10].

The close relationship between both phenomena consists in the facts that when the magnetic flux Φ_{A-B} is an integer multiple of the quantum of flux $\Phi_0=\frac{2\pi}{|e|}$ associated with the electric charge $|e|$, the $A-B$ effect vanishes, and when Φ_{A-B} also equals the magnetic flux of the monopole, Φ_D, the Dirac quantization condition ($D.Q.C.$) follows. In this note we want to emphasize this relation at a perhaps deeper level, namely through the relationship between the fiber bundles ξ_{A-B} (trivial) and ξ_D (non-trivial) in which both phenomena occur. After some basic material in section **2.**, in section **3.** we exhibit the bundle morphism $\xi_{A-B}\to\xi_D$ induced by the inclusion ι between the corresponding base spaces, and in section **4.** we use ι to construct the pull-back bundle $\iota^*(\xi_D)$, which in turn is proved, in section **5.**, to be isomorphic to ξ_{A-B} i.e.

$$\xi_{A-B}\cong\iota^*(\xi_D). \tag{1}$$

This is the main result of the present paper, since it exhibits a deep geometric relation between the $A-B$ effect and the magnetic monopole. Of course, the pull-back of the first Chern class c_1 of ξ_D, $\iota^*(c_1)$, vanishes in ξ_{A-B}, what is proved in section **6.** In section **7.** we show that the pull-back of the Dirac connection from ξ_D to ξ_{A-B} leads to the vanishing of the $A-B$ effect when the $D.Q.C.$ holds, thus setting on purely geometric grounds, one of the basic relations between $A-B$ and D. Section **8** is devoted to final comments.

We use the natural system of units $\hbar=c=1$.

Copyright © 2018 Isaac Scientific Publishing

2 Basics

In Ref. [8], the $U(1)$-bundle associated with the $A-B$ effect [1] with an infinitesimally thin and infinitely long solenoid was shown to be the *product* -and therefore trivial- *bundle*

$$\xi_{A-B}: S^1 \to (T_0^2)^* \xrightarrow{pr_1} (D_0^2)^* \tag{2}$$

where $S^1 = U(1) = \{z \in \mathbb{C},\ |z| = 1\}$ is the structure group, $(D_0^2)^*$ is the punctured open disk in two dimensions, $(T_0^2)^* = (D_0^2)^* \times S^1$ is the open solid 2-torus minus a circle, and pr_1 is the projection in the first entry. One has the homeomorphisms $(D_0^2)^* \cong (\mathbb{R}^2)^* = \mathbb{R}^2 \setminus \{0\} \cong \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The reason for (2) is that, in the above conditions, by symmetry reasons the space available to the electrically charged particles ("electrons") moving around the solenoid is $(\mathbb{R}^2)^*$ which is of the same homotopy type as the circle S^1. Then the set of isomorphism classes of $U(1)$-bundles over $(\mathbb{R}^2)^*$ consists of only one element [11]: the class of the product (trivial) bundle $(T_0^2)^*$.

On the other hand, the fiber bundles associated with Dirac monopoles [2],[3] of magnetic charge $g = \#k$ with k an integer and $\#$ a number depending on units, are the Hopf bundles [10],[12]

$$\xi_D^{(k)}: S^1 \to P_k^3 \xrightarrow{\pi_k} S^2 \tag{3}$$

where $P_0^3 = S^2 \times S^1$ (the trivial bundle), $P_k^3 \cong P_{-k}^3$, S^2 is the 2-sphere with $S^2 \cong \mathbb{R}^2 \cup \{\infty\} \cong \mathbb{C} \cup \{\infty\}$. In particular, we are interested in the case $k = 1$ for which $P_1^3 \cong S^3$: the 3-sphere given by

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2,\ |z_1|^2 + |z_2|^2 = 1\}, \tag{4}$$

$\pi_3 \equiv \pi$ is the Hopf map [13]

$$\pi: S^3 \to S^2,\ (z_1, z_2) \mapsto \pi(z_1, z_2) = \begin{cases} z_1/z_2, & z_2 \neq 0 \\ \infty, & z_2 = 0 \end{cases}. \tag{5}$$

We denote this *non-trivial* bundle ξ_D:

$$\xi_D^{(1)} \equiv \xi_D: S^1 \to S^3 \xrightarrow{\pi} S^2. \tag{6}$$

The global connection on ξ_D corresponding to $g = \frac{1}{2}$ ($\# = \frac{1}{2}$ and $k = 1$) is the 1-form $\omega \in \Omega^1 S^3 \otimes u(1)$, with $u(1) = Lie(U(1)) = i\mathbb{R}$, given by [14]

$$\omega = \frac{i}{2}(d\chi + cos\theta d\varphi), \tag{7}$$

where χ, θ and φ are the Euler angles in S^2 or $\mathbb{R}^3$ ($\theta \in [0, \pi]$ and $\chi, \varphi \in [0, 2\pi)$). The differential of ω is the 2-form

$$d\omega = -\frac{i}{2} sin\theta d\theta \wedge d\varphi = -F \in \Omega^2 S^3 \otimes u(1) \tag{8}$$

where F is the field strength

$$F = i|\boldsymbol{B}| sin\theta d\theta \wedge d\varphi \tag{9}$$

with

$$\boldsymbol{B} = (\frac{1}{2})\frac{\boldsymbol{r}}{r^3} \tag{10}$$

the magnetic field of the monopole in $\mathbb{R}^3 \setminus \{0\}$ (see below).

ω can be read from the squared length element on S^3:

$$dl_{S^3}^2(\chi, \theta, \varphi) = \frac{1}{4}(d\theta^2 + sin^2\theta d\varphi^2 + (d\chi + cos\theta d\varphi)^2) \tag{11}$$

which, in turn, can be obtained from the identification of S^3 with the group $SU(2)$ with elements

$$\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}(\varphi+\chi)} cos\frac{\theta}{2} & e^{\frac{i}{2}(\varphi-\chi)} sin\frac{\theta}{2} \\ -e^{-\frac{i}{2}(\varphi-\chi)} sin\frac{\theta}{2} & e^{-\frac{i}{2}(\varphi+\chi)} cos\frac{\theta}{2} \end{pmatrix}. \tag{12}$$

 Copyright © 2018 Isaac Scientific Publishing

Covering S^2 with the open sets U_+ and U_- respectively defined by $\theta \in [0, \pi)$ (the south pole S excluded) and $\theta \in (0, \pi]$ (the north pole N excluded), considering the pull-back of ω to $S^2 \setminus \{N, S\}$ with the local sections

$$s_N : U_+ \setminus \{N\} \to S^3,\ s_N(\hat{n}) = (cos\frac{\theta}{2}, sin\frac{\theta}{2}e^{i\varphi}), \tag{13a}$$

$$s_S : U_- \setminus \{S\} \to S^3,\ s_S(\hat{n}) = (cos\frac{\theta}{2}e^{i\varphi}, sin\frac{\theta}{2}), \tag{13b}$$

with $\hat{n} = (sin\theta cos\varphi, sin\theta sin\varphi, cos\theta)$, using the inclusion

$$j : S^3 \to \mathbb{R}^4,\ j(z_1, z_2) = (x_1, x_2, x_3, x_4)$$

$$= (cos(\frac{\varphi+\chi}{2})cos\frac{\theta}{2}, sin(\frac{\varphi+\chi}{2})cos\frac{\theta}{2}, cos(\frac{\varphi-\chi}{2})sin\frac{\theta}{2}, sin(\frac{\varphi-\chi}{2})sin\frac{\theta}{2}), \tag{14}$$

and defining the 1-form $\tilde{\omega} \in \Omega^1\mathbb{R}^4 \otimes u(1)$ through

$$\tilde{\omega} = i(x^1dx^2 - x^2dx^1 - x^3dx^4 + x^4dx^3), \tag{15}$$

one can prove that $j^*(\tilde{\omega}) = \omega$ and that $s^*_{N,S}(\omega)$ are the usual local 1-forms $A_\pm$ on S^2, namely

$$A_+(\theta, \varphi) = s^*_N(\omega)(\theta, \varphi) = (j \circ s_N)^*(\tilde{\omega})(\theta, \varphi) = -\frac{i}{2}(1 - cos\theta)d\varphi, \tag{16a}$$

$$A_-(\theta, \varphi) = s^*_S(\omega)(\theta, \varphi) = (j \circ s_S)^*(\tilde{\omega})(\theta, \varphi) = \frac{i}{2}(1 + cos\theta)d\varphi. \tag{16b}$$

The corresponding $u(1)$-valued 3-vector potentials are

$$\boldsymbol{A}_+ = -i\frac{1 - cos\theta}{2rsin\theta}\hat{\varphi},\ \boldsymbol{A}_- = +i\frac{1 + cos\theta}{2rsin\theta}\hat{\varphi}, \tag{17a}$$

defined also at $\theta = 0$ ($\boldsymbol{A}_+$) and $\theta = \pi$ ($\boldsymbol{A}_-$):

$$\boldsymbol{A}_+(\theta = 0) = \boldsymbol{A}_-(\theta = \pi) = \boldsymbol{0} \tag{17b}$$

and on a 2-sphere of arbitrary radius $r > 0$. Clearly, the rotor of $\boldsymbol{A}_+$ and $\boldsymbol{A}_-$ gives the magnetic field $\boldsymbol{B}$.

The first Chern class of ξ_D (taking S^2 with unit radius) is given by

$$c_1(\xi_D) = \frac{i}{2\pi}[F] \tag{18}$$

where $[F]$ is the cohomology class of F in $H^2(S^2)$: cohomology of the 2-sphere in dimension 2. The integral of $\frac{i}{2\pi}F$ over S^2 gives the first Chern number of ξ_D:

$$\frac{i}{2\pi}\int_{S^2} F = 1. \tag{19}$$

This means that the magnetic charge is a measure of the topological non-triviality of the bundle ξ_D i.e. of the space where it “lives". In other words, the monopole charge is not a property of the gauge field $A_\pm$ itself, but of the $U(1)$-bundle on which the monopole is a connection.

3 Bundle Morphism $\xi_{A-B} \to \xi_D$

Using the homeomorphisms $(D^2_0)^* \cong \mathbb{C}^*$ and $S^2 \cong \mathbb{C} \cup \{\infty\}$, it can be easily verified that (ι,$\bar{\iota}$) given by

$$\iota : \mathbb{C}^* \to \mathbb{C} \cup \{\infty\},\ \iota(z) = z \tag{20}$$

and

$$\bar{\iota} : \mathbb{C}^* \times S^1 \to S^3,\ \bar{\iota}(z, e^{i\varphi}) = \frac{(z, 1)}{||(z, 1)||}e^{i\varphi} \tag{21}$$

Copyright © 2018 Isaac Scientific Publishing

with $||(z,1)|| = \sqrt{1+|z|^2}$, and (ψ_{A-B},ψ_D) the right actions

$$\psi_{A-B} : (\mathbb{C}^* \times S^1) \times S^1 \to \mathbb{C}^* \times S^1,\ \psi_{A-B}((z,e^{i\alpha}),e^{i\beta}) = (z,e^{i(\alpha+\beta)}) \tag{22}$$

and

$$\psi_D : S^3 \times S^1 \to S^3,\ \psi_D((z_1,z_2),e^{i\lambda}) = (z_1 e^{i\lambda}, z_2 e^{i\lambda}) \tag{23}$$

is the unique *bundle morphism*

$$\xi_{A-B} \to \xi_D \tag{24}$$

induced by the inclusion ι i.e.

$$\pi \circ \bar{\iota} = \iota \circ pr_1 \tag{25}$$

and

$$\psi_D \circ (\bar{\iota} \times Id_{S^1}) = \bar{\iota} \circ \psi_{A-B} \tag{26}$$

namely, with lower and upper parts of Diagram 1 commuting.

$$\begin{array}{ccc} (\mathbb{C}^* \times S^1) \times S^1 & \stackrel{\bar{\iota}\times Id_{S^1}}{\longrightarrow} & S^3 \times S^1 \\ \psi_{A-B} \downarrow & & \downarrow \psi_D \\ \mathbb{C}^* \times S^1 & \stackrel{\bar{\iota}}{\longrightarrow} & S^3 \\ pr_1 \downarrow & & \downarrow \pi \\ \mathbb{C}^* & \stackrel{\iota}{\longrightarrow} & \mathbb{C} \cup \{\infty\} \end{array}$$

Diagram 1

In fact:

$$\pi \circ \bar{\iota}(z,e^{i\varphi}) = \pi(\frac{(z,1)}{||(z,1)||} e^{i\varphi}) = z,$$

$$\iota \circ pr_1(z,e^{i\varphi}) = \iota(z) = z;$$

$$\psi_D \circ (\bar{\iota} \times Id_{S^1})((z,e^{i\varphi}),e^{i\lambda}) = \psi_D(\bar{\iota}(z,e^{i\varphi}),e^{i\lambda}) = \frac{(z,1)}{||(z,1)||} e^{i(\varphi+\lambda)},$$

$$\bar{\iota} \circ \psi_{A-B}((z,e^{i\varphi}),e^{i\lambda}) = \bar{\iota}(z,e^{i(\varphi+\lambda)}) = \frac{(z,1)}{||(z,1)||} e^{i(\varphi+\lambda)}.$$

4 Pull-back of ξ_D by ι: $\iota^*(\xi_D)$

The total space of the *induced* or *pull-back* bundle [14] of ξ_D by ι, $\iota^*(\xi_D) : S^1 \to P_{\iota^*(\xi_D)} \stackrel{pr_1}{\longrightarrow} \mathbb{C}^*$, is defined by

$$P_{\iota^*(\xi_D)} = \{(z,(z_1,z_2)) \in \mathbb{C}^* \times S^3,\ \iota(z) = \pi(z_1,z_2)\} \tag{27}$$

and must be such that both the upper and lower parts of Diagram 2 commute i.e. such that (ι, pr_2) is a bundle morphism $\iota^*(\xi_D) \to \xi_D$. In Diagram 2, pr_2 is the projection in the second entry, and

$$\psi_{\iota^*(\xi_D)} : P_{\iota^*(\xi_D)} \times S^1 \to P_{\iota^*(\xi_D)},\ \psi_{\iota^*(\xi_D)}((z,(z_1,z_2)),e^{i\lambda}) = (z,(z_1,z_2)e^{i\lambda}) \tag{28}$$

is the right action of S^1 on $P_{\iota^*(\xi_D)}$.

$$\begin{array}{ccc} P_{\iota^*(\xi_D)} \times S^1 & \stackrel{pr_2 \times Id_{S^1}}{\longrightarrow} & S^3 \times S^1 \\ \psi_{\iota^*(\xi_D)} \downarrow & & \downarrow \psi_D \\ P_{\iota^*(\xi_D)} & \stackrel{pr_2}{\longrightarrow} & S^3 \\ pr_1 \downarrow & & \downarrow \pi \\ \mathbb{C}^* & \stackrel{\iota}{\longrightarrow} & \mathbb{C} \cup \{\infty\} \end{array}$$

Diagram 2

 Copyright © 2018 Isaac Scientific Publishing

From

$$\iota \circ pr_1 = \pi \circ pr_2 \tag{29}$$

one has:

$$\iota \circ pr_1((z,(z_1,z_2)) = \iota(z) = z,$$
$$\pi \circ pr_2((z,(z_1,z_2)) = \pi(z_1,z_2) = z_1/z_2,$$

so $z_1 = z_2 z$ and $||(z_1,z_2)|| = 1$ implies $(z_1,z_2) = \frac{(z,1)}{||(z,1)||}e^{i\varphi}$. Then,

$$P_{\iota^*(\xi_D)} = \{(z, \frac{(z,1)}{||(z,1)||}e^{i\varphi}),\ z \in \mathbb{C}^*,\ \varphi \in [0,2\pi)\} \subset \mathbb{C}^* \times S^3. \tag{30}$$

On the other hand, it holds

$$\psi_D \circ (pr_2 \times Id_{S^1}) = pr_2 \circ \psi_{\iota^*(\xi_D)}. \tag{31}$$

In fact:

$$\psi_D \circ (pr_2 \times Id_{S^1})((z,(z_1,z_2)),e^{i\lambda}) = \psi_D((z_1,z_2)e^{i\lambda}) = (z_1e^{i\lambda}, z_2e^{i\lambda}),$$
$$pr_2 \circ \psi_{\iota^*(\xi_D)}((z,(z_1,z_2)),e^{i\lambda}) = pr_2((z,(z_1,z_2)e^{i\lambda})) = (z_1,z_2)e^{i\lambda} = (z_1e^{i\lambda}, z_2e^{i\lambda}).$$

5 Bundle Isomorphism: $\iota^*(\xi_D) \xrightarrow{\cong} \xi_{A-B}$

In this section we exhibit a “natural" isomorphism between the $A-B$ bundle and the pull-back by the inclusion $\iota : \mathbb{C}^* \to \mathbb{C} \cup \{\infty\}$ (i.e. $\iota : (D_0^2)^* \to S^2$ up to homeomorphisms) of the Dirac bundle ξ_D corresponding to unit magnetic charge, thus establishing a deep relation between the two systems ($A-B$: experimentally observed, and D: only theoretical, up to now).

The homeomorphism between the total spaces of the bundles is given by

$$\Psi : P_{\iota^*(\xi_D)} \to \mathbb{C}^* \times S^1,\ \Psi(z, \frac{(z,1)}{||(z,1)||}e^{i\varphi}) = (z, e^{i\varphi}). \tag{32}$$

It is clear that Ψ is continuous, one-to-one and onto, with continuous inverse Ψ^{-1}. It is easily verified that Diagram 3, corresponding to this isomorphism, commutes in its upper and lower parts i.e.

$$pr_1 \circ \Psi = Id_{\mathbb{C}^*} \circ pr_1 \tag{33}$$

and

$$\psi_{A-B} \circ (\Psi \times Id_{S^1}) = \Psi \circ \psi_{\iota^*(\xi_D)}. \tag{34}$$

$$\begin{array}{ccc} P_{\iota^*(\xi_D)} \times S^1 & \overset{\Psi \times Id_{S^1}}{\longrightarrow} & (\mathbb{C}^* \times S^1) \times S^1 \\ \psi_{\iota^*(\xi_D)} \downarrow & & \downarrow \psi_{A-B} \\ P_{\iota^*(\xi_D)} & \overset{\Psi}{\longrightarrow} & \mathbb{C}^* \times S^1 \\ pr_1 \downarrow & & \downarrow pr_1 \\ \mathbb{C}^* & \overset{Id_{\mathbb{C}^*}}{\longrightarrow} & \mathbb{C}^* \end{array}$$

Diagram 3

In fact:

$$pr_1 \circ \Psi(z, \frac{(z,1)}{||(z,1)||}e^{i\varphi}) = pr_1(z, e^{i\varphi}) = z,$$
$$Id_{\mathbb{C}^*} \circ pr_1(z, \frac{(z,1)}{||(z,1)||}e^{i\varphi}) = Id_{\mathbb{C}^*}(z) = z;$$

$$\psi_{A-B} \circ (\Psi \times Id_{S^1})((z,(z_1,z_2)),e^{i\lambda}) = \psi_{A-B}(\Psi((z,(z_1,z_2)),e^{i\lambda})) = \Psi(z,(z_1,z_2))e^{i\lambda} = \Psi(z, \frac{(z,1)}{||(z,1)||}e^{i\varphi})e^{i\lambda}$$
$$= (z, e^{i\varphi})e^{i\lambda} = (z, e^{i(\varphi+\lambda)}),$$
$$\Psi \circ \psi_{\iota^*(\xi_D)}((z,(z_1,z_2)),e^{i\lambda}) = \Psi(z,(z_1,z_2)e^{i\lambda}) = \Psi(z, \frac{(z,1)}{||(z,1)||}e^{i(\varphi+\lambda)}) = (z, e^{i(\varphi+\lambda)}).$$

Copyright © 2018 Isaac Scientific Publishing

6 Chern Classes

ξ_{A-B} is the pull-back of ξ_D by the inclusion $\iota : (D_0^2)^* \to S^2$; however, since ξ_{A-B} is trivial, then all its Chern classes must vanish. Then, in particular, we must verify the vanishing of the pull-back of c_1.

$\xi_{A-B} = \iota^*(\xi_D)$ passes to cohomology [15] in the form

$$\iota^* : H^*(S^2) \to H^*((D_0^2)^*) \tag{35a}$$

i.e.

$$\iota^* : H^k(S^2) \to H^k((D_0^2)^*),\ k = 0, 1, 2 \tag{35b}$$

where

$$H^*(S^2) = (H^0(S^2), H^1(S^2), H^2(S^2)) \cong (\mathbb{R}, 0, \mathbb{R}) \tag{36}$$

and

$$H^*((D_0^2)^*) = (H^0((D_0^2)^*), H^1((D_0^2)^*), H^2((D_0^2)^*)) \cong (\mathbb{R}, \mathbb{R}, 0) \tag{37}$$

are the cohomology groups of the 2-sphere and the punctured disk respectively. $H^*((D_0^2)^*) \cong H^*(S^1)$ by homotopy invariance. Since $c_1 \in H^2(S^2)$, then

$$\iota^*(c_1) = 0. \tag{38}$$

7 Pull-back of the Dirac Connection and Vanishing of the $A - B$ Effect

In terms of the cartesian coordinates in $\mathbb{R}^3$, $(x, y, z) = r(sin\theta cos\varphi, sin\theta sin\varphi, cos\theta)$ with $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$ which implies $(x, y, z) \neq (0, 0, z)$, the monopole potentials $A_\pm$ of equations (16a) and (16b) are given by

$$A_\pm(x, y, z) = (A_\pm)_x dx + (A_\pm)_y dy \tag{39}$$

with

$$(A_\pm)_x(x, y, z) = \pm\frac{i}{2}(\frac{y}{x^2 + y^2})(1 \mp \frac{z}{\sqrt{x^2 + y^2 + z^2}}),\ (A_\pm)_y(x, y, z) = \mp\frac{i}{2}(\frac{x}{x^2 + y^2})(1 \mp \frac{z}{\sqrt{x^2 + y^2 + z^2}}). \tag{40}$$

(Notice that $[(A_\pm)_x] = [(A_\pm)_y] = [L]^{-1}$ since $[x] = [y] = [z] = [L]$ while $[A_\pm] = [L]^0$, L: length.)

To pull-back by ι these 1-forms to $(D_0^2)^*$ we must first restrict $A_\pm$ to $z = 0$ and then perform the pull-back operation, which reduces to the identity:

$$\iota^*(A_\pm(x, y, 0)) = \pm\frac{i}{2}(\frac{ydx - xdy}{x^2 + y^2}) := ia_\pm(x, y) \tag{41}$$

with

$$a_\pm(x, y) = \mp\frac{1}{2}(\frac{xdy - ydx}{x^2 + y^2}) \tag{42}$$

the real-valued $A - B$ potential 1-forms. Clearly, $a_\pm$ are closed ($da_\pm = 0$) but not exact since $a_\pm = \mp\frac{1}{2}d\varphi$ only for $\varphi \in (0, 2\pi)$. If we surround the thin solenoid in the $A - B$ side with closed curves $\gamma_\pm$ with $\gamma_- = -\gamma_+$, then the surrounded magnetic flux is

$$\Phi_{A-B} = \int_{\gamma_+} a_+ + \int_{\gamma_-} a_- = \int_{\gamma_+} a_+ + \int_{\gamma_-}(-a_+) = \int_{\gamma_+} a_+ - \int_{\gamma_+}(-a_+) = 2\int_{\gamma_+} a_+ = 2\int_{\gamma_+}(-\frac{1}{2}d\varphi) = -2\pi, \tag{43}$$

which coincides, up to a sign, with the flux of the monopole:

$$\Phi_D = \int_{S^2} \boldsymbol{B} = (\frac{1}{2})\int_{S^2}\frac{\hat{r}\cdot\hat{r}}{r^2} = (\frac{1}{2})4\pi = 2\pi. \tag{44}$$

But this implies that the $A - B$ effect vanishes if and only if the value of the electric charge $|e|$ is an integer: the $D.Q.C.$ for the present case where $g = \frac{1}{2}$. In fact, with $\Phi_0 = \frac{2\pi}{|e|}$ the quantum of magnetic flux associated with the charge $|e|$, the phase change of the wave function in the $A - B$ experiment due to the presence of magnetic flux is

$$e^{-i|e|\Phi_{A-B}} = e^{-2\pi i\frac{\Phi_{A-B}}{\Phi_0}} = e^{2\pi i\frac{\Phi_D}{\Phi_0}} = e^{i|e|(\frac{1}{2})4\pi} = e^{2\pi i|e|} = 1 \Leftrightarrow |e| = n \in \mathbb{Z}. \tag{45}$$

(For arbitrary g, the $D.Q.C.$ would be $|e|g = \frac{n}{2}$.)

 Copyright © 2018 Isaac Scientific Publishing

8 Final Comments

It is well known that the $A - B$ effect and the Dirac monopole are closed related [16]; in particular the disappearance of the Dirac string simultaneously with the vanishing of the $A - B$ effect when appropriate conditions of the magnetic fluxes are fulfilled [17]. In the present paper, the above relation has been described in the context of the fiber bundles associated with both phenomena, respectively ξ_{A-B} (trivial) and ξ_D (non-trivial Hopf bundle). The remarkable fact is that ξ_{A-B} turns out to be the pull-back of ξ_D by the inclusion ι of the corresponding base spaces, which allows to discuss the above relation in a purely geometric language. It would be interesting to investigate if this bundle theoretic relation exists in non-abelian cases.

Acknowledgments. The author thanks Gregory L. Naber and Sebastián Nájera Valencia for useful and enlightening discussions.

References

1. Y. Aharonov, D. Bohm, Physical Review **15**, 485 (1959).
2. P.A.M. Dirac, Proceedings of the Royal Society **A133**, 60 (1931).
3. P.A.M. Dirac, Physical Review **74**, 817 (1948).
4. R.G. Chambers, Physical Review Letters **5**, 3 (1960).
5. M. Peshkin, A. Tonomura, *The Aharonov-Bohm Effect*, (Springer, Berlin, 1989).
6. J. Preskill, Annual Review of Nuclear and Particle Science **34**, 451 (1984).
7. J. Polchinski, International Journal of Modern Physics A**19**, 145 (2004).
8. M. Aguilar, M. Socolovsky, International Journal of Theoretical Physics **41**, 839 (2002).
9. M. Socolovsky, *Encyclopedia of Mathematical Physics*, eds. J.P. Francoise, G.L. Naber, T.S. Sun; Elsevier; 191 (2006).
10. G.L. Naber, *Topology, Geometry, and Gauge Fields. Foundations*, (Springer-Verlag, N.Y., 1997).
11. N. Steenrod, *The Topology of Fibre Bundles*, (Princeton University Press, N.J., 1951).
12. M. Socolovsky, Aportaciones Matemáticas, Notas de Investigación (Sociedad Matemática Mexicana) **6**, 141 (1992).
13. H. Hopf, Mathematische Annalen **104**, 637 (1931).
14. A. Trautman, International Journal of Theoretical Physics **16**, 561 (1977).
15. D. Husemoller, *Basic Bundle Theory and K-Cohomology Invariants*, Lecture Notes in Physics **726**, (Springer, Berlin Heidelberg, 2008).
16. T.T. Wu, C.N. Yang, Physical Review D **12**, 3845 (1975).
17. A. Rajantie, Contemporary Physics **53**, 195 (2012).

Copyright © 2018 Isaac Scientific Publishing