Electrodynamics in Accelerated Frames

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Abstract. In the current paper we present a generalization of the transforms of the electromagnetic field from the frame co-moving with an accelerated particle into an inertial frame of reference. The solution is of great interest for real time applications, because earth-bound laboratories are inertial only in approximation. We conclude by deriving the general form of the relativistic Doppler effect and of the relativistic aberration formulas for the case of accelerated motion.

Keywords: Accelerated motion, general coordinate transformations, accelerated particles, planar electromagnetic waves, relativistic Doppler effect, relativistic aberration, Bremsstrahlung

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1. Introduction

Real life applications include accelerating and rotating frames more often than the idealized case of inertial frames. Our daily experiments happen in the laboratories attached to the rotating, continuously accelerating Earth. Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2-12]. In a recent pair of papers [13-14], we have presented the equations of electrodynamics in an accelerated /rotating frame as viewed from the point of view of the inertial frame of the laboratory. In the current paper, we are presenting the equations of electrodynamics in an accelerated frame as viewed from the accelerated frame. There is also great interest in producing a general solution that deals with arbitrary orientation of acceleration in the case of rectilinear motion, so we produced the equations for the general case as well. The main idea of this paper is to generate a standard blueprint for a general solution that gives equivalent of the Lorentz transforms for the case of the transforms between an inertial frame and an accelerated frame.

2. Accelerated Rectilinear Motion – the Transforms of the Electromagnetic Field

2.1. Transforms between the Accelerated Frame and the Inertial Frame

In this section we will derive the transforms between the accelerated frame and the inertial frame for the electromagnetic tensor. Let S represent an inertial system of coordinates and $S'(\tau)$ an accelerated one. According to reference [1] the transformation for the particular case of accelerated motion along the x-axis from $S'(\tau)$ into S is:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = Phy\_rectilinear \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} + \begin{bmatrix} \dfrac{c^2}{g}(\cosh \dfrac{g\tau}{c} - 1) \\ 0 \\ 0 \\ \dfrac{c}{g}\sinh \dfrac{g\tau}{c} \end{bmatrix} \tag{2.1}$$

where:
"c" is the speed of light in vacuum, "g" is the proper acceleration, τ is the proper time and

$$Phy\_rectilinear = \begin{bmatrix} \cosh\dfrac{g\tau}{c} & 0 & 0 & c\sinh\dfrac{g\tau}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{1}{c}\sinh\dfrac{g\tau}{c} & 0 & 0 & \cosh\dfrac{g\tau}{c} \end{bmatrix} \tag{2.2}$$

The electromagnetic potential, by virtue of being a 4-vector transforms the same way:

$$\begin{pmatrix} A_x \\ A_y \\ A_z \\ c\varphi \end{pmatrix} = Phy\_rectilinear \begin{pmatrix} A_x^{'} \\ A_y^{'} \\ A_z^{'} \\ c\varphi^{'} \end{pmatrix} \tag{2.3}$$

In the inertial frame, the differential Maxwell equations in vacuum, in the absence of electric charge, are [1]:

$$-E_x = \frac{\partial A_x}{\partial t} + c^2 \frac{\partial \varphi}{\partial x} \tag{2.4}$$

$$-E_y = \frac{\partial A_y}{\partial t} + c^2 \frac{\partial \varphi}{\partial y} \tag{2.5}$$

$$-E_z = \frac{\partial A_z}{\partial t} + c^2 \frac{\partial \varphi}{\partial z} \tag{2.6}$$

$$\mathbf{B} = curl\mathbf{A} \tag{2.7}$$

Let's start with (2.4):

$$\frac{\partial A_x}{\partial t} = \frac{\partial A_x^{'}}{\partial t} \cosh\frac{g\tau}{c} + \frac{\partial \varphi^{'}}{\partial t} c\sinh\frac{g\tau}{c}$$

$$\frac{\partial A_x^{'}}{\partial t} = \frac{\partial A_x^{'}}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial A_x^{'}}{\partial t'} \frac{\partial t'}{\partial t} = \frac{\partial A_x^{'}}{\partial x'}(-c\sinh\frac{g\tau}{c}) + \frac{\partial A_x^{'}}{\partial t'} \cosh\frac{g\tau}{c}$$

$$\frac{\partial \varphi^{'}}{\partial t} = \frac{\partial \varphi^{'}}{\partial x'}(-c\sinh\frac{g\tau}{c}) + \frac{\partial \varphi^{'}}{\partial t'} \cosh\frac{g\tau}{c} \tag{2.8}$$

$$\frac{\partial A_x}{\partial t} = \frac{\partial A_x^{'}}{\partial x'}(-c\sinh\frac{g\tau}{c}\cosh\frac{g\tau}{c}) + \frac{\partial A_x^{'}}{\partial t'}\cosh^2\frac{g\tau}{c} + \frac{\partial \varphi^{'}}{\partial x'}(-c^2\sinh^2\frac{g\tau}{c}) + \frac{\partial \varphi^{'}}{\partial t'}c\sinh\frac{g\tau}{c}\cosh\frac{g\tau}{c}$$

In a similar manner:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}(\frac{A_x^{'}}{c}\sin\frac{g\tau}{c} + \varphi^{'}\cos\frac{g\tau}{c})$$

$$\frac{\partial A_x^{'}}{\partial x} = \frac{\partial A_x^{'}}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial A_x^{'}}{\partial t'} \frac{\partial t'}{\partial x} = \frac{\partial A_x^{'}}{\partial x'}\cos\frac{g\tau}{c} + \frac{\partial A_x^{'}}{\partial t'}\frac{-\sin\dfrac{g\tau}{c}}{c}$$

$$\frac{\partial \varphi^{'}}{\partial x} = \frac{\partial \varphi^{'}}{\partial x'}\cos\frac{g\tau}{c} + \frac{\partial \varphi^{'}}{\partial t'}\frac{-\sin\dfrac{g\tau}{c}}{c} \tag{2.9}$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial A_x^{'}}{\partial x'}\frac{\sin\dfrac{g\tau}{c}\cos\dfrac{g\tau}{c}}{c} - \frac{\partial A_x^{'}}{\partial t'}\frac{\sin^2\dfrac{g\tau}{c}}{c^2} + \frac{\partial \varphi^{'}}{\partial x'}\cos^2\frac{g\tau}{c} - \frac{\partial \varphi^{'}}{\partial t'}\frac{\sin\dfrac{g\tau}{c}\cos\dfrac{g\tau}{c}}{c}$$

Substitute (2.8), (2.9) into (2.4):

$$-E_x = \frac{\partial A_x^{'}}{\partial t'} + c^2 \frac{\partial \varphi^{'}}{\partial x'} \tag{2.10}$$

$$E_x = E_x^{'}$$

Moving on to (2.5)

$$\frac{\partial A_y}{\partial t} = \frac{\partial A_y'}{\partial t} = \frac{\partial A_y'}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial A_x'}{\partial t'}\frac{\partial t'}{\partial t} = \frac{\partial A_y'}{\partial x'}(-c\sinh\frac{g\tau}{c}) + \frac{\partial A_y'}{\partial t'}\cosh\frac{g\tau}{c}$$

$$(2.11)$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y'}(\frac{A_x'}{c}\sinh\frac{g\tau}{c} + \varphi'\cosh\frac{g\tau}{c}) = \frac{\partial A_x'}{\partial y'}\frac{\sinh\frac{g\tau}{c}}{c} + \frac{\partial \varphi'}{\partial y'}\cosh\frac{g\tau}{c}$$

Substitute (2.11) into (2.5):

$$E_y = -(\frac{\partial A_y'}{\partial t'} + c^2\frac{\partial \varphi'}{\partial y'})\cosh\frac{g\tau}{c} + \left(\frac{\partial A_y'}{\partial x'} - \frac{\partial A_x'}{\partial y'}\right)c\sinh\frac{g\tau}{c}$$

$$E_y' = -(\frac{\partial A_y'}{\partial t'} + c^2\frac{\partial \varphi'}{\partial y'})$$

$$(2.12)$$

$$E_y = E_y'\cosh\frac{g\tau}{c} + \left(\frac{\partial A_y'}{\partial x'} - \frac{\partial A_x'}{\partial y'}\right)c\sinh\frac{g\tau}{c}$$

In a similar manner we obtain from (2.6):

$$E_z = E_z'\cosh\frac{g\tau}{c} + \left(\frac{\partial A_z'}{\partial x'} - \frac{\partial A_x'}{\partial z'}\right)c\sinh\frac{g\tau}{c} \qquad (2.13)$$

From (2.7) we obtain:

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z'}{\partial y'} - \frac{\partial A_y'}{\partial z'} = B_x' \qquad (2.14)$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial A_y'}{\partial x} - \frac{\partial A_x}{\partial y'} = (\frac{\partial A_y'}{\partial x'} - \frac{\partial A_x'}{\partial y'})\cosh\frac{g\tau}{c} - \frac{\partial A_y'}{\partial t'}\frac{\sinh\frac{g\tau}{c}}{c} -$$

$$-\frac{\partial \varphi'}{\partial y'}c\sinh\frac{g\tau}{c} = (\frac{\partial A_y'}{\partial x'} - \frac{\partial A_x'}{\partial y'})\cosh\frac{g\tau}{c} - \frac{\sinh\frac{g\tau}{c}}{c}(\frac{\partial A_y'}{\partial t'} + c^2\frac{\partial \varphi'}{\partial y'}) = \qquad (2.15)$$

$$= B_z'\cosh\frac{g\tau}{c} + E_y'\frac{\sinh\frac{g\tau}{c}}{c}$$

$$B_z' = \frac{\partial A_y'}{\partial x'} - \frac{\partial A_x'}{\partial y'} \qquad (2.16)$$

Therefore:

$$E_y = E_y'\cosh\frac{g\tau}{c} + B_z'c\sinh\frac{g\tau}{c} \qquad (2.17)$$

In a similar manner we obtain:

$$B_y = B_y'\cosh\frac{g\tau}{c} - E_z'\frac{\sinh\frac{g\tau}{c}}{c} \qquad (2.18)$$

$$B_y' = \frac{\partial A_x'}{\partial z'} - \frac{\partial A_z'}{\partial x'} \qquad (2.19)$$

$$E_z = E_z'\cosh\frac{g\tau}{c} - B_y'c\sinh\frac{g\tau}{c} \qquad (2.20)$$

Putting everything together:

$$E_x = E_x^{'}$$

$$E_y = E_y^{'} \cosh \frac{g\tau}{c} + B_z^{'} c \sinh \frac{g\tau}{c}$$

$$E_z = E_z^{'} \cosh \frac{g\tau}{c} - B_y^{'} c \sinh \frac{g\tau}{c}$$

$$B_x = B_x^{'}$$

$$\tag{2.21}$$

$$B_y = B_y^{'} \cosh \frac{g\tau}{c} - E_z^{'} \frac{\sinh \dfrac{g\tau}{c}}{c}$$

$$B_z = B_z^{'} \cosh \frac{g\tau}{c} + E_y^{'} \frac{\sinh \dfrac{g\tau}{c}}{c}$$

Notice the resemblance with the standard Lorentz transforms in [1], for example:

$$E_x = E_x^{'}$$

$$E_y = \gamma(E_y^{'} + V \frac{H_z^{'}}{c})$$

$$E_z = \gamma(E_z^{'} - V \frac{H_y^{'}}{c})$$

$$H_x = H_x^{'}$$

$$\tag{2.22}$$

$$H_y = \gamma(H_y^{'} - V \frac{E_z^{'}}{c})$$

$$H_z = \gamma(H_z^{'} + V \frac{E_y^{'}}{c})$$

2.2 Consequences

2.2.1. Maxwell laws in accelerated frame

$$\mathbf{B}' = curl\mathbf{A}'$$

$$\mathbf{E}' = -\frac{\partial \mathbf{A}'}{\partial t'} - \nabla \varphi'$$

$$\tag{2.23}$$

The above follows immediately from (2.14), (2.15) and (2.19).

2.2.2. The gauge invariance condition in a uniformly accelerated frame

$$div\mathbf{A}' + \frac{\partial \varphi'}{\partial t'} = div\mathbf{A} + \frac{\partial \varphi}{\partial t} = 0 \tag{2.24}$$

$$div\mathbf{A}' = \frac{\partial A_x^{'}}{\partial x'} + \frac{\partial A_y^{'}}{\partial y'} + \frac{\partial A_z^{'}}{\partial z'} = \frac{\partial A_x^{'}}{\partial x'} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2.25}$$

$$\frac{\partial A_x^{'}}{\partial x'} = \frac{\partial A_x}{\partial x} \cosh^2 \frac{g\tau}{c} + \frac{\partial A_x}{\partial t} \frac{\sinh \dfrac{g\tau}{c} \cosh \dfrac{g\tau}{c}}{c} - \frac{\partial \varphi}{\partial x} c \sinh \frac{g\tau}{c} \cosh \frac{g\tau}{c} - \frac{\partial \varphi}{\partial t} \sinh^2 \frac{g\tau}{c} \tag{2.26}$$

$$\frac{\partial \varphi'}{\partial t'} = -\frac{\partial A_x}{\partial x} \sinh^2 \frac{g\tau}{c} - \frac{\partial A_x}{\partial t} \frac{\sinh \dfrac{g\tau}{c} \cosh \dfrac{g\tau}{c}}{c} + \frac{\partial \varphi}{\partial x} c \sinh \frac{g\tau}{c} \cosh \frac{g\tau}{c} + \frac{\partial \varphi}{\partial t} \cosh^2 \frac{g\tau}{c} \tag{2.27}$$

$$\frac{\partial A_x^{'}}{\partial x'} + \frac{\partial \varphi'}{\partial t'} = \frac{\partial A_x}{\partial x} + \frac{\partial \varphi}{\partial t} \tag{2.28}$$

Equality (2.28) results into:

$$div\mathbf{A}' + \frac{\partial \varphi'}{\partial t'} = div\mathbf{A} + \frac{\partial \varphi}{\partial t} = 0 \tag{2.29}$$

Equalities (2.23) result into Maxwell's wave equations having the same exact form in the accelerated frame as the equations in the inertial frames with the immediate consequence that light speed in vacuum in a uniformly accelerated frame is "c". Indeed, (2.23) results into:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}'}{\partial t'^2} - \nabla^2 \mathbf{E}' = 0$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}'}{\partial t'^2} - \nabla^2 \mathbf{B}' = 0$$

$$(2.30)$$

The above means that electromagnetic waves propagate in the accelerated frame, in vacuum, at the same speed as they propagate in inertial frames. We will re-derive this interesting result in a different way, alongside with several other interesting consequences, in the next section.

2.2.3. The Lorentz force in an accelerated frame

In the inertial frame, the Lorentz force has the expression:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{v} = v_x \mathbf{e}_x$$

$$(2.31)$$

$$\mathbf{F} = q(E_x \mathbf{e}_x + E_y \mathbf{e}_y + E_z \mathbf{e}_z + \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ v_x & 0 & 0 \\ B_x & B_y & B_z \end{bmatrix}) = q(E_x \mathbf{e}_x + (E_y - v_x B_z)\mathbf{e}_y + (E_z + v_x B_y)\mathbf{e}_z)$$

We know that:

$$dx = dx'\cosh\frac{g\tau}{c} + cdt'\sinh\frac{g\tau}{c}$$

$$dt = \frac{1}{c}\sinh\frac{g\tau}{c}dx' + dt'\cosh\frac{g\tau}{c}$$

$$(2.32)$$

$$v_x = \frac{dx}{dt} = \frac{\dfrac{dx'}{dt'}\cosh\dfrac{g\tau}{c} + c\sinh\dfrac{g\tau}{c}}{\dfrac{1}{c}\sinh\dfrac{g\tau}{c}\dfrac{dx'}{dt'} + \cosh\dfrac{g\tau}{c}} = \frac{v_x'\cosh\dfrac{g\tau}{c} + c\sinh\dfrac{g\tau}{c}}{\dfrac{v_x'}{c}\sinh\dfrac{g\tau}{c} + \cosh\dfrac{g\tau}{c}}$$

The formula:

$$v_x = \frac{v_x'\cosh\dfrac{g\tau}{c} + c\sinh\dfrac{g\tau}{c}}{\dfrac{v_x'}{c}\sinh\dfrac{g\tau}{c} + \cosh\dfrac{g\tau}{c}}$$

$$(2.33)$$

ties the speed v_x' of the particle in the accelerated frame to its measured speed in the inertial frame v_x. Using the above, we obtain:

$$E_y - v_x B_z = E_y'\cosh\frac{g\tau}{c} + B_z'c\sinh\frac{g\tau}{c} - (\frac{E_y'}{c}\sinh\frac{g\tau}{c} + B_z'\cosh\frac{g\tau}{c})\frac{v_x'\cosh\dfrac{g\tau}{c} + c\sinh\dfrac{g\tau}{c}}{\dfrac{v_x'}{c}\sinh\dfrac{g\tau}{c} + \cosh\dfrac{g\tau}{c}} =$$

$$= \frac{E_y' - v_x'B_z'}{\dfrac{v_x'}{c}\sinh\dfrac{g\tau}{c} + \cosh\dfrac{g\tau}{c}}$$

$$(2.34)$$

Similarly:

$$E_z + v_x B_y = \frac{E_z' + v_x'B_y'}{\dfrac{v_x'}{c}\sinh\dfrac{g\tau}{c} + \cosh\dfrac{g\tau}{c}}$$

$$(2.35)$$

So, we can write:

$$\mathbf{F}_{\parallel}^{'} = \mathbf{F}_{\parallel} = qE_x\mathbf{e}_x$$

$$\mathbf{F}_{\perp}^{'} = \frac{\mathbf{F}_{\perp}}{\dfrac{v_x^{'}}{c}\sinh\dfrac{g\tau}{c} + \cosh\dfrac{g\tau}{c}} = q\frac{(E_y^{'} - v_x^{'}B_z^{'})\mathbf{e}_y + (E_z^{'} + v_x^{'}B_y^{'})\mathbf{e}_z}{\dfrac{v_x^{'}}{c}\sinh\dfrac{g\tau}{c} + \cosh\dfrac{g\tau}{c}} \qquad (2.36)$$

Expressions (2.36) represent the transformation of the Lorentz force between the inertial and the accelerated frame.

2.2.4. Bremsstrahlung

Bremsstrahlung is the electromagnetic radiation produced by the deceleration of a charged particle. The moving particle loses kinetic energy, which is converted into a photon, it is the process of producing the energy radiation. Bremsstrahlung has a continuous spectrum, which becomes more intense and whose peak intensity shifts toward higher frequencies as the change of the energy of the decelerated particles increases. The term is frequently used in the more narrow sense of radiation from electrons (from whatever source) slowing in matter. In astrophysics, Bremsstrahlung refers to radiation emitted from zones of the universe characterized by a high concentration of plasma. The radiation in this case is created by charged particles that are free; i.e., not part of an ion, atom or molecule, both before and after the deflection (acceleration) that caused the emission. In any case, the total radiated power is given by [15]

$$P = \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}(\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2)$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\vec{\beta} = \frac{\vec{v}}{c} \qquad (2.37)$$

$$\dot{\vec{\beta}} = \frac{\vec{g}}{c}$$

In the case where velocity is parallel to acceleration (for example, linear motion), the formula simplifies to [15]:

$$P = \frac{q^2 g^2 \gamma^6}{6\pi\varepsilon_0 c^3} \qquad (2.38)$$

For the case of acceleration perpendicular to the velocity (as in the case of synchrotrons), the formula simplifies to:

$$P = \frac{q^2 g^2 \gamma^4}{6\pi\varepsilon_0 c^3} \qquad (2.39)$$

In either case, we do not observe any significant X-rays radiated from the free electrons in the Earth atmosphere due to several factors:

- the speed of the electrons is low (γ is small, very close to unity),
- the deceleration "g" is very small due to the absence of fields or matter that could affect the free electrons,
- the electron density per unit of volume is small,
- the presence of c^{-3}.

By contrast, this is not the case in astrophysics, where we have observed significant emission from certain galaxies' intra-cluster medium due to thermal bremsstrahlung. This radiation is in the energy range of X-rays and can be easily observed with space-based telescopes such as Chandra X-ray Observatory, XMM-Newton, ROSAT, ASCA, EXOSAT, Suzaku, RHESSI and future missions like IXO [16] and Astro-H [17]. The reasons for observing such effects are:

- much higher charge density,
- much larger speeds and accelerations (due to the presence of strong magnetic fields).

In addition to the changes in frequency (energy), we also observe light polarization effects, due to the presence of the magnetic fields mentioned above.

3. Planar Wave Transformation and Speed of Light in a Uniformly Accelerated Frame

In this section we apply the formalism derived in the previous paragraph in order to obtain the transform of a planar wave. Assume that a planar wave is propagating along the y' axis in the accelerated frame $S'(\tau)$. The wave has the electric component $\mathbf{E}'_x$ and the magnetic component $\mathbf{B}'_z$ along the x' and z' axes, respectively. The components equations are (see fig.1):

$$\mathbf{E}'_x = E'_{0x}\cos(\omega't'-k'_y y'+\varphi')\mathbf{e}_x$$
$$\mathbf{B}'_z = B'_{0z}\cos(\omega't'-k'_y y'+\varphi')\mathbf{e}_z$$

(3.1)

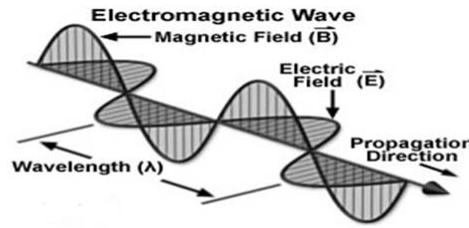**Figure 1**.The planar electromagnetic wave

Inverting transforms (2.1) we obtain:

$$y' = y$$
$$t' = -\frac{x}{c}\sinh\frac{g\tau}{c} + t\cosh\frac{g\tau}{c} - \frac{a}{c}\sinh\frac{g\tau}{c} + b\cosh\frac{g\tau}{c}$$
$$a = \frac{c^2}{g}(\cosh\frac{g\tau}{c} - 1)$$
$$b = \frac{c}{g}\sinh\frac{g\tau}{c}$$

(3.2)

Substituting (3.2) into (3.1) we obtain:

$$E'_x = E'_{0x}\cos[(\omega'\cosh\frac{g\tau}{c})t - (\frac{\omega'}{c}\sinh\frac{g\tau}{c})x - k'_y y + \varphi' + \omega'(b\cosh\frac{g\tau}{c} - \frac{a}{c}\sinh\frac{g\tau}{c})]$$

(3.3)

On the other hand, in frame S, the wave equation is:

$$\mathbf{E}_x = E_{0x}\cos(\omega t - k_x x - k_y y - k_z z + \varphi)\mathbf{e}_x$$

(3.4)

Since $\mathbf{E}_x = \mathbf{E}'_x$ it follows that:

$$E'_{0x} = E_{0x}$$
$$\omega = \omega'\cosh\frac{g\tau}{c}$$
$$k_x = \frac{\omega'}{c}\sinh\frac{g\tau}{c} = \frac{\omega}{c}\tanh\frac{g\tau}{c}$$
$$k_y = k'_y$$
$$k_z = 0$$
$$\varphi = \varphi' + \omega'(b\cosh\frac{g\tau}{c} - \frac{a}{c}\sinh\frac{g\tau}{c})$$

(3.5)

The formula $\omega = \omega' \cosh \dfrac{g\tau}{c}$ represents the Doppler effect due to acceleration. We see that the pulsation decreases in time by the factor $\cosh \dfrac{g\tau}{c}$ in a frame that is uniformly accelerated in the same direction of the propagation of the electromagnetic wave.

From (3.8) we obtain:

$$\frac{\omega^2}{c^2} = k^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \tanh^2 \frac{g\tau}{c} + k_y'^2 \qquad (3.6)$$

Therefore, in frame $S'(\tau)$ the wave vector is:

$$k' = k_y' = \frac{\omega}{c \cosh \dfrac{g\tau}{c}} \qquad (3.7)$$

We can now calculate the phase light speed in the accelerated frame:

$$v_p' = \frac{\omega'}{k'} = \frac{\omega}{k} = c \qquad (3.8)$$

So, the light speed in the accelerated frame equals the light speed in the inertial frame, c.

We can now proceed to calculating the amplitude and the phase transformation between the inertial and the accelerated frame:

$$\varphi = \varphi' + \omega'(b \cosh \frac{g\tau}{c} - \frac{a}{c} \sinh \frac{g\tau}{c}) = \varphi' + \frac{\omega c}{g} \tanh \frac{g\tau}{c}$$

$$\varphi' = \varphi - \frac{\omega c}{g} \tanh \frac{g\tau}{c} \qquad (3.9)$$

The magnetic field component transforms as:

$$B_{0z} = B_{0z}' \cosh \frac{g\tau}{c} \qquad (3.10)$$

So, the absolute value of the Poynting vector transforms as:

$$S = E_{0x} B_{0z} = S' \cosh \frac{g\tau}{c} \qquad (3.11)$$

So the electromagnetic flux in the accelerated frame decreases by with respect to the flux in the inertial frame. Finally, we can calculate the aberration in frame S induced by the acceleration:

$$\cos \theta = \frac{k_x}{\sqrt{k_x^2 + k_y^2}} = \tanh \frac{g\tau}{c} \qquad (3.12)$$

4. General Case of Uniform Acceleration in an Arbitrary Direction

In a prior paper we have shown [12] that the particular transformation (2.1) can be generalized for the case of arbitrary direction constant acceleration $\mathbf{g} = (g_x, g_y, g_z)$ to:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = (Tr^{-1} * Phy\_rectilinear * Tr) \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} + N \qquad (4.1)$$

where:

$$Tr = Rot(\mathbf{e}_z)_{-90^0} * Rot(\mathbf{e}_y)_{90^0 - \varphi} * Rot_y \qquad (4.2)$$

Introducing the triplet $(a, b, c) = (-\dfrac{g_z}{g}, 0, \dfrac{g_x}{g})$ the following expressions hold:

$$Rot_{y} = \begin{bmatrix} c & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.3}$$

$$Rot(\mathbf{e}_{y})_{90^{0}-\varphi} = \begin{bmatrix} \cos(90^{0}-\varphi) & 0 & -\sin(90^{0}-\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(90^{0}-\varphi) & 0 & \cos(90^{0}-\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sin\varphi & 0 & -\cos\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.4}$$

$Rot(\mathbf{e}_{y})_{90^{0}-\varphi} * Rot_{y}$ aligns $\mathbf{g}$ with $\mathbf{e}_{y}$. The second step is comprised by another rotation around the z-axis by -90^{0} that aligns $\mathbf{g}$ with $\mathbf{e}_{x}$ (fig.2):

$$Rot(\mathbf{e}_{z})_{-90^{0}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.5}$$
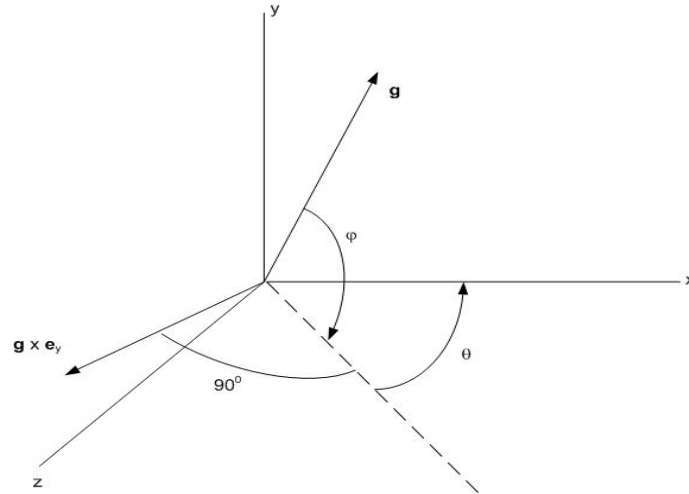

Figure 2 General acceleration

Tr^{-1} reverses all the effects of Tr. Expression (4.1) gives the solution for the general case, of arbitrary acceleration direction. The net effect is that the derivative operators become more complicated:

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t'} \tag{4.6}$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x'} \tag{4.7}$$

$$\begin{bmatrix} \dfrac{\partial x}{\partial x'} & \dfrac{\partial x}{\partial y'} & \dfrac{\partial x}{\partial z'} & \dfrac{\partial x}{\partial t'} \\ \dfrac{\partial y}{\partial x'} & \dfrac{\partial y}{\partial y'} & \dfrac{\partial y}{\partial z'} & \dfrac{\partial y}{\partial t'} \\ \dfrac{\partial z}{\partial x'} & \dfrac{\partial z}{\partial y'} & \dfrac{\partial z}{\partial z'} & \dfrac{\partial z}{\partial t'} \\ \dfrac{\partial t}{\partial x'} & \dfrac{\partial t}{\partial y'} & \dfrac{\partial t}{\partial z'} & \dfrac{\partial t}{\partial t'} \end{bmatrix} = Tr^{-1} * Phy\_rectilinear * Tr \tag{4.8}$$

Using (4.8) in (4.6), (4.7) gives the general forms of the transforms (2.21) for the electromagnetic field tensor. Next, we will show a very nice way of getting the general transforms. We start by writing (2.21) in the form:

$$
\begin{bmatrix} E_x & 0 & 0 & 0 \\ * & \dfrac{E_y}{c} & B_z & * \\ B_y & * & * & \dfrac{E_z}{c} \\ 0 & 0 & 0 & B_x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\dfrac{g\tau}{c} & \sinh\dfrac{g\tau}{c} & 0 \\ 0 & -\sinh\dfrac{g\tau}{c} & \cosh\dfrac{g\tau}{c} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E'_x & 0 & 0 & 0 \\ \dfrac{E'_z}{c} & \dfrac{E'_y}{c} & B'_z & B'_y \\ B'_y & B'_z & \dfrac{E'_y}{c} & \dfrac{E'_z}{c} \\ 0 & 0 & 0 & B'_x \end{bmatrix} \tag{4.9}
$$

The elements marked with asterisks represent entities without any meaning. We do not care about them. Then, the matrix for the general transform is simply:

$$
Tr^{-1} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\dfrac{g\tau}{c} & \sinh\dfrac{g\tau}{c} & 0 \\ 0 & -\sinh\dfrac{g\tau}{c} & \cosh\dfrac{g\tau}{c} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * Tr \tag{4.10}
$$

5. Conclusions

We constructed the general transforms from the frame $S'(\tau)$ co-moving with an accelerated particle in rectilinear motion into an inertial frame of reference S. The solution is of great interest for real life applications, because our earth-bound laboratories are inertial only in approximation; in real life, the laboratories are accelerated. We produced a blueprint for generalizing the solutions for the arbitrary cases and we concluded with an application that explains the general case of planar electromagnetic waves. A very interesting consequence is the fact that light speed in vacuum in the accelerated frames is "c". A second interesting consequence is that acceleration induces aberration.

Acknowledgments

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