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The Linear Stability of Collinear Equilibrium Points and Resonances

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Abstract The stability properties, dynamical processes and factors affecting these are very important aspects to describing the behaviour of a dynamical system because these play a significant role in the study of their past evolution. The present article discuss about the existence of collinear equilibrium points, their computation and stability analysis in the Chermnykh-Like problem under the influence of perturbations in the form of radiation pressure, oblateness and a disc. In the presence of the disc, there exists a new collinear equilibrium point in addition to the three points of the classical problem. We examine the linear stability of the collinear equilibrium points with respect to disc's outer radius b instead of mass parameter μ and it is found that all the collinear equilibrium points are unstable except L_3 which is stable for $b \in (1.3312, 1.5275)$ provided that remaining parameters are fixed. Further, we obtain stability regions and perturbed mass ratio in the case of three main resonances for L_3 under appropriate approximations. We analyze the effect of the perturbations numerically and it is observed that they significantly affect the motion of infinitesimal mass. The results are limited up to the regular symmetric disc and the radiation effect of the bigger primary but further it can be extended for more generalized cases.

Keywords: Chermnykh-Like problem, collinear equilibrium points, disc, linear stability, resonances.

1 Introduction

A significant development in the field of nonlinear dynamics and recent advancement in technologies to observe the solar system have attracted a number of researchers toward the study about dynamics of the celestial bodies in the form of different celestial problems. Among these, a well known problem is Chermnykh-like problem which is also known as modified Chermnykh's problem. Chermnykh problem was first time studied by [1] hence, the name. This problem consists of the motion of a particle in the gravity field of a uniform rotating dumbbell. Chermnykh-like problem generalizes two classical problem of the celestial mechanics known as two fixed center problem and the restricted problem of three body. Existence of a disc or belt like structure in the problems of celestial mechanics is a common phenomenon. For example, in the solar system there are asteroid belt and Kuiper belt [2], [3], in the extra- solar planetary systems the disc of dust [4], [5], [6] and circumbinary rings in case of binary system [7]. The disc or belt-like structure affects the dynamical evolution of these systems. It can change the location of equilibrium points and also the orbital behaviors. The influence of the belt for planetary system is observed by [2], [3] and found that probability of equilibrium points is larger, near the inner part of the belt than outer one. Different aspects of the problem with and without disc are studied by many authors as [8], [9], [10], [7], [11], [12], [13], [14], [15], [16] etc., under some assumptions on mass parameter and angular velocity of the system. They have found that at the mass parameter $\mu = 0.5$, there is a deviation in classical results [10], whereas in the presence of a belt, there exist new equilibrium points [7], [13], called Jiang-Yeh points [14], [17], [18]. Moreover, it is also observed that analysis of linear stability is different [10].

The classical problem has no more existence when the massive bodies interact under the influence of perturbations in the form of radiation pressure and oblateness of the primaries. In the solar system as well as in the extra solar planetary system, there are several dynamics problems where it is inadequate to consider only gravitational field. For example, when a radiating body i.e. star in the space, acts upon an infinitesimal mass in a cloud of dust or gas then in addition to gravitational force due to star, radiation pressure force [19] and drag forces also work there. Thus, problem is modified by superimposing a repulsive field due to radiation whose source is same as of gravitational field. On the other hand, unlike to classical problem, it is assumed that masses are not spherically symmetrical in homogeneous layer because there are several celestial bodies (Saturn, Jupiter, Earth etc.) which are sufficiently oblate. In literature, a number of researchers as [20], [21], [22], [23], [24], [25], [13], [26], [14] etc. have studied different aspects of restricted three body problem (RTBP), by taking one or both primary as source of radiation or oblate spheroid or both and have discussed their effects on the motion of infinitesimal mass. [27] have found non-linear stability zones around triangular equilibrium point in the RTBP with oblateness. [11], [12] have studied generalized photogravitational Chermnykh-like problem with P-R drag and found that triangular points are stable under Routh's condition but collinear equilibrium points are unstable whereas, [28] have discussed about linear stability of triangular equilibrium point and resonances. [15] analyze the influence of radiation pressure on the nature of orbits whereas, [16] discuss the combined effects of oblateness, radiation and power-law profile.

Authors are interested in the problem due to a number of recent applications in addition to past one, in different areas such as celestial mechanics, chemistry and extra solar planetary system [29], [4], [5], [2]. Since, the collinear equilibrium points play a crucial role in space dynamics. For example, collinear point L_1 of the Sun-Earth system is home to the SOHO spacecraft and L_2 point is the home of WMAP Spacecraft and James Wave Space Telescope, moreover, many important missions of NASA are considered in the basin of these collinear points [30]. Hence, authors are much interested in the analysis of linear stability of the collinear equilibrium points.

Present paper is organized as follows: existence and determination of the collinear equilibrium points are described in section-3 after formulation of the model as in section-2. Analysis of linear stability and resonance cases, in the vicinity of collinear equilibrium points under the influence of perturbations, are presented in section-5 and section-4, respectively. All the numerical as well as algebraic computation have performed with the help of MATHEMATICA [31] software package. For numerical treatment, we have used following values of parameters: Mass parameter $(\mu) = 9.53728 \times 10^{-4}$ of the Sun-Jupiter system; mass reduction factor $(q_1) = 0.9985$ and oblateness coefficients $A_2 = 0.03$; total mass of the disc $(M_b) = 0.02$ [14] which is obtained by taking disc's inner radius $(a) = 1.30$ and outer radius $(b) = 1.33$, respectively with thickness of the disc $(h) = 0.0001$ and control factor of density profile $(c) = 1910.83$ of the disc, whereas to observed the effects of perturbations, we have taken appropriate parametric values in the neighborhood of above mentioned values. Finally, paper is concluded in section- 6.

2 Equations of Motion

We consider Chermnykh-like problem under the influence of perturbations in the form of radiation pressure (first primary), oblateness (second primary) and a disc with power law density profile which is rotating about the common center of mass of the system. Units of distance and mass are taken as the distance between both the primaries and sum of their masses, respectively whereas, unit of time is taken as time period of the rotating frame. Suppose $P(x, y, 0)$, $A(-\mu, 0, 0)$ and $B(1-\mu, 0, 0)$ be the co-ordinates of infinitesimal mass, first primary and second primary respectively, with respect to a rotating frame, where $\mu = \frac{M_P}{M_S+M_P}$ is the mass parameter (M_S and M_P are masses of the Sun and the Planet, respectively).

Thus, equations of motion of the infinitesimal mass in xy-plane, are written as in [13]:

$$\ddot{x} - 2n\dot{y} = U_x, \tag{1}$$

$$\ddot{y} + 2n\dot{x} = U_y, \tag{2}$$

$$\text{where} \quad U_x = n^2 x - \frac{(1-\mu)q_1(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} - \frac{3\mu A_2(x+\mu-1)}{2r_2^5} + \frac{x}{r} f_b(r), \tag{3}$$

$$U_y = n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3\mu A_2 y}{2r_2^5} + \frac{y}{r} f_b(r), \tag{4}$$

with $r_1 = \sqrt{(x+\mu)^2 + y^2}$, $r_2 = \sqrt{(x+\mu-1)^2 + y^2}$, $r = \sqrt{x^2+y^2}$. Mean motion of the system $n = \sqrt{q_1 + \frac{3}{2}A_2 - 2f_b(r)}$ with mass reduction factor $q_1 = (1 - \frac{F_p}{F_g})$ [19], where F_p and F_g are the radiation pressure and gravitational forces of the radiating body, respectively. Oblateness $A_2 = \frac{R_e^2 - R_p^2}{5R^2}$ [32], where R_e and R_p are equatorial and polar radii of oblate body, respectively and R is distance between both the primary. Gravitational force due to the disc is given as in [10]:

$$f_b(r) = -2\int_{r'} \frac{\rho(r')r'}{r}\left[\frac{E(\xi)}{r-r'} + \frac{F(\xi)}{r+r'}\right]dr', \tag{5}$$

where $F(\xi)$ and $E(\xi)$ are elliptic integrals of first and second kind, respectively with $\xi = 2\frac{\sqrt{rr'}}{r+r'}$ and r' is the disc reference radius. It is supposed that the power law density profile of the disc having thickness $h \approx 10^{-4}$, is $\rho(r) = \frac{c}{r^p}$, where $p \in \mathbb{N}$, (in particular, we take $p = 3$) and c is a constant which depends on total mass of the disc. Series expansion of elliptic integrals $F(\xi)$ and $E(\xi)$ over $a \le r' \le b$ with an appropriate approximation yields

$$f_b(r) = -\pi ch\left[\frac{2(b-a)}{abr^2} + \frac{3\log(\frac{b}{a})}{8r^3}\right], \tag{6}$$

where a, b are inner and outer radii of the radially symmetric disc, respectively.

3 Collinear Equilibrium Points

Equilibrium point is a point at which velocity and hence, acceleration of the infinitesimal body vanishes. Thus, keeping in mind $\dot{x} = 0 = \dot{y} = \ddot{x} = \ddot{y}$, we obtain the equilibrium point of the problem by solving equations (1-2) i.e. $U_x = 0 = U_y$, for x and y simultaneously. In other words,

$$U_x = n^2x - \frac{q_1(1-\mu)(x+\mu)}{r_1^3} - \left(1+\frac{3A_2}{2r_2^2}\right)\frac{\mu(x+\mu-1)}{r_2^3} - \pi ch\left[\frac{2(b-a)}{abr^3} + \frac{3\log(\frac{b}{a})}{8r^4}\right]x = 0, \tag{7}$$

$$U_y = n^2y - \frac{(1-\mu)q_1y}{r_1^3} - \left(1+\frac{3A_2}{2r_2^2}\right)\frac{\mu y}{r_2^3} - \pi ch\left[\frac{2(b-a)}{abr^3} + \frac{3\log(\frac{b}{a})}{8r^4}\right]y = 0. \tag{8}$$

From equation (8), it is obvious that either $y = 0$ or remaining factor is equal to zero. If $y = 0$ then equilibrium points are called as collinear equilibrium points which lie on the line joining primaries. If $y \neq 0$ then these points known as triangular equilibrium points.

As we are interested in collinear points, therefore, $y = 0$ and hence, equation (8) is unimportant. Let $U_x(y = 0) = K(x)$, then from equation (7), we get

$$\begin{aligned} K(x) = n^2x - \frac{(1-\mu)q_1(x+\mu)}{|x+\mu|^3} - \frac{\mu(x+\mu-1)}{|x+\mu-1|^3} - \frac{3\mu A_2(x+\mu-1)}{2|x+\mu-1|^5} \\ -\pi ch\left[\frac{2(b-a)}{ab|x|^3} + \frac{3\log(\frac{b}{a})}{8|x|^4}\right]x = 0. \end{aligned} \tag{9}$$

Three possible positions (say x), of the infinitesimal mass on the line joining the primaries are as follows: (1) $(1-\mu) < x < \infty$, (2) $-\mu < x < (1-\mu)$ and (3) $-\infty < x < -\mu$. For the simplicity, we have divided interval (2) in to two sub intervals: $0 < x < (1-\mu)$ and $-\mu < x < 0$. Hence, there are four cases to analyze the collinear equilibrium points.

In case (1), where $x \in (1-\mu, \infty)$, we have $x+\mu-1 > 0$ and $x+\mu > 0$. Therefore, from equation (9), we find that $\lim_{x\to(1-\mu)^+} K(x) < 0$, $\lim_{x\to\infty^-} K(x) > 0$ (Fig. 1a) and $K'(x) > 0$ (Fig. 1b) which shows that the function $K(x)$ is an increasing function of x having distinct sign in the interval $1-\mu < x < \infty$ this implies that there exists a point x_{L_2} (say), in $(1-\mu, \infty)$ such that $K(x_{L_2}) = 0$. Similarly, in case (2), when $x \in (0, 1-\mu)$, then $x+\mu-1 < 0$ and $x+\mu > 0$. Thus, from equation (9) we get, $\lim_{x\to0^+} K(x) < 0$, $\lim_{x\to(1-\mu)^-} K(x) > 0$ (Fig. 2a) and $K'(x) > 0$ for all $x \in (0, 1-\mu)$ (Fig. 2b) and hence, there exists a point (say) $x_{L_1} \in (0, 1-\mu)$ such that $K(x_{L_1}) = 0$. Next, when $x \in (-\mu, 0)$, then

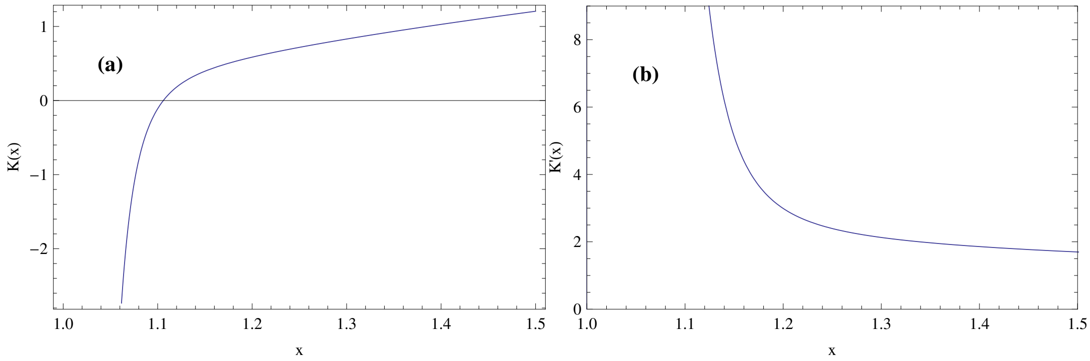

Figure 1. Collinear equilibrium point L_2: (a) $K(x)$ vs x and (b) $K'(x)$ vs x.

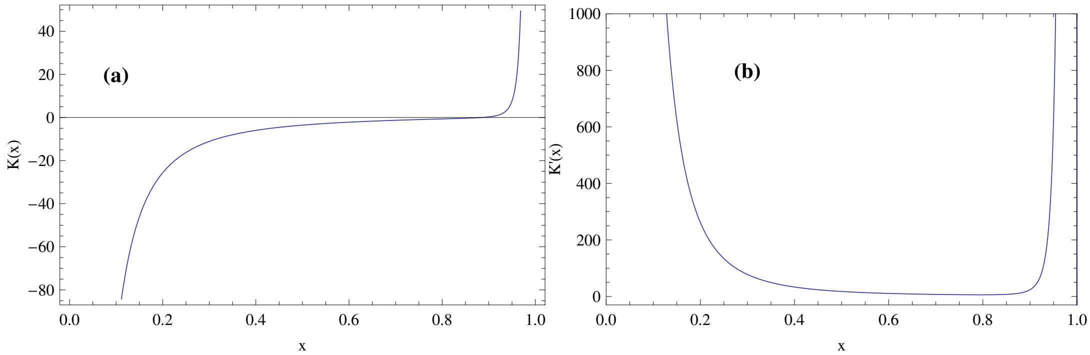

Figure 2. Collinear equilibrium point L_1: (a) $K(x)$ vs x and (b) $K'(x)$ vs x.

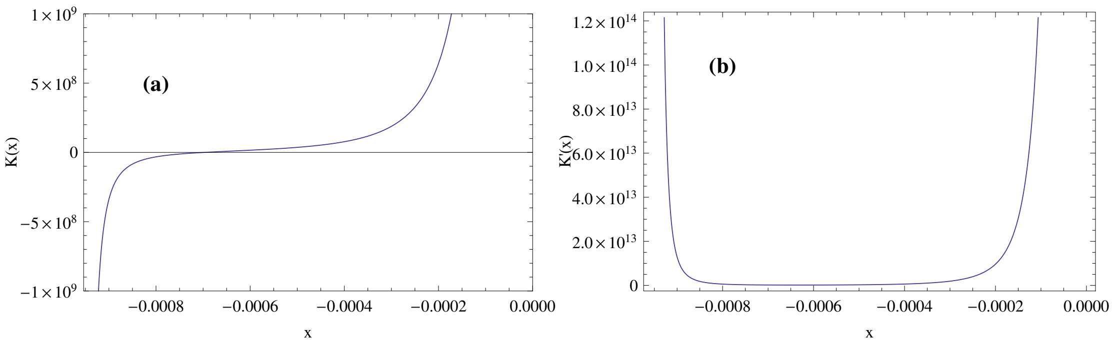

Figure 3. Collinear equilibrium point $JY1$: (a) $K(x)$ vs x and (b) $K'(x)$ vs x.

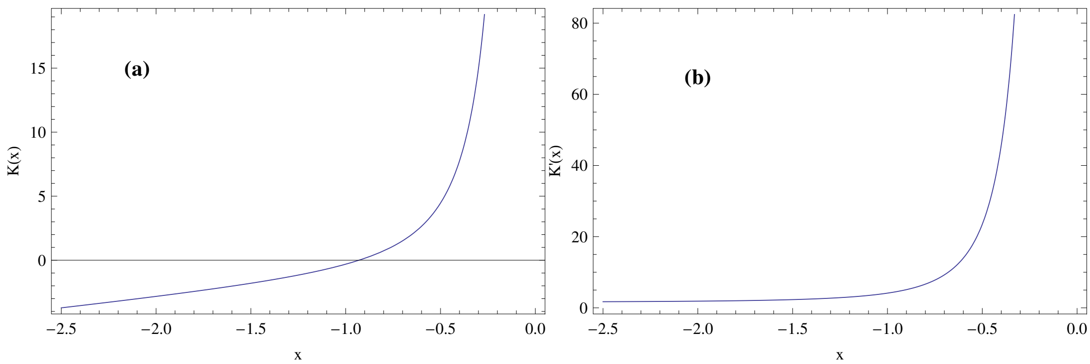

Figure 4. Collinear equilibrium point L_3: (a) $K(x)$ vs x and (b) $K'(x)$ vs x.

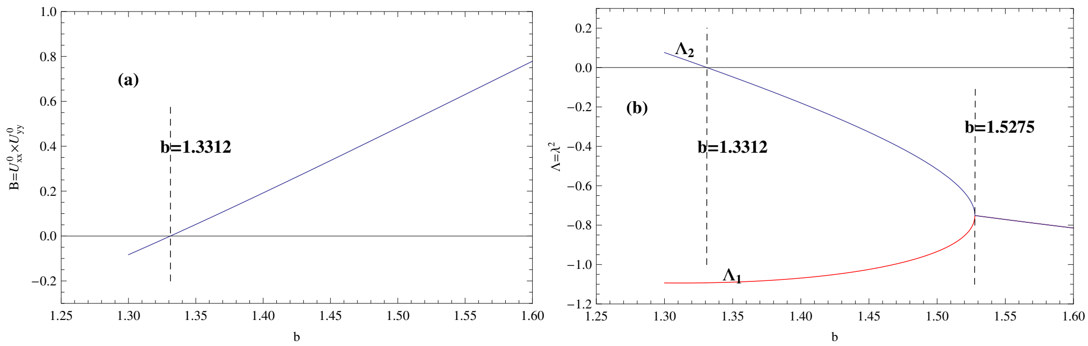

Figure 5. Sign of $B = U^0_{xx}U^0_{yy}$ and $\Lambda_{1,2}$ in case of L_3: (a) $U^0_{xx}U^0_{yy}$ vs b and (b) $\Lambda_{1,2}$ vs b.

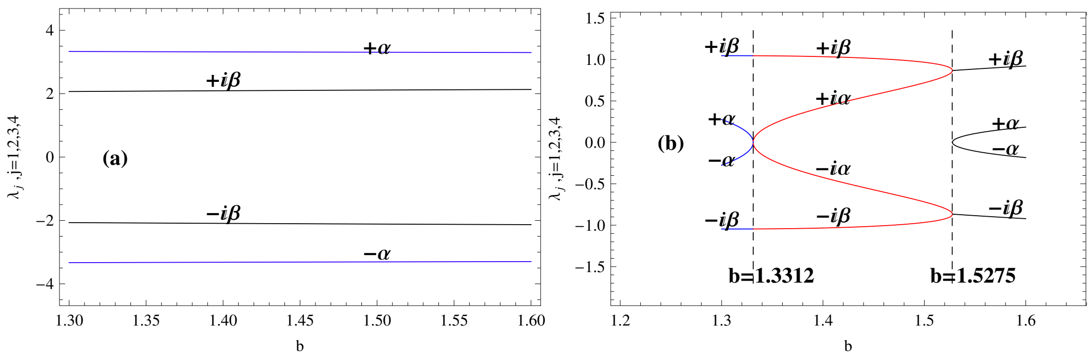

Figure 6. Real and imaginary components of the roots of the characteristic equation in space $\lambda_j - b$ for collinear equilibrium points (a) L_1 and (b) L_3.

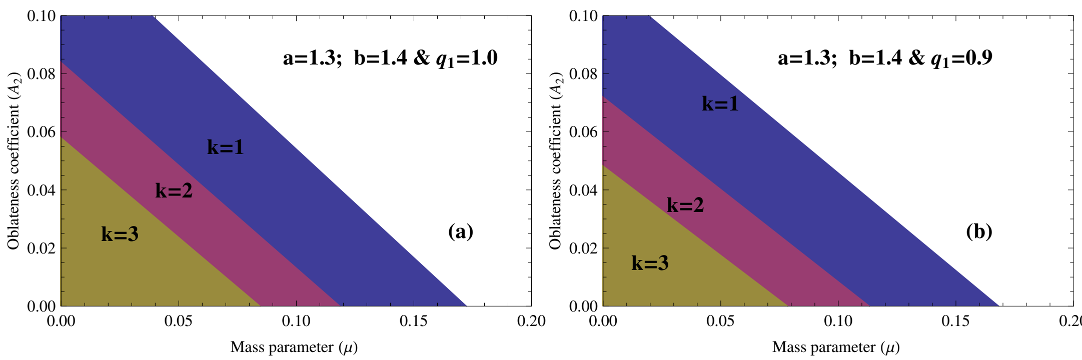

Figure 7. Stability region of collinear equilibrium point L_3 in the space $\mu - A_2$ and the resonance curves $\omega_1 - \kappa\omega_2 = 0$, $\kappa = 1, 2, 3$. (a) without radiation effect and (b) with radiation effect.

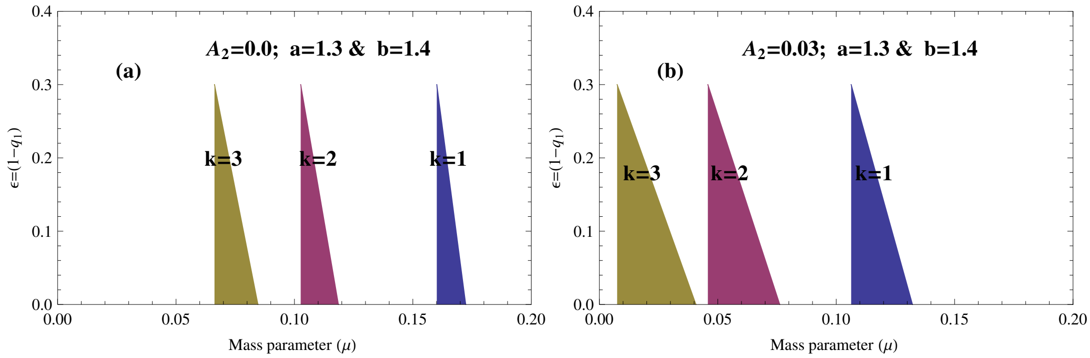

Figure 8. Stability region of collinear equilibrium point L_3 in the space $\mu - \epsilon$ and the resonance curves $\omega_1 - \kappa\omega_2 = 0$, $\kappa = 1, 2, 3$. (a) without oblateness and (b) with oblateness.

we again have $x + \mu - 1 > 0$ and $x + \mu > 0$ but x is negative and less than μ. Therefore, from equation (9) we have, $\lim_{x\to 0^-} K(x) < 0$, $\lim_{x\to -\mu^-} K(x) > 0$ (Fig. 3a) and $K'(x) > 0$ for all $x \in (-\mu, 0)$ (Fig. 3b). Hence, there exists a point (say) $x = x_{JY1} \in (-\mu, 0)$ for which $K(x_{JY1}) = 0$. Finally, in case (3) where $x \in (-\infty, -\mu)$, we have $x + \mu - 1 < 0$ and $x + \mu < 0$. Thus, again from equation (9), we obtain $\lim_{x\to -\mu^-} K(x) > 0$, $\lim_{x\to -\infty^+} K(x) < 0$ (Fig. 4a) and $K'(x)$ have different sign for all $x \in (-\infty, -\mu)$ (Fig. 4b). So, there exists a point (say) $x_{L_3} \in (-\infty, -\mu)$ such that $K(x_{L_3}) = 0$. The numerical values of the collinear equilibrium points are computed at different values of parameter and results are displayed in Table 1.

From Table 1, it is clear that the positions of collinear points L_1, L_2 and L_3 have tendency to move towards origin with increment in the value of b, but when q_1 increases, L_1 and L_3 (except L_3 for $a = b = 1.30$) move away from the origin, whereas L_2 shifts toward origin. Again, when A_2 increases L_1 and L_3 move toward origin but L_2 goes away from origin. On the other hand, new collinear point $JY1$ depends significantly on the disc's outer radius b and has tendency to move away from the center of mass of the system, whereas influence of q_1 and A_2 on $JY1$ is very less. A slight change in the position of $JY1$ with q_1 is seen after fifth places of the decimal, whereas changes in the position of $JY1$ with A_2 is not seen even up to ninth places of the decimal (Table 1). Hence, we can say that the existence of $JY1$ is due to the presence of the disc in the system.

4 Stability Analysis

In the previous section we have obtained equilibrium points. The next task is to examine the stability of these points. For this, in general, it is enough to observe the shape of the effective potential and find out, if the points occur at hills, valley or saddles. However, this criterion fails due to presence of velocity dependent term in the effective potential. Instead, we must perform a linear stability test in the neighborhood of each equilibrium point. The motion of infinitesimal body about an equilibrium point is said to be stable if given a small displacement with a very small velocity to the infinitesimal body then it should oscillate for considerable time around that equilibrium point. If the infinitesimal body departs from that point for time $t > 0$ then the motion is said to be unstable. In other words, to insure the stability, displacement and velocity should be bounded functions of time in the vicinity of equilibrium point. The stability analysis of collinear equilibrium points is very important for the space observations. As, one of the important consideration of NASA missions, is the stability of collinear points, specially L_1 and L_2 points of the Earth-Sun system. Therefore, we are interested in discussing the collinear equilibrium points under the frame of Sun-Jupiter system in the presence perturbations.

Since, all collinear equilibrium points are unstable with respect to the mass parameter μ, alike in classical case i.e. when perturbations are ignored (because of existence of at least one positive real root or at least one complex root with positive real part of the characteristic equation in the vicinity of collinear equilibrium points) but the effect of μ in addition to other parameters (q_1, A_2 and b) on the stability property, is remains there. In this article, we have analyzed the stability of the points with respect to the

Table 1. Positions of collinear equilibrium points at different values of parameters q_1, A_2 and b.

Points	q_1	A_2	$b = 1.30$	$b = 1.32$	$b = 1.34$	$b = 1.36$
x_{L_2}	1.0	0.0	1.06883	1.06354	1.05920	1.05558
		0.02	1.10426	1.10080	1.09783	1.09526
		0.03	1.11073	1.10742	1.10455	1.10205
	0.9985	0.0	1.06886	1.06357	1.05922	1.05560
		0.02	1.10430	1.10083	1.09786	1.09528
		0.03	1.11076	1.10745	1.10458	1.10207
	0.9	0.0	1.07138	1.06552	1.06076	1.05685
		0.02	1.10661	1.10278	1.09953	1.09674
		0.03	1.11306	1.10942	1.10629	1.10358
x_{L_1}	1.0	0.0	0.93237	0.93105	0.92975	0.92846
		0.02	0.89435	0.89370	0.89305	0.89241
		0.03	0.88646	0.88585	0.88526	0.88469
	0.9985	0.0	0.93234	0.93102	0.92971	0.92842
		0.02	0.89432	0.89366	0.89301	0.89237
		0.03	0.88462	0.88581	0.88522	0.88465
	0.9	0.0	0.93005	0.92861	0.92719	0.92579
		0.02	0.89177	0.89105	0.89035	0.88966
		0.03	0.88368	0.88302	0.88238	0.88176
x_{L_3}	1.0	0.0	-1.00040	-0.99450	-0.98909	-0.98411
		0.02	-0.99060	-0.98510	-0.98005	-0.97539
		0.03	-0.98584	-0.98053	-0.97565	-0.97114
	0.9985	0.0	-1.00040	-0.99449	-0.98968	-0.98409
		0.02	-0.99058	-0.98508	-0.98002	-0.97536
		0.03	-0.98581	-0.98050	-0.97561	-0.97111
	0.9	0.0	-1.00041	-0.99388	-0.98792	-0.98247
		0.02	-0.98954	-0.98350	-0.97798	-0.97292
		0.03	-0.98428	-0.97847	-0.97315	-0.96828
x_{JY1}	1.0	0.0	—	-0.000663038	-0.000720185	-0.000750235
		0.02	—	-0.000663038	-0.000720185	-0.000750235
		0.03	—	-0.000663038	-0.000720185	-0.000750235
	0.9985	0.0	—	-0.000663170	-0.000720303	-0.000753440
		0.02	—	-0.000663170	-0.000720303	-0.000753440
		0.03	—	-0.000663170	-0.000720303	-0.000753440
	0.9	0.0	—	-0.000665215	-0.000728381	-0.000757767
		0.02	—	-0.000665215	-0.000728381	-0.000757767
		0.03	—	-0.000665215	-0.000728381	-0.000757767

disc's outer radius b and obtained the range of b for which L_3 is stable, at fixed values of the remaining parameters. Moreover, we have also, determined the values of mass parameter (in the form of perturbed mass ratio: Table 3), at different values of the parameters and at the values of b, for which stability of L_3 is maintained.

To analyze the linear stability of equilibrium points, firstly, we linearize the equations of motion of the infinitesimal mass as in [33], in the neighborhood of equilibrium point. Let (x_e, y_e) be the coordinate of equilibrium point. Consider a small change in this coordinate such as $x = x_e + X, y = y_e + Y$, where $X = Pe^{\lambda t}$, $Y = Qe^{\lambda t}$ are very small quantities and P, Q are constants and λ is parameter to be determined. Substituting these coordinates into equations (1) and (2). Keeping in mind that displacements are sufficiently small, we have expanded U_x and U_y by Taylor series neglecting the second and higher order terms, we get

$$\ddot{X} - 2n\dot{Y} = XU^0_{xx} + YU^0_{xy}, \tag{10}$$
$$\ddot{Y} + 2n\dot{X} = XU^0_{yx} + YU^0_{yy}. \tag{11}$$

These are linearized differential equations of motion in the vicinity of equilibrium point. Superfix 0 denotes the corresponding values at equilibrium point. The second order partial derivatives of U with respect to space coordinates are obtained with the help of equations (3) and (4). In order to determine P and Q, we have substituted $X = Pe^{\lambda t}$ and $Y = Qe^{\lambda t}$ into equations (10) and (11) and simplifying them, we have

$$(\lambda^2 - U^0_{xx})P + (-2n\lambda - U^0_{xy})Q = 0, \tag{12}$$
$$(2n\lambda - U^0_{yx})P + (\lambda^2 - U^0_{yy})Q = 0. \tag{13}$$

Since, for nontrivial solutions of P and Q, determinant of the coefficient matrix of the above linear system must vanish i.e. we have

$$\begin{vmatrix} \lambda^2 - U^0_{xx} & -2n\lambda - U^0_{xy} \\ 2n\lambda - U^0_{yx} & \lambda^2 - U^0_{yy} \end{vmatrix} = 0$$

On simplifying the above determinant, we get a biquadratic equation

$$\lambda^4 + A\lambda^2 + B = 0, \tag{14}$$

$$\text{where} \quad A = 4n^2 - \left(U^0_{xx} + U^0_{yy}\right) \quad \text{and} \tag{15}$$
$$B = U^0_{xx}U^0_{yy} - (U^0{}_{xy})^2. \tag{16}$$

If we take $\varLambda = \lambda^2$ then we have

$$\varLambda^2 + A\varLambda + B = 0. \tag{17}$$

The equation (17) is quadratic in $\varLambda$ and is known as characteristic equation.

Now, for the collinear points $y = 0$ consequently, $U_{xy} = U_{yx} = 0$; $r_1 = |x + \mu|$, $r_2 = |x + \mu - 1|$ and $r = |x|$. Thus, we have $B = U^0_{xx}U^0_{yy}$ where,

$$U^0_{xx} = n^2 + 2F_0 + \frac{9ch\pi \log(\frac{b}{a})}{8r_0^4}, \tag{18}$$
$$U^0_{yy} = n^2 - F_0 - \frac{3ch\pi \log(\frac{b}{a})}{8r_0^4} \tag{19}$$

$$\text{with} \quad F_0 = \frac{q_1(1-\mu)}{r^3_{1,0}} + \left(1 + \frac{3A_2}{2r^2_{2,0}}\right)\frac{\mu}{r^3_{2,0}} + \frac{2ch\pi(b-a)}{abr_0^3} \tag{20}$$

Subscript 0 indicates the value is at equilibrium point. In order to ensure the stability of the equilibrium points, small displacements X and Y must be bounded and periodic functions of time. In other words, the four roots λ_i, $i = 1, 2, 3, 4$ of the equation (14) must be purely imaginary i.e. two roots $\Lambda_{1,2}$ of equation (17) must be of negative sign. Now, from the characteristic equation (17), we have

$$\Lambda_{1,2} = \lambda^2 = \frac{[-A \pm \sqrt{A^2 - 4B}]}{2} \tag{21}$$

and hence,

$$\lambda_{1,2} = \pm\sqrt{\frac{[-A - \sqrt{A^2 - 4B}]}{2}}, \quad \lambda_{3,4} = \pm\sqrt{\frac{[-A + \sqrt{A^2 - 4B}]}{2}} \tag{22}$$

From equation (21) or (22), it is clear that the sign of B plays a crucial role to decide the nature of roots of the characteristic equation and hence, the nature of orbits of infinitesimal mass in the vicinity of collinear equilibrium points. There are two cases.

Case I: When $B < 0$

Negative sign of B guarantees that there are two real roots of opposite sign (Fig. 5a,b) i.e.

$$\Lambda_1 = \lambda^2 = \alpha^2; \quad \Lambda_2 = \lambda^2 = -\beta^2. \tag{23}$$

Hence, four roots $\lambda_{1,2,3,4}$ can be written as:

$$\lambda_{1,2} = \pm\alpha; \quad \lambda_{3,4} = \pm i\beta. \tag{24}$$

Hence, from equation (24) it is clear that the orbits corresponding to the roots λ_1 and λ_2 are of the exponential type however, these are periodicly corresponding to the roots λ_3 and λ_4 with period $\frac{2\pi}{\beta}$ (because $e^{i\theta} = \cos\theta + i\sin\theta$). Therefore, the general solution of the equations (10) and (11) can be written as :

$$X = \sum_{j=1}^{4} P_j e^{\lambda_j t}, \quad Y = \sum_{j=1}^{4} Q_j e^{\lambda_j t}, \tag{25}$$

where P_j and Q_j, $j = 1, 2, 3, 4$ are constants and related by the linear equation (12) or (13) i.e.

$$Q_j = \left(\frac{\lambda_j^2 - U_{xx}^0}{2n\lambda_j}\right) P_j, (j = 1, 2, 3, 4), \tag{26}$$

and $\lambda_{1,2,3,4}$ are given by (24). If at time $t = 0$, initial conditions are $X = X_0$, $Y = Y_0$, $\dot{X} = \dot{X}_0$ and $\dot{Y} = \dot{Y}_0$ then four arbitrary constant P_j and hence, constants Q_j are determined from the solution of the four simultaneous linear equations

$$\sum_{j=1}^{4} P_j = X_0, \quad \sum_{j=1}^{4} Q_j = Y_0, \quad \sum_{j=1}^{4} \lambda_j P_j = \dot{X}_0, \quad \text{and} \quad \sum_{j=1}^{4} \lambda_j Q_j = \dot{Y}_0. \tag{27}$$

Although, the complete solution of equations (10) and (11) is given by equation (25), because of real eigen values in case of $B < 0$, the solution (25) contains exponential terms hence, it becomes unstable.

As a specific example consider the stability of the $L_1 = 0.88552$ at $b = 1.33$ and for other parametric values mentioned earlier. If we take small displacements $X_0 = Y_0 = 10^{-5}$ and initial velocity $\dot{X}_0 = \dot{Y}_0 = 0$ as in [33] then the resulting eigen values are $\lambda_{1,2} = \pm 3.33$ and $\lambda_{3,4} = \pm 2.08i$, which indicate that this point is unstable due to a positive real eigen value. After, solving for P_j and Q_j, we have obtained the orbits of infinitesimal mass in the vicinity of L_1 which are given as:

$$X(t) = 10^{-6}\left(5.41e^{3.33t} + 6.77e^{-3.33t} - 2.18\cos 2.08t + 2.16\sin 2.08t\right) \tag{28}$$

$$Y(t) = 10^{-6}\left(-2.60e^{3.33t} + 3.25e^{-3.33t} + 9.31\cos 2.08t + 9.40\sin 2.08t\right) \tag{29}$$

The second term in each of these equations eventually dominate which result in exponential growth in X and Y. The e-folding time scale for growth is $\frac{1}{\lambda_+} = \frac{1}{3.33}$ orbital periods of the mass $\mu = 9.53728 \times 10^{-4}$. We have plotted the nonzero values of real and imaginary parts of the roots of the equation (14) for $b \in (1,\ 2)$. In this case, roots are always of the form $\pm\alpha$ and $\pm i\beta$, where α and β are real numbers (Fig. 6a). A similar observation is obtained for other points i.e. for L_2, L_3 and JY_1 for $b \in (1,\ 2)$ whereas, L_3 shows different nature for $b \in (1.3312,\ 1.5275)$ (Fig. 6b). Thus, we have noticed that all collinear equilibrium points are unstable except L_3 which is stable for $b \in (1.3312,\ 1.5275)$.

We can understand the occurrence of the unstable collinear points by considering changes in the gravitational attraction as well as centrifugal force at different points along x-axis. If we move away from both the massive bodies, gravity drops off and the centrifugal acceleration increases. In other words, there are two equilibrium points outside of both the primaries on the line joining them and two equilibrium point are in between them. The force on infinitesimal mass is directed away along the x-axis from these equilibrium points and hence, all collinear points are unstable. However, with special conditions it is possible to find stable solution i.e. periodic orbits in the basin of the collinear equilibrium points [34].

Case II: When $B > 0$

From equation (21), it is seen that when B is positive then two roots $\Lambda_{1,2}$ have negative signs (Fig. 5a,b), i.e.

$$\Lambda_1 = \lambda^2 = -\alpha^2; \quad \Lambda_2 = \lambda^2 = -\beta^2. \tag{30}$$

Hence, four roots $\lambda_{1,2,3,4}$ must be pure imaginary i.e.

$$\lambda_{1,2} = \pm i\alpha; \quad \lambda_{3,4} = \pm i\beta, \tag{31}$$

where, α and β are real quantities. Hence, from equation (31), it is clear that the orbits corresponding to two different roots $i\alpha$ and $i\beta$ are of periodic type with periods $T_\alpha = \frac{2\pi}{\alpha}$ and $T_\beta = \frac{2\pi}{\beta}$. In other words, resulting motion of infinitesimal mass is composed from two different motions, one is known as short periodic motion with period $\frac{2\pi}{\alpha}$ which is very close to orbital period of mass $\mu = 9.53728 \times 10^{-4}$ and the other is long periodic motion with period $\frac{2\pi}{\beta}$ known as liberation about the equilibrium point. The resulting motion of infinitesimal mass is given by the equation (25) together with (26), where λ_j, $j = 1, 2, 3, 4$ are given by (31). The amplitudes of the motion are obtained by evaluating the constant P_j and Q_j, with the help of initial conditions. Thus, we can think of the motion of infinitesimal mass as a long-period motion of an epicenter around the equilibrium point with the execution of a short-period motion of infinitesimal mass around the epicenter.

For example, we have considered the stability of the $L_3 = -0.97801$ point for $b = 1.4 \in (1.3312,\ 1.5275)$. We have taken initial displacements and velocities as in case-I i.e. when $B < 0$ and obtained the four roots $\pm\alpha i = \pm 0.42i$ and $\pm\beta i = \pm 1.03i$, respectively which are pure imaginary and hence, L_3 is linearly stable for $b \in (1.3312,\ 1.5275)$. The solution for perturbed motion in the vicinity of L_3 is given as:

$$X(t) = 10^{-6}\left(5.78\cos\omega_1 t - 2.24\sin\omega_1 t + 4.22\cos\omega_2 t + 0.92\sin\omega_2 t\right) \tag{32}$$

$$Y(t) = 10^{-6}\left(8.18\cos\omega_1 t + 21.10\sin\omega_1 t + 1.89\cos\omega_2 t - 8.61\sin\omega_2 t\right), \tag{33}$$

where, characteristic frequencies $\omega_{1,2}$ at different values of perturbing parameters q_1, A_2 at $b = 1.4$ are given in Table 2.

The above solution is of the oscillatory type with fundamental periods $\frac{2\pi}{\omega_1}$ and $\frac{2\pi}{\omega_2}$. Figure (6a,b) shows, how the nature of the roots varies with the disc's outer radius b. In case of point L_3, it is noticed that the roots are purely imaginary for $b \in (1.3312,\ 1.5275)$, whereas these are real and imaginary for $1.3 < b < 1.3312$ and $b > 1.5275$. Thus, the motion of infinitesimal mass in the vicinity of L_3 is stable for specific range of disc outer radius provided that other parameters are fixed as in earlier case. From, Table 2, it is noticed that frequencies ω_1 and ω_2 both, are increasing functions of A_2, whereas ω_1 and ω_2 are decreasing and increasing function of functions of q_1, respectively.

Table 2. Characteristic frequencies $\omega_{1,2}$ at $b = 1.4$ and different values of parameters q_1 and A_2.

frequency	A_2	$q_1 = 1$	$q_1 = 0.9985$	$q_1 = 0.9$	$q_1 = 0.8$
ω_1	0.00	0.15495	0.15597	0.21742	0.27712
	0.01	0.25018	0.25091	0.29958	0.35417
	0.02	0.31538	0.31603	0.36131	0.41572
	0.03	0.36671	0.36733	0.41154	0.46735
	0.04	0.40926	0.40987	0.45381	0.51147
	0.05	0.44546	0.44606	0.48996	0.54937
	0.06	0.47669	0.47729	0.52112	0.58182
	0.09	0.54855	0.54913	0.59156	0.65174
ω_2	0.00	1.14941	1.14787	1.04453	0.93377
	0.01	1.16332	1.16163	1.05522	0.93949
	0.02	1.17833	1.17672	1.06743	0.94678
	0.03	1.19480	1.19315	1.08125	0.95590
	0.04	1.21261	1.21090	1.09674	0.96710
	0.05	1.23173	1.23003	1.11392	0.98062
	0.06	1.25214	1.25042	1.13280	0.99662
	0.09	1.32039	1.31864	1.19891	1.05953

5 Resonance Cases

Since, four roots of the characteristic equation (14) are

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{-A \pm \sqrt{A^2 - 4B}}{2}} = \pm i\omega_{1,2} \tag{34}$$

which are purely imaginary for the values of b lying in the range (1.3312, 1.5275) and provides two frequencies ω_1 and ω_2. Consequently, question about resonance phenomenon arises. Since, three main resonances described in [35], [27] and [28] are obtained as:

$$\frac{\omega_1}{\omega_2} = \kappa, \quad \kappa = 1, 2, 3. \tag{35}$$

Equation (35) with the help of equation (34) gives

$$A^2 - BK = 0, \quad K = \left(\frac{\kappa^2 + 1}{\kappa}\right), \tag{36}$$

where symbols have there usual meaning. Substituting the values of A and B and simplifying resulting equation under some appropriate approximation in context of the order of parameters, one can obtain a quadratic equation in μ which gives the perturbed mass ratio $0 < \mu_\kappa < \frac{1}{2}$ for these three main resonance cases in the vicinity of collinear point $L_3 = -0.96306$ for the values of $b = 1.4 \in (1.3312,\ 1.5275)$. Since, $q_1 \in (0,\ 1]$, therefore, let $q_1 = 1 - \epsilon$, $\epsilon << 1$ and $A_2 << 1$. Substituting these values and using Taylor,s series expansion up to second order terms in μ and then simplifying, we have resulting quadratic equation

$$B_1\mu^2 + B_2\mu + B_3 = 0, \tag{37}$$

where

$$B_1 = -0.98105 + 2.22434\epsilon + 0.10218A_2 - 1.96209K - 0.11583\epsilon A_2 + 0.20435A_2K$$
$$+4.44868\epsilon K - 0.23167\epsilon A_2K, \tag{38}$$
$$B_2 = -2.18401 + 4.21348\epsilon - 5.82914A_2 + 3.67945K + 6.64668\epsilon A_2 - 1.67732A_2K$$
$$-7.62942\epsilon K + 1.86438\epsilon A_2K, \tag{39}$$
$$B_3 = -1.21551 + 1.93409\epsilon - 6.61501A_2 - 0.17055K + 5.26284\epsilon A_2 + 5.36025A_2K$$
$$+0.59773\epsilon K - 4.68429\epsilon A_2K. \tag{40}$$

Table 3. Perturbed mass ratio $\mu_\kappa(A_2,\, q_1)$ at $b = 1.4$ and different values of parameters q_1 and A_2.

q_1	κ	$\mu_\kappa(0.0,\, q_1)$	$\mu_\kappa(0.01,\, q_1)$	$\mu_\kappa(0.02,\, q_1)$	$\mu_\kappa(0.03,\, q_1)$	$\mu_\kappa(0.04,\, q_1)$	$\mu_\kappa(0.05,\, q_1)$
1.0	1	0.17233	0.15896	0.14558	0.13221	0.111883	0.10546
	2	0.11857	0.10444	0.09030	0.07617	0.06204	0.04791
	3	0.08459	0.06999	0.05538	0.04078	0.02617	0.01157
0.9985	1	0.17227	0.15887	0.14548	0.13209	0.11868	0.10529
	2	0.18489	0.10433	0.09018	0.07602	0.06187	0.04771
	3	0.08450	0.06987	0.05524	0.04061	0.02598	0.01135
0.9	1	0.16826	0.15336	0.13847	0.12358	0.10869	0.09380
	2	0.11326	0.09752	0.08179	0.06605	0.050316	0.03458
	3	0.07848	0.06221	0.04595	0.02968	0.01342	——
0.8	1	0.16418	0.14777	0.13136	0.11495	0.09854	0.08213
	2	0.10795	0.09061	0.07327	0.05593	0.03859	0.02125
	3	0.07236	0.05444	0.03652	0.01859	0.00067	——

Solving equations (37-40), for $\mu \in (0,\, \frac{1}{2}]$, we have perturbed mass ratio μ_κ, $\kappa = 1, 2, 3$ for three main resonances, which are given as

$$\mu_1 = 0.17233 - 0.04073\epsilon - 1.33738A_2 - 1.51858\epsilon A_2, \tag{41}$$

$$\mu_2 = 0.11857 - 0.05307\epsilon - 1.41308A_2 - 1.60483\epsilon A_2, \tag{42}$$

$$\mu_3 = 0.08459 - 0.06115\epsilon - 1.46051A_2 - 1.65895\epsilon A_2, \tag{43}$$

Obviously, perturbed mass ratio depend significantly on parameter $q_1 = (1 - \epsilon)$ and A_2. These expressions of the perturbed mass ratio have similar form to that of [36], [37], [38], [39] and [28], obtained in case of triangular equilibrium points under specific assumptions. Numerically, the perturbed mass ratio are obtained at different values of q_1 and A_2 (Table 3). It is observed that these are decreasing functions of A_2 and κ whereas, increasing functions of q_1. Stability region at different values of q_1 and A_2 are obtained with the help of equations (41-43). Figures (7-8) show the stability region at $b = 1.4$ corresponding to three main resonance curves in the spaces $\mu - A_2$ and $\mu - q_1$ with $0 \le A_2 \le 0.1$ and $0 \le \epsilon = (1 - q_1) \le 0.4$, respectively. From these figures, it is clear that in the absence of radiation pressure (Fig. 7a), stability region decreases with the increment in the values of oblateness whereas in the presence of radiation effect (Fig. 7b), stability region decreases more for all $\kappa = 1, 2, 3$. In the absence as well as in the presence of oblateness (Fig. 8a,b), stability region decreases with the increment in the value of $\epsilon = 1 - q_1$ but extent of decrement is greater in later case. In other words, stability region increases with radiation pressure for all cases $\kappa = 1, 2, 3$.

As per, numerical as well as graphical observations, it is noticed that perturbations i.e. radiation pressure, oblateness and disc, affects the motion of small object as asteroid, satellite or spacecraft in the Sun-planet system. Since, presence of the disc with a specific range of disc's outer radius $1.3312 < b < 1.5275$ make the collinear equilibrium point L_3 stable. The opposing nature of radiation pressure relative to gravitational force of the Sun, minimizes the strength of its gravity field which results that there is an increment in the stability region for the motion of infinitesimal mass. The oblateness increases gravity field due to equatorial bulge which implies that stability region get contracted.

6 Conclusion

We have studied Chermnykh-like problem in the context of collinear equilibrium points and its linear stability, under the influence of perturbations in the form of radiation pressure, oblateness and a disc which is rotating around the common center of mass of the system. In the presence of the disc, there exists a new collinear equilibrium point $JY1$ [7] in addition to three points L_1, L_2 and L_3 of the classical problem. The dependence of $JY1$ on the radiation pressure and oblateness is of negligible order in compare to the disc outer radius (Table 1).

We have analyzed linear stability of the collinear equilibrium points with respect to the mass parameter μ and it is found that similar to the classical problem, all the collinear points are unstable due to existence of at least one root with positive real part (contents are not given so as to minimize the length of the article) but the effect of other parameters on the stability property can not be ignored i.e. a slight influence of all the parameters is obvious.

Moreover, presence of the disc in system, may change the number as well as stability property of the equilibrium points [10]. Therefore, linear stability of the collinear points are studied with respect to the disc's outer radius b and noticed that all the points are, generally unstable but L_3 is stable for certain range of disc's outer radius $1.3312 < b < 1.5275$ provided that other parameters are fixed. For this specific range of b, frequencies of infinitesimal mass at L_3 is obtained (Table 2) and it is notice that $\omega_{1,2}$ increase with oblateness, whereas radiation pressure has different effect (ω_1 increases with radiation pressure but ω_2 decreases).

Again, perturbed mass ratio of the problem is obtained for three main resonances cases on the similar basis of triangular equilibrium points [27], [28] and it is observed that perturbed mass ratio is decreasing function of oblateness A_2, κ and radiation pressure (Table 3). Moreover, stability region of the infinitesimal mass are found for fixed value of disc's outer radius.

As per, analysis of numerical results as well as graphical observations, it is concluded that in the presence of perturbations, motion of infinitesimal mass in the vicinity of collinear point L_3 is affected significantly. Moreover, presence of the disc like structure in the system may change the scenario of the motion. Due to radiation pressure, strength of gravity field of the Sun reduces and hence, stability region increases. On the other hand, oblateness of the secondary increases its gravity field due to equatorial bulge and hence, stability region decreases. The results are important to study the motion of infinitesimal mass under the influence of perturbations in a more generalized dynamical system. In future, these analysis would also be helpful to describe the proposed model in presence of Poyinting- Roberston drag, solar wind drag etc.

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