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The Grunsky Coefficients as a Model of Universal Teichmüller Space

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Abstract Some models of the universal Teichmüller space that are given by its holomorphic embedding into appropriate Banach spaces play a crucial role in various applications of this space. We provide a new model of this space as a domain formed by the Grunsky coefficients of basic univalent functions with quasiconformal extension.

Keywords: Universal Teichmüller space, Grunsky coefficients, Schwarzian derivative, holomorphic embedding

In the memory of Igor Zhuravlev

1 Introduction

The universal Teichmüller space $\mathbf{T}$ is intrinsically connected with complex geometric function theory. This connection is based on a special role of univalent functions with quasiconformal extension and it has created many deep results in both fields. It is well-known that the space $\mathbf{T}$ (like other Teichmüller spaces) with its canonical complex structutre is pseudo-convex.

Recall that $\mathbf{T}$ is the space of quasisymmetric homeomorphisms of the unit circle $S^1 = \partial\mathbb{D}$ factorized by Möbius maps, and its complex structure is inherited from the ball of Beltrami coefficients

$$\mathrm{Belt}(\mathbb{D})_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\mathbb{D}^* = 0, \ \|\mu\|_\infty < 1\},$$

letting $\mu, \nu \in \mathrm{Belt}(\mathbb{D})_1$ be equivalent if the corresponding quasiconformal homeomorphisms w^μ, w^ν of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ coincide on S^1. Here $\mathbb{D} = \{|z| < 1\}, \ \mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : \ |z| > 1\}$. This construction determines the space $\mathbf{T}$ as the quotient space formed by the corresponding equivalence classes $[\mu]$, with holomorphic factorizing map $\phi_{\mathbf{T}} : \mu \to [\mu]$.

The known holomorphic embeddings of $\mathbf{T}$ into Banach spaces play a crucial role in applications. The *Bers embedding* [2] involves the Schwarzian derivatives

$$S_f(z) = \Big(\frac{f''(z)}{f'(z)}\Big)' - \frac{1}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2$$

of univalent holomorphic funcions $f(z)$ in the disk $\mathbb{D}^*$ (or in the half-plane) and represents the space $\mathbf{T}$ via a bounded domain in the Banach space $\mathbf{B}$ of hyperbolically bounded holomorphic functions φ in $\mathbb{D}$ with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\mathbb{D}^*}(|z|^2 - 1)^2|\varphi(z)|. \tag{1}$$

The *Becker embedding* [1] models $\mathbf{T}$ as a bounded domain in the space of logarithmic derivatives $\mathbf{b}_f = (\log f')' = f''/f'$ with norm $\|b_f\| = \sup_{\mathbb{D}^*}(|z|^2 - 1)|\mathbf{b}_f(z)|$.

Due to [6], [8], both these models of $\mathbf{T}$ are not starlike, and moreover, the space $\mathbf{T}$ has points which cannot be joined to a distinguished point even by curves of a considerably general form, in particular, by polygonal lines with the same finite number of rectilinear segments.

Two other known models of $\mathbf{T}$ were introduced by Zhuravlev [17] and by Takhtajan-Teo [15] and represent this space as the manifolds in the Bloch and Hilbert spaces, respectively; both manifolds have uncountable many components.

We establish here that the Grunsky coefficients $c_{mn}(f)$ of original univalent functions in the disk $\mathbb{D}^*$ allows one to create a natural holomorphic embedding of the space $\mathbf{T}$ into some Banach space and give a new model of $\mathbf{T}$.

Theorem. *The universal Teichmüller space is biholomorphically equivalent to a bounded domain in a subspace of l_∞ determined by the Grunsky coefficients of univalent functions in the disk.*

This result is also interesting independently because of a strong rigidity of the Grunsky coefficients of univalent functions caused by a mutual implicit connection of these coefficients.

2 Background: Basic Features of the Grunsky Inequality

Due to the cassical Grunsky theorem [5], a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ is extended to a univalent holomorphic function on the disk $\mathbb{D}_r^* = \{z \in \widehat{\mathbb{C}} : |z| > r\}$ if and only if its Grunsky coefficients c_{mn} defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} c_{mn} r^{m+n} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\mathbb{D}_r^*)^2, \tag{2}$$

satisfy the inequality

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\; c_{mn} r^{m+n} x_m x_n \Big| \le 1, \tag{3}$$

for any sequence $\mathbf{x} = (x_n)$ from the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\| = \big(\sum_1^\infty |x_n|^2\big)^{1/2}$. Here the principal branch of the logarithmic function is chosen and $0 < r < \infty$. We shall use the case $r = 1$. Then $|c_{mn}| \le 1/\sqrt{mn}$ for all m, n.

Consider the univalent functions in $\mathbb{D}^*$ with hydrodynamical normalization

$$f(z) = z + b_1 z^{-1} + \dots : \mathbb{D}^* \to \widehat{\mathbb{C}} \setminus \{0\}.$$

and denote their collection by Σ. Note that $c_{mn} = c_{nm}$ for all $m, n \ge 1$ and $c_{m1} = b_m$ for any $m \ge 1$.

Denote by Σ_k the subset of Σ formed by the functions with k-quasiconformal extensions to $\mathbb{D} = \{|z| < 1\}$, and let $\Sigma^0 = \bigcup_k \Sigma_k$.

The Grunsky theorem was strengthened for the functions admitting quasiconformal extensions by several authors. Due to [11], the Grunsky norm

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\; c_{mn}\; x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\} \tag{4}$$

is dominated by the Teichmüller norm $k(f) = \tanh \tau_{\mathbf{T}}(\mathbf{0}, S_f)$, via $\varkappa(f) \le k(f)$ (there is also a stronger estimate given in [8]), and the equality $\varkappa(f) = k(f)$ holds only on a sparse set of $f \in \Sigma^0$. On the other hand, due to Pommerenke and Zhuravlev, if $f \in \Sigma$ satisfies the inequality $\varkappa(f) < k$ with some constant $k < 1$, then this function has a quasiconformal extension to $\widehat{\mathbb{C}}$ with dilatation $k_1 = k_1(k) \ge k$ (see [14], [16], [11, pp. 82-84]). The explicit estimates for $k_1(k)$ are established in [9], [12]. We shall use the result from [9] presented below as Lemma 1.

Each Grunsky coefficient $c_{mn}(f^\mu)$ of $f^\mu \in \Sigma^0$ depends holomorphically on the initial $m + n - 1$ Taylor coefficients b_j of this function. Since for every finite $M,\ N$,

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \sqrt{mn}\; c_{mn} x_m x_n \Big|^2 \le \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2 \tag{5}$$

(see [15, p. 61]), then for any $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ the map

$$h_{\mathbf{x}}(\mu) = \sum_{m,n=1}^{\infty} \sqrt{mn}\, c_{mn}(f^{\mu}) x_m x_n : \ \mathbf{T} \to \mathbb{D} \tag{6}$$

with fixed $\mathbf{x} = (x_n) \in S(l^2)$ is holomorphic. Due to [9], the upper envelope

$$\sup_{\mathbf{x}} |h_{\mathbf{x}}(f^{\mu})| = \varkappa(f^{\mu})$$

is a Lipschitz-continuous plurisubharmonic function on $\mathbf{T}$.

On the other hand, both functions $\varkappa(f)$ and $k(f)$ are only lower semicontinuous in topology on Σ^0 induced by the locally uniform convergence on $\mathbb{D}^*$, in contrast to the case of a stronger convergence in the norm (1) (or equivalently, in the Teichmüller distance).

The technique of the Grunsky inequalities was generalized to finitely bordered Riemann surfaces (in particular, to finitely connected plane domains). The quasiconformal aspect of this theory was extended in [8] to univalent functions in arbitrary unbounded quasiconformal disks D^* containing infinity. In this case, one must deal with the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \frac{c_{mn}}{\eta(z)^m \ \eta(\zeta)^n}, \tag{7}$$

where η denotes a conformal map of D^* onto the disk $\mathbb{D}^*$ so that $\eta(\infty) = \infty, \ \eta'(\infty) > 0$, and accordingly fix in the space $\mathbf{T}$ the base point D^*. Accordingly, one must use the Beltrami coefficients from the ball

$$\mathrm{Belt}(D)_1 = \{\mu \in L_{\infty}(\mathbb{C}) : \ \mu|D^* = 0, \ \|\mu\|_{\infty} < 1\}.$$

Now again $c_{mn} = c_{nm}$ but the connection $c_{m1} = b_m$ does not need to hold.

A theorem of Milin [13] extending the Grunsky univalence criterion for the disk $\mathbb{D}^*$ to multiply connected domains D^* states that a holomorphic function $f(z) = z + \mathrm{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ can be continued to a univalent function in the whole domain D^* if and only if the coefficients c_{mn} in (7) satisfy the same inequality (3) (with $r = 1$). Then the expression (4) with such c_{mn} defines the Grunsky norm for functions $f \in \Sigma(D^*)$ and the corresponding functions (6) map holomorphically the universal Teichmüller space with the base point D^* onto the unit disk (moving this point to the origin). The underlying inequality (5) remains.

Consider in the complementary domain $D = \widehat{\mathbb{C}} \setminus \overline{D^*}$ the subspace $A_1(\mathbb{D})$ of $L_1(D)$ formed by holomorphic functions, and let A_1^2 be its subset consisting of the squares of holomorphic functions in D. Define for $f \in \Sigma(D^*)$ the quantity

$$\alpha_D(f) = \sup_{\psi \in A_1^2, \|\psi\|_{A_1(D)}=1} \Big| \iint_D \mu_0(z)\psi(z)\mathrm{d}x\mathrm{d}y \Big|,$$

where μ_0 is an extremal Beltrami coefficient among extensions of f to D. Then we have the following estimate.

Lemma 1. [7] *Let D^* be a quasidisk containing $z = \infty$. Then for every function $f \in \Sigma^0(D^*)$, its Grunsky and Teichmüller norms are related by*

$$\varkappa(f) \le k(f) \le \varkappa(f)/\alpha_D(f). \tag{8}$$

Note that the right inequality in (8) is given in [7] for functions f having a unique extremal extension. Since, due to [4], the Schwarzians S_f of such functions fill a dense open set in the space $\mathbf{T}$ and both Teichmüller and Grunsky norms are continuous on this space, Lemma 1 holds in its general form presented above.

3 Proof of Theorem

Step 1: Auxiliary construction. First observe that Grunsky coefficients of the functions $f \in \Sigma^0$ span a $\mathbb{C}$-linear space $\mathcal{L}^0$ of sequences $\mathbf{c} = (c_{mn})$ which satisfy the symmetry relation $c_{mn} = c_{nm}$ and

$$|c_{mn}| \le C(\mathbf{c})/\sqrt{mn}, \quad C(\mathbf{c}) = \text{const} < \infty \quad \text{for all} \ \ m, n \ge 1,$$

with finite norm

$$\|\mathbf{c}\| = \sup_{m,n} \ \sqrt{mn}\ |c_{mn}| + \sup_{\mathbf{x}=(x_n)\in S(l^2)} \Big| \sum_{m,n=1}^{\infty} \ \sqrt{mn}\ c_{mn} x_m x_n \Big|. \tag{9}$$

This quantity obeys all the requirements to the norm: its subadditivity is trivial; the homogeneity property $\|t\mathbf{c}\| = |t|\|\mathbf{c}\|$ is immediate for the first term; for the second term one can write, letting $t = re^{i\theta}$,

$$\sum_{m,n=1}^{\infty} \ \sqrt{mn}\ tc_{mn} x_m x_n = \sum_{m,n=1}^{\infty} \ \sqrt{mn}\ rc_{mn} e^{i\theta/2} x_m e^{i\theta/2} x_n,$$

so both sides have the same supremum over $S(l^2)$. Finally, if $\|\mathbf{c}\| = 0$, each term in (9) must vanish, hence all $c_{mn} = 0$.

Each sequence $\mathbf{c} = (c_{mn})$ from $\mathcal{L}^0$ determines the corresponding double power series

$$g_{\mathbf{c}}(z, \zeta) = \sum_{m,n=1}^{\infty} \ c_{mn} z^{-m} \zeta^{-n} \tag{10}$$

convergent absolutely on the bidisk $(\mathbb{D}^*)^2$ so that $g_{\mathbf{c}}(z, \zeta) dz d\zeta$ is a holomorphic bilinear differential on $\mathbb{D}^*$, and vice versa. We define on the linear space of these series the same norm (9) and identify it with $\mathcal{L}^0$.

Note that the set $\mathbf{T}_\infty$ of collections $\mathbf{c}(f)$ corresponding to $f \in \Sigma^0$ is placed in the ball $\{\|\mathbf{c}\|_{\mathcal{L}} < 2\}$. The classical estimates for univalent functions in Σ (following from the area theorem) yield that the boundary of $\mathbf{T}_\infty$ has common points with the sphere $\{\|\mathbf{c}\|_{\mathcal{L}} = 2\}$ only at $\mathbf{c}(f_\epsilon)$, where $f_\epsilon(z) = z + \epsilon z^{-1}$ with $|\epsilon| = 1$.

We denote the closure of span $\mathcal{L}^0$ by $\mathcal{L}$ and observe that the limits of convergent sequences $\{\mathbf{c}^{(p)} = (c_{mn}^{(p)})\} \subset \mathcal{L}$ in the norm (9) also are represented by the double series (10) convergent absolutely in the unit bidisk.

First we verify this for uniformly bounded sequences for which

$$\sup_{\mathbf{x}=(x_n)\in S(l^2)} \Big| \sum_{m,n=1}^{\infty} \ \sqrt{mn}\ c_{mn}^{(p)}\ x_m x_n \Big| \le a < \infty \tag{11}$$

for all p, generated by the functions

$$f_p(z) = z + c_{11}^{(p)} z^{-1} + c_{21}^{(p)} z^{-2} + \dots \in \mathcal{L}.$$

In this case, if $a < 1$ is sufficiently small, then $\varkappa(f_p) < \|\mathbf{c}^{(p)}\|_{\mathcal{L}} < a$, and Lemma 1 yields that each f_p has a $k(a)$-quasiconformal extension to $\widehat{\mathbb{C}}$ with $k(a) < 1$.

Since the coefficients $c_{mn}^{(p)}$ are uniformly bounded, the corresponding series

$$g_{\mathbf{c}^{(p)}}(z, \zeta) = -\log[(f_p(z) - f_p(\zeta))/(z - \zeta)] = \sum_{m,n=1}^{\infty} \ c_{mn}^{(p)} z^{-m} \zeta^{-n}$$

are bounded locally uniformly on the bidisk $(D^*)^2$ so their sequence is compact in the topology of locally uniform convergence. Then the limit

$$\lim_{p\to\infty} g_{\mathbf{c}^{(p)}}(z, \zeta) = g_\infty(z, \zeta) = -\log[(f_\infty(z) - f_\infty(\zeta))/(z - \zeta)]$$

corresponds to the map $f_\infty(z) = \lim_{p\to\infty} f_p(z)$, and by the general properties of quasiconformal maps, this map also is conformal in $\mathbb{D}^*$ and extends $k(a)$-quasiconformally to $\mathbb{D}$. Hence $\mathbf{c}(f_\infty) \in \mathcal{L}$.

If in (11) $a = 1$, one obtains using the lower semicontinuity of the Grunsky norm $\varkappa(f)$ o Σ that the limit function f_∞ is univalent on $\mathbb{D}*$ and $\varkappa(f_\infty) = 1$.

The same arguments work also for arbitrary points of $\mathcal{L}$ which are the finite linear combinations of $\mathbf{c}(f),\ f \in \Sigma^0$ and then to their arbitrary combinations, acting with formal expansions

$$f_p(z) = z + b_1^{(p)} z + b_2^{(p)} + \dots .$$

(where unnecessarily $b_j^{(p)} = c_{j1}^{(p)}$).

The space $\mathcal{L}$ is straightforwardly generalized to arbitrary quasidisks $D^* \ni \infty$ (taking the series (10) convergent near $z = \infty$). We denote the corresponding space of Grunsky coefficients on such a quasidisk by $\mathcal{L}(D^*)$.

Step 2: Openness. Our main goal is to prove that the map $\chi:\ \mathbf{T} \to \mathcal{L}$ defined by

$$\chi(S_f) = \mathbf{c}(S_f) := (c_{mn}(f)) \tag{12}$$

establishes a biholomorphic isomorphism in the norms (1) and (9); hence the image $\chi(\mathbf{T}) = \mathbf{T}_\infty$ is a bounded domain (i.e., open and connected) in the space $\mathcal{L}$.

We first establish that the origin $\mathbf{c} = \mathbf{0}$ of $\mathcal{L}$ is an inner point of the image $\mathbf{T}_\infty$, and moreover, that each point of the unit ball $B(\mathbf{0}, 1) = \{\mathbf{c}:\ \|\mathbf{c}\|_{\mathcal{L}} < 1\}$ is a collection of the Grunsky coefficients $c_{mn}(f)$ of some function $f \in \Sigma^0$.

First observe that if a point $\mathbf{c} \in \mathbf{T}_\infty$ corresponds to a function $f \in \Sigma^0$, then

$$f_{\mathbf{c}}(z) = z + c_{11} z^{-1} + c_{21} z^{-2} + \dots$$

and

$$g_{\mathbf{c}}(z,\zeta) = -\log[(f_{\mathbf{c}}(z) - f_{\mathbf{c}}(\zeta))/(z-\zeta)].$$

A simple calculation yields that near any finite ζ,

$$\frac{\partial^2 g_{\mathbf{c}}(z,\zeta)}{\partial z \partial \zeta} = \frac{1}{6} S_{f_{\mathbf{c}}}(z) + O(z-\zeta);$$

hence, the Schwarzian of $f_{\mathbf{c}}$ is given by

$$S_{f_{\mathbf{c}}}(z) = 6 \lim_{\zeta \to z} \partial^2 g_{\mathbf{c}}(z,\zeta)/\partial z \partial \zeta. \tag{13}$$

It is holomorphic in the disk $\mathbb{D}^*$, being represented there by the series

$$S_{f_{\mathbf{c}}}(z) = 6 \sum_{m+n=2}^{\infty} \frac{mnc_{mn}}{z^{m+n+2}} \tag{14}$$

following from (2) and (13). This representation yields the convergence of the Schwarzians in weak topology on $\mathbf{T}$ (induced by their locally uniform convergence in the disk $\mathbb{D}^*$).

Now, using the hydrodynamical normalization of $f_{\mathbf{c}}$ and noting that this function is restored from $S_{f_{\mathbf{c}}}$ via the ratio $f_{\mathbf{c}}(z) = w_2(z)/w_1(z)$ of two linearly independent solutions of the differential equation $2w'' + S_{f_{\mathbf{c}}}(z) w = 0$ with expansions

$$w_1(z) = \frac{1}{z} + \frac{a_3^*}{z^3} + \dots\ , \quad w_2(z) = 1 + \frac{a_2^{**}}{z^2} + \dots\ ,$$

one obtains that $c_{mn}(f_{\mathbf{c}})$ must coincide with the given coordinates c_{mn} of the point $\mathbf{c}$, and

$$\sup_{\mathbf{x}=(x_n)\in S(l^2)} \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ c_{mn} x_m x_n \Big| < 1. \tag{15}$$

For an arbitrary point $\mathbf{c} \in \mathbf{T}_\infty$, one can formally define the corresponding Schwarzian $S_{f_\mathbf{c}}$ by(14) and get similar to above a function $f_\mathbf{c}$ which is locally univalent in $\mathbb{D}^*$, in particular, near $z = \infty$. Now the needed inequality (15) follows from the assumption $\|\mathbf{c}\|_\mathcal{L} < 1$. This inequality ensures that $f_\mathbf{c}(z)$ is univalent in the disk $\mathbb{D}^*$ and has quasiconformal extensions to $\mathbb{C}$.

Note that univalence of the functions $f_\mathbf{c}$ near $z = \infty$ (to define their Grunsky coefficients) was underlying in the previous arguments. For sufficiently small $\varepsilon > 0$ and all $\mathbf{c}$ with $\{\mathbf{c} : \|\mathbf{c}\|_\mathcal{L} < \varepsilon\}$, such univalence of $f_\mathbf{c}$ can be derived also from the argument principle.

Now consider a generic point $\mathbf{c}(f^{\mu_0}) = (c_{mn}(f^{\mu_0}))$ of $\mathbf{T}_\infty$ and take $\mathbf{c}^\mu = (c_{mn}(f^\mu))$ with μ close to μ_0 in $\text{Belt}(\mathbb{D})_1$. Using the chain rule for Beltrami coefficients

$$w^\nu = w^{\sigma(\nu)} \circ (f^{\nu_0})^{-1}$$

with

$$\sigma(\nu) \circ f^{\nu_0} = \frac{\nu - \nu_0}{1 - \overline{\nu}_0 \nu} \frac{\partial_z f^{\nu_0}}{\overline{\partial_z f^{\nu_0}}},$$

one can write

$$\log \frac{f^\mu(z) - f^\mu(\zeta)}{z - \zeta} = \log \frac{f^{\sigma(\mu)}(w) - f^{\sigma(\mu)}(\omega)}{w - \omega} + \log \frac{f^{\mu_0}(z) - f^{\mu_0}(\zeta)}{z - \zeta} \tag{16}$$

with $w = f^{\mu_0}(z)$. The second term in (16) is fixed, and $\sigma(\mu) \in \text{Belt}(f^{\mu_0}(\mathbb{D})$.

Using the representation (7), we expand the first term into a series of the generalized Grunsky coefficients $\mathbf{c} = (c_{mn}(f^{\sigma(\mu)}))$ in the domain $D^* = f^{\mu_0}(\mathbb{D}^*)$.

Taking into account the chain rule for the Schwarzian derivatives

$$S_{f_2 \circ f_1} = (S_{f_2} \circ f_1)(f_1')^2 + S_{f_1},$$

one actually can argue with these coefficients similar to the above special case and define now for any point $\mathbf{c} \in \mathcal{L}(f^{\mu_0})$ with small $\|\mathbf{c}\|$ a holomorphic function $f_\mathbf{c}(w)$ in the domain $f^{\mu_0}(\mathbb{D}^*)$ with expansion

$$f_\mathbf{c}(w) = w + b_1 w^{-1} + b_2 w^{-2} + \dots$$

near $w = \infty$. In the same way, one obtains that such points fill some neighborhood of the origin of the space $\mathcal{L}(f^{\mu_0}(D^*))$. This implies by (16) that the corresponding points $\mathbf{c}(f^\mu)$ must fill a whole neighborhood of $\mathbf{c}(f^{\mu_0})$ in $\mathcal{L}$.

Step 3: Holomorphy of map (12). We use the following Earle lemma [3] on holomorphy in the functional spaces with sup-norms.

Lemma 2. *Let E, T be open subsets of complex Banach spaces X, Y and $B(E)$ be a Banach space of holomorphic functions on E with sup-norm. If $\varphi(x,t)$ is a bounded map $E \times T \to B(E)$ such that $t \mapsto \varphi(x,t)$ is holomorphic for each $x \in E$, then the map φ is holomorphic.*

Note that holomorphy of $\varphi(x,t)$ in t for fixed x implies the existence of complex directional derivatives

$$\varphi_t'(x,t) = \lim_{\zeta \to 0} \frac{\varphi(x, t + \zeta v) - \varphi(x,t)}{\zeta} = \frac{1}{2\pi i} \int\limits_{|\xi|=1} \frac{\varphi(x, t + \xi v)}{\xi^2} \mathrm{d}\xi,$$

while the boundedness of φ in sup-norm provides the uniform estimate

$$\|\varphi(x, t + c\zeta v) - \varphi(x,t) - \varphi_t'(x,t) c v\|_{B(E)} \le M|c|^2,$$

for sufficiently small $|c|$ and $\|v\|_Y$.

Since the Grunsky coefficients $c_{mn}(f)$ depend holomorphically on the Schwarzians $S_f \in \mathbf{T}$, the same holds for the corresponding series (10). Applying Lemma 2, one derives holomorphy of function $\chi(S_f) = \mathbf{c}(f)$ in the sup-norm (9) defined on the space $\mathcal{L}$.

Conversely, using the inequality (5), one simply derives that for every fixed pair $(z, \zeta) \in \mathbb{D}^* \times \mathbb{D}^*$ the function $l(\mathbf{c}) = g_{\mathbf{c}}(z, \zeta)$ defines a linear continuous functional on the coefficients $\mathbf{c} = (c_{mn}) \in \mathcal{L}$ (its continuity with respect to the second term of norm (9) easily follows from (5)). Then by (14), the corresponding Schwarzian $S_{f_{\mathbf{c}}}(z) = S(z; \mathbf{c})$ represents for a fixed z a holomorphic function $\mathbf{T}_\infty \to \mathbb{C}$, and Lemma 2 implies that this Schwarzian is also holomorphic as an element of the space $\mathbf{B}$. This provides the holomorphy of the inverse map χ^{-1}.

We have established that the map χ is a biholomorphis homeomorphism between the domain $\mathbf{T}$ in $\mathbf{B}$ and the open set $\mathbf{T}_\infty = \chi(\mathbf{T}) \subset \mathcal{L}$. Thus the topological properties of $\mathbf{T}$ such as path-wise connectedness and contractibility are carried to $\mathbf{T}_\infty$. This completes the proof of the theorem.

4 Additional Remarks

1. The arguments used in the proof of the theorem are valid only for univalent functions $f_{\mathbf{c}}$, which correspond to $\mathbf{c} \in \mathcal{L}$ satisfying $\|\mathbf{c}\|_{\mathcal{L}} < 1$, and illustrate that norm (9) (especially, its second term) relates to the Grunsky inequalities intrinsically.

For example, one can vary in a given series (10) only one coefficient (coordinate) $c_{m_0 n_0}$ with $m_0, n_0 > 1$ replacing it by $c'_{m_0 n_0} = c_{m_0 n_0} + \varepsilon_{m_0 n_0}$ with small $|\varepsilon_{m_0 n_0}|$ so that $|c'_{m_0 n_0}| \leq 1/\sqrt{m_0 n_0}$. This does not affect the convergence of the series on $(\mathbb{D}^*)^2$, but such variations are not compatible with norm (9). It provides a function $f_{\mathbf{c}'}(z)$ which is univalent only in some disk $\{|z| > r'\}$ with $r' > 1$.

2. The nonstarlikeness of Teichmüller spaces established in [6], [8] causes obstructions to some problems in the Teichmüller space theory and its applications to geometric complex analysis. Together with the connection (13) between the Grunsky coefficients and the corresponding Schwarzians (in weak topology), these results of [6], [8]) imply that *the domain* $\mathbf{T}_\infty$ *also is not starlike*.

In particular, due to [8], the space $\mathbf{T} \subset \mathbf{B}$ is not starlike in the directions rS_{f_0}, $0 < r < \infty$, determined by conformal maps f_0 of the disk $\mathbb{D}^*$ onto the complements of appropriate convex polygons $P_0 \Subset \mathbb{C}$. Their images $\chi(S_{f_0})$ are placed in our domain $\mathbf{T}_\infty \subset \mathcal{L}$ described above; so, using the connection (13), one can establish explicitly the values r for which the corresponding points $\mathbf{c}_r = (c_{mn}(rS_{f_0})) \in \mathcal{L}$ escape the domain $\mathbf{T}_\infty$. Note that these values of r affect the first term of norm (9).

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