Existence of Solutions for Implicit Fractional Differential Systems with Coupled Nonlocal Conditions

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Abstract This article concerns the existence of solutions for implicit fractional differential systems with coupled nonlocal conditions of functional type. We use a vectorial version of Krasnoselskii's fixed point theorem in generalized metric space to overcome the lack of complete continuity of the associated integral operators. Moreover, the sufficient conditions for the existence will be weakened on the subinterval in which our nonlocal conditions act. An example is presented to illustrate the theory.

Keywords: Fractional differential systems, implicit differential equation, coupled nonlocal conditions, fixed point, vector-valued norm, spectral radius of a matrix.

1 Introduction and Preliminaries

In this article, we study the implicit fractional differential systems

$$
\begin{cases}
{}^{C}D_{0^+}^{\beta}x_1(t) = f_1(t, x_1(t), x_2(t), \cdots, x_n(t)) + g_1(t, {}^{C}D_{0^+}^{\beta}x_1(t), \cdots, {}^{C}D_{0^+}^{\beta}x_n(t)) \\[2mm]
{}^{C}D_{0^+}^{\beta}x_2(t) = f_2(t, x_1(t), x_2(t), \cdots, x_n(t)) + g_2(t, {}^{C}D_{0^+}^{\beta}x_1(t), \cdots, {}^{C}D_{0^+}^{\beta}x_n(t)) \\[2mm]
\cdots \\[2mm]
{}^{C}D_{0^+}^{\beta}x_n(t) = f_n(t, x_1(t), x_2(t), \cdots, x_n(t)) + g_n(t, {}^{C}D_{0^+}^{\beta}x_1(t), \cdots, {}^{C}D_{0^+}^{\beta}x_n(t))
\end{cases}
\tag{1.1}
$$

for $t \in [0, 1]$, subject to the coupled nonlocal conditions

$$
\begin{cases}
x_1(0) = \alpha_{11}[x_1] + \alpha_{12}[x_2] + ... + \alpha_{1n}[x_n] \\[2mm]
x_2(0) = \alpha_{21}[x_1] + \alpha_{22}[x_2] + ... + \alpha_{2n}[x_n] \\[2mm]
\cdots \\[2mm]
x_n(0) = \alpha_{n1}[x_1] + \alpha_{n2}[x_2] + ... + \alpha_{nn}[x_n]
\end{cases}
\tag{1.2}
$$

where $f_i, g_i : [0, 1] \times R^n \to R \ (i = 1, 2, \cdots, n)$ are continuous functions, $\alpha_{ij} : C[0, 1] \to R \ (i, j = 1, 2, \cdots, n)$ are continuous linear functionals, ${}^{C}D_{0^+}^{\beta}$ is the Caputo fractional derivative of order $0 < \beta \le 1$.

It is well known that fractional derivatives provide a more excellent tool for the description of memory and hereditary properties of various materials and processes than integer derivatives in various fields of applied sciences and engineering ([1-6]). Recently, considerable attention has been given to a special class of fractional differential equations, that is, implicit fractional differential equations ([7-11]). For example, in [7], the authors considered the following initial value problem of implicit fractional order differential equation

$$
\begin{cases}
{}^{C}D^{\alpha}y(t) = f(t, y(t), {}^{C}D^{\alpha}y(t)), \quad t \in [0, b], \quad b > 0, \\
y(0) = y_0,
\end{cases}
$$

where $0 < \alpha \le 1$, $f : [0, b] \times R \times R \to R$ is a given function. By using fixed point theory and approximation method, some existence and uniqueness results were obtained.

In [8], by using Banach contraction principle, Schauder's fixed point theorem and nonlinear alternative of Leray-Schauder type, the authors investigated the following boundary value problem of the form

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t), {}^C D^\alpha y(t)), & t \in [0, T], \quad T > 0, \\ y(0) = y_0, \quad y(T) = y_1, \end{cases}$$

where $1 < \alpha \le 2, y_0, y_1 \in R, f : [0, T] \times R \times R \to R$ is a given function.

On the other hand, nonlocal problems for differential equations were extensively discussed in the literature by various methods (see [14-19, 22, 23]). For instance, in [14], the authors deal with the existence, localization and multiplicity of positive solutions to nonlocal problems for first order differential systems

$$\begin{cases} u' = f(t, u), & t \in [0, 1], \\ u(0) = \alpha[u], \end{cases}$$

where $f : [0, 1] \times R_+^n \to R^n$ is a continuous function and $\alpha : C([0, 1], R^n) \to R^n$ is a linear continuous operator.

In [18], the authors considered the following first order differential systems

$$\begin{cases} u_1'(t) = f_1(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ u_2'(t) = f_2(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \cdots \\ u_n'(t) = f_n(t, u_1(t), u_2(t), \cdots, u_n(t)) \end{cases}$$

for a.e. $t \in [0, 1]$, subject to the coupled nonlocal conditions

$$\begin{cases} u_1(0) = \alpha_{11}[u_1] + \alpha_{12}[u_2] + ... + \alpha_{1n}[u_n] \\ u_2(0) = \alpha_{21}[u_1] + \alpha_{22}[u_2] + ... + \alpha_{2n}[u_n] \\ \cdots \\ u_n(0) = \alpha_{n1}[u_1] + \alpha_{n2}[u_2] + ... + \alpha_{nn}[u_n] \end{cases}$$

where $f_i : [0, 1] \times R^n \to R$ $(i = 1, 2, \cdots, n)$ are L^1-Carathéodory functions, $\alpha_{ij} : C[0, 1] \to R$ $(i, j = 1, 2, \cdots, n)$ are continuous linear functionals.

Unfortunately, implicit integer order differential equations with nonlocal conditions, to our knowledge, were rarely investigated except for [12,13]. The paper [12] obtained an absolutely continuous solution for an implicit first order equation under nonlocal conditions, via Schauder's fixed point theorem and Kolmogorov's compactness criterion in L^1. In [13], the author used the Kransnoselskii's fixed point theorem for implicit first order systems with nonlocal conditions. As far as we know, there is no paper to studying implicit fractional order differential equations with nonlocal conditions. The purpose of this paper is to fill this gap and investigate (1.1) subject to the coupled nonlocal conditions (1.2). The main features of this paper are as follows. First, the equation considered here is an implicit fractional order differential equation and the boundary conditions are coupled nonlocal conditions. Second, the main tool used here is vectorial version of Krasnoselskii's fixed point theorem in generalized metric space to overcome the lack of complete continuity arising from implicit equations.

The paper is organized as follows. In the rest of this section, some preliminaries are stated. Section 2 contains the main results and their proof. Finally, in section 3, an example is worked out to demonstrate the main result.

For brevity, rewrite the problem (1.1) (1.2) into the vector form

$$\begin{cases} {}^C D_{0^+}^\beta x(t) = f(t, x(t)) + g(t, {}^C D_{0^+}^\beta x(t)) \\ x(0) = \alpha[x], \end{cases} \tag{1.3}$$

where $x = (x_1, x_2, \cdots, x_n)^T, f = (f_1, f_2, \cdots, f_n)^T, g = (g_1, g_2, \cdots, g_n)^T$ and

$$\alpha[x] = (\alpha_1[x], \alpha_2[x], \cdots, \alpha_n[x])^T, \tag{1.4}$$

$$\alpha_i[x] = \alpha_{i1}[x_1] + \alpha_{i2}[x_2] + \cdots + \alpha_{in}[x_n], \ i = 1, 2, \cdots, n. \tag{1.5}$$

Note that α is a continuous mapping from $C[0,1]^n$ to R^n.

Denote $u(t) =^C D_{0^+}^\beta x(t)$, we obtain

$$x(t) = h(t) + x(0),$$

where $x(0) = (x_1(0), x_2(0), \cdots, x_n(0))^T$ and $h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s) ds$. The condition $x(0) = \alpha[x]$ gives

$$x(0) = \alpha[h] + \alpha[x(0)].$$

In view of (1.4) and (1.5), we obtain

$$\alpha[x(0)] = (\alpha_1[x(0)], \alpha_2[x(0)], \cdots, \alpha_n[x(0)])^T$$

and

$$\begin{aligned}
\alpha_i[x(0)] &= \alpha_{i1}[x_1(0)] + \alpha_{i2}[x_2(0)] + \cdots + \alpha_{in}[x_n(0)] \\
&= x_1(0)\alpha_{i1}[1] + x_2(0)\alpha_{i2}[1] + \cdots + x_n(0)\alpha_{in}[1].
\end{aligned}$$

Hence $\alpha[x(0)] = \alpha[1]x(0)$, where $\alpha[1] = (a_{ij}[1])_{1 \le i,j \le n}$.

Here and through the paper, the vectors in R^n are seen as column vectors.

Assume that $I - \alpha[1]$ is nonsingular, where I is the unit matrix of order n. Then

$$x(0) = (I - \alpha[1])^{-1}\alpha[h].$$

Therefore, $x(t) = (I - \alpha[1])^{-1}\alpha[h] + h(t)$. Let

$$(Fu)(t) = f(t, (I - \alpha[1])^{-1}\alpha[h] + h(t)),$$

$$(Gu)(t) = g(t, u(t)).$$

Then the problem (1.1) is equivalent to

$$u = Fu + Gu. \tag{1.6}$$

Let $T = F + G$, then $T : C[0,1]^n \to C[0,1]^n$. Therefore, the system (1.6) can be regarded as a fixed point problem for the operator T.

We now give some notations, definitions and basic results which are used in this paper. By a vector-valued metric on a set X we mean a mapping $d : X \times X \to R^n$ such that

(1) $d(x,y) \ge 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X$.

The meaning of " $\le$ " is the natural componentwise order relation of R^n, namely, if $x, y \in R^n, x = (x_1, x_2, \cdots, x_n)^T, y = (y_1, y_2, \cdots, y_n)^T$, by "$x \le y$" we mean $x_i \le y_i$ for $i = 1, 2, \cdots, n$.

We call the pair (X, d) a generalized metric space. For such a space, convergence and completeness are similar to those in usual metric spaces. An operator $T : X \to X$ is said to be a generalized contraction, if there exists a matrix M with $\rho(M) < 1$ such that

$$d(T(x), T(y)) \le Md(x,y)$$

for all $x, y \in X$, where $\rho(M)$ is the spectral radius of M.

For any nonnegative square matrix M, the following properties are equivalent (see $[20 - 21]$):

(1) $\rho(M) < 1$;

(2) $M^k \to 0$ as $k \to \infty$;

(3) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + \cdots$;

(4) $I - M$ is nonsingular and $(I - M)^{-1}$ is a nonnegative matrix.

As we know, for both discrete and continuous nonlocal conditions, it's important to consider the interval on which the condition acts, namely, the support of the conditions. Throughout this paper, we assume that the support of the condition $x(0) = \alpha[x]$ is the subinterval $[0, t_0]$ of $[0, 1]$, with the property $\alpha[x] = 0$ whenever $x(t) \equiv 0$ on $[0, t_0]$.

Denote the vector-norm on R^n

$$\|x\| = \begin{pmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{pmatrix} \tag{1.7}$$

for $x \in R^n, x = (x_1, x_2, \cdots, x_n)^T$.

Denote by $M_{n \times n}(R)$, $M_{n \times n}(R_+)$ the sets of all square matrices of order n with elements in R, R_+, respectively, and the inequality

$$\|Mx\| \leq |M|\|x\|$$

also holds for $M = (a_{ij})_{1 \leq i,j \leq n} \in M_{n \times n}(R), |M| = (|a_{ij}|)_{1 \leq i,j \leq n} \in M_{n \times n}(R_+)$, and $x \in R^n$.

Consider a special norm on $C[0, 1]$,

$$|x|_{C[0,1]} = \max\{|x|_{C[0,t_0]}, |x|_{C[t_0,1]}\},$$

where $|x|_{C[0,t_0]} = \max\limits_{t \in [0,t_0]} |x(t)|, |x|_{C[t_0,1]} = \max\limits_{t \in [t_0,1]} |x(t)|e^{-\theta(t-\eta)}$ for given $\theta > 0, \eta < t_0$ which will be specified later.

Note that $\|\alpha[x]\| \leq |\alpha|\|x\|_{C[0,t_0]}$, here $|\alpha| = (|\alpha_{ij}|)_{1 \leq i,j \leq n}$, $|\alpha_{ij}|$ is the norm of the functional α_{ij}.

Now we state the following lemma and theorem that will be used in the sequel.

Lemma 1.1. ([20]) *If $A \in M_{n \times n}(R_+)$ is a matrix with $\rho(A) < 1$, then $\rho(A + B) < 1$ for every matrix $B \in M_{n \times n}(R_+)$ whose elements are small enough.*

Theorem 1.2. ([21]) *Let $(X, \|\cdot\|)$ be a generalized Banach space, D be a nonempty closed bounded convex subset of X and $T : D \to X$ such that*

(i) $T = F + G$ with $F : D \to X$ completely continuous and $G : D \to X$ a generalized contraction, i.e. there exists a matrix $M \in M_{n \times n}(R_+)$ with $\rho(M) < 1$, such that $\|G(x) - G(y)\| \leq M\|x - y\|$ for all $x, y \in D$;

(ii) $F(x) + G(y) \in D$ for all $x, y \in D$.

Then T has at least one fixed point in D.

2   Main Results

In this section we assume that

$$|f_i(t, u)| \leq \begin{cases} a_{i1}|u_1| + a_{i2}|u_2| + \cdots + a_{in}|u_n| + a_i, \, t \in [0, t_0], \\ b_{i1}|u_1| + b_{i2}|u_2| + \cdots + b_{in}|u_n| + b_i, \, \, t \in [t_0, 1], \end{cases} \tag{2.1}$$

for all $u \in R^n$,

$$|g_i(t, u) - g_i(t, v)| \leq c_{i1}|u_1 - v_1| + c_{i2}|u_2 - v_2| + \cdots + c_{in}|u_n - v_n|, \tag{2.2}$$

for all $u, v \in R^n$ and $t \in [0, 1]$. Here $a_{ij}, b_{ij}, c_{ij}, a_i, b_i$ $(i, j = 1, 2, \cdots, n)$ are nonnegative numbers.

Rewrite (2.1) and (2.2) in the vector form

$$\|f(t, u)\| \leq \begin{cases} A\|u\| + S, \, t \in [0, t_0], \\ B\|u\| + L, \, t \in [t_0, 1], \end{cases}$$

$$\|g(t, u) - g(t, v)\| \leq C\|u - v\|,$$

where $A = (a_{ij})_{1 \le i,j \le n}, B = (b_{ij})_{1 \le i,j \le n}, C = (c_{ij})_{1 \le i,j \le n} \in M_{n \times n}(R_+)$, and $S = (a_1, a_2, \cdots, a_n)^T$, $L = (b_1, b_2, \cdots, b_n)^T \in R_+^n$.

The following theorem is our main results.

Theorem 2.1. *Assume that (2.1) and (2.2) hold. If the spectral radius of the matrix $M_0 + C$ is less than one, then the problem (1.1) and (1.2) has at least one solution, where $M_0 = \frac{t_0^\beta}{\beta \Gamma(\beta)} A(\|(I - \alpha[1])^{-1}\|\|\alpha\| + I)$.*

Proof. Firstly, the fact that F is completely continuous follows in a routine way from Assoli-Arzelà theorem.

Secondly, it follows from (2.2) that

$$\|(Gu)(t) - (Gv)(t)\| = \|g(t, u(t)) - g(t, v(t))\|$$

$$\le C\|u(t) - v(t)\|$$

$$\le C\|u - v\|_{C[0,t_0]}, \ t \in [0, t_0]$$

for any $u, v \in C[0,1]^n$. Taking the supremum on $[0, t_0]$, one can obtain that

$$\|Gu - Gv\|_{C[0,t_0]} \le C\|u - v\|_{C[0,t_0]}. \tag{2.3}$$

On the other hand, from (2.2) it follows that

$$\|(Gu)(t) - (Gv)(t)\| = \|g(t, u(t)) - g(t, v(t))\|$$

$$\le C\|u(t) - v(t)\|$$

$$= e^{\theta(t-\eta)} e^{-\theta(t-\eta)} C\|u(t) - v(t)\|$$

$$\le e^{\theta(t-\eta)} C\|u(t) - v(t)\|_{C[t_0,1]}, \ t \in [0, t_0]$$

for any $u, v \in C[0,1]^n$. Dividing by $e^{\theta(t-\eta)}$ and taking the supremum on $[t_0, 1]$, we obtain

$$\|Gu - Gv\|_{C[t_0,1]} \le C\|u - v\|_{C[t_0,1]}. \tag{2.4}$$

From (2.3) and (2.4), it is easy to see that

$$\|Gu - Gv\|_{C[0,1]} \le C\|u - v\|_{C[0,1]}. \tag{2.5}$$

According to $\rho(M_0 + C) < 1$ and $C < M_0 + C$, we have $\rho(C) < 1$. Hence G is a generalized contraction.

Lastly, we look for a nonempty, bounded, closed and convex subset D of $C[0,1]^n$ such that $F(D) + G(D) \subset D$. In fact, for any $u \in C[0,1]^n$, $t \in [0, t_0]$, one can obtain that

$$\|h(t)\| = \|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s) ds\|$$

$$\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \|u(s)\| ds$$

$$\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} ds \|u\|_{C[0,t_0]}$$

$$= \frac{t^\beta}{\beta \Gamma(\beta)} \|u\|_{C[0,t_0]}$$

$$\le \frac{t_0^\beta}{\beta \Gamma(\beta)} \|u\|_{C[0,t_0]}.$$

This together with (2.1) guarantees that

$$\|(Fu)(t)\| = \|f(t, (I - \alpha[1])^{-1}\alpha[h] + h(t))\|$$
$$\leq A\|(I - \alpha[1])^{-1}\alpha[h] + h(t)\| + S$$
$$\leq M_0\|u\|_{C[0,t_0]} + S, \ t \in [0, t_0].$$

Taking the maximum on $[0, t_0]$, we obtain

$$\|Fu\|_{C[0,t_0]} \leq M_0\|u\|_{C[0,t_0]} + S. \tag{2.6}$$

On the other hand, one can obtain that

$$\|\int_0^{t_0} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s)ds\| \leq \int_0^{t_0} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\|u(s)\|ds$$
$$\leq \int_0^{t_0} \frac{(t_0-s)^{\beta-1}}{\Gamma(\beta)}\|u(s)\|ds$$
$$\leq \int_0^{t_0} \frac{(t_0-s)^{\beta-1}}{\Gamma(\beta)}ds\|u\|_{C[0,t_0]}$$
$$= \frac{t_0^{\beta}}{\beta\Gamma(\beta)}\|u\|_{C[0,t_0]}, \ t \in [t_0, 1],$$

$$\|\int_{t_0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} u(s)ds\| \leq \int_{t_0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\|u(s)\|ds$$
$$= \int_{t_0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}e^{\theta(s-\eta)}e^{-\theta(s-\eta)}\|u(s)\|ds$$
$$\leq \int_{t_0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}e^{\theta(s-\eta)}ds\|u\|_{C[t_0,1]}$$
$$= \int_0^{t-t_0} \frac{\sigma^{\beta-1}}{\Gamma(\beta)}e^{-\theta\sigma}d\sigma e^{\theta(t-\eta)}\|u\|_{C[t_0,1]}$$
$$= \frac{1}{\theta^{\beta}}\int_0^{\theta(t-t_0)} \frac{\tau^{\beta-1}}{\Gamma(\beta)}e^{-\tau}d\tau e^{\theta(t-\eta)}\|u\|_{C[t_0,1]}$$
$$\leq \frac{1}{\theta^{\beta}}e^{\theta(t-\eta)}\|u\|_{C[t_0,1]}, \ t \in [t_0, 1].$$

Then

$$\|(Fu)(t)\| \leq B\|(I - \alpha[1])^{-1}\alpha[h] + h(t)\| + L$$
$$\leq B\|(I - \alpha[1])^{-1}\alpha[h] + \int_0^{t_0} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds\| + B\|\int_{t_0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds\| + L$$
$$\leq \frac{t_0^{\beta}}{\beta\Gamma(\beta)}B(|(I - \alpha[1])^{-1}||\alpha| + I)\|u\|_{C[0,t_0]} + L + \frac{1}{\theta^{\beta}}e^{\theta(t-\eta)}B\|u\|_{C[t_0,1]}, \ t \in [t_0, 1].$$

Dividing by $e^{\theta(t-\eta)}$ and taking the supremum on $[t_0, 1]$, we obtain

$$\|Fu\|_{C[t_0,1]} \leq (\frac{t_0^{\beta}}{\beta\Gamma(\beta)}B(|(I - \alpha[1])^{-1}||\alpha| + I)\|u\|_{C[0,t_0]} + L)e^{-\theta(t_0-\eta)} + \frac{1}{\theta^{\beta}}B\|u\|_{C[t_0,1]}.$$

Now taking advantage of the choice of $\eta < t_0$, we assume (choosing large enough $\theta > 0$) that

$$e^{-\theta(t_0-\eta)}B(|(I - \alpha[1])^{-1}||\alpha| + I)\|u\|_{C[0,t_0]} \leq A(|(I - \alpha[1])^{-1}||\alpha| + I)\|u\|_{C[0,t_0]},$$

$$e^{-\theta(t_0-\eta)}L \leq S.$$

By deduction, one can obtain that

$$\|Fu\|_{C[t_0,1]} \le M_0\|u\|_{C[0,t_0]} + S + \frac{1}{\theta^\beta}B\|u\|_{C[t_0,1]}. \tag{2.7}$$

Now, (2.6) and (2.7) imply that

$$\|Fu\|_{C[0,1]} \le (M_0 + \frac{1}{\theta^\beta}B)\|u\|_{C[0,1]} + S. \tag{2.8}$$

On the other hand, it follows from (2.5) that

$$\|Gu\|_{C[0,1]} \le C\|u\|_{C[0,1]} + P, \tag{2.9}$$

for every $u \in C[0,1]^n$, where

$$P = \|G(0)\|_{C[0,1]}.$$

Now we look for $R = (R_1, R_2, \ldots, R_n) \in R_+^n$ such that $\|Fu + Gv\|_{C[0,1]} \le R$ for $u, v \in C[0,1]^n$ with $\|u\|_{C[0,1]} \le R$ and $\|v\|_{C[0,1]} \le R$. To this end, according to (2.8) and (2.9), it is sufficient to show $(M_0 + \frac{1}{\theta^\beta}B + C)R + S + P \le R$. Equivalently

$$S + P \le (I - M_0 - \frac{1}{\theta^\alpha}B - C)R. \tag{2.10}$$

Since $\rho((M_0 + C) < 1$ and the entries of $\frac{1}{\theta^\beta}B$ are as small as desired for $\theta > 0$ large enough, according to Lemma 1.1, we can choose θ such that

$$\rho(M_0 + \frac{1}{\theta^\beta}B + C) < 1.$$

Then, $I - M_0 - \frac{1}{\theta^\beta}B - C$ is invertible and its inverse $(I - M_0 - \frac{1}{\theta^\beta}B - C)^{-1}$ is a nonnegative matrix , (2.10) is equivalent to

$$R \ge (I - M_0 - \frac{1}{\theta^\beta}B - C)^{-1}(S + P).$$

Therefore, $F(D) + G(D) \subset D$ if we consider $D = \{u \in C[0,1]^n : \|u\|_{C[0,1]} \le R\}$. Thus the problem (1.1) (1.2) have at least one solution. $\square$

3   An Example

Example 3.1. *Consider the following nonlocal problem*

$$\begin{cases} {}^C D_{0+}^{\frac{1}{2}}x(t) = \frac{1}{4}\sin x + \frac{1}{4}y + p(t) + \sin(\frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}{}^C D_{0+}^{\frac{1}{2}}x(t) + \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}{}^C D_{0+}^{\frac{1}{2}}y(t)) \\[2mm] {}^C D_{0+}^{\frac{1}{2}}y(t) = \frac{1}{4}x + \frac{1}{4}\cos y + q(t) + \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}\cos({}^C D_{0+}^{\frac{1}{2}}x(t)) + \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}\cos({}^C D_{0+}^{\frac{1}{2}}y(t)) \\[2mm] x(0) = \int_0^{\frac{1}{2}} 2x(s) + y(s)ds \\[2mm] y(0) = \int_0^{\frac{1}{2}} x(s) + 2y(s)ds \end{cases} \tag{3.1}$$

where $p, q \in C[0,1]$.

This problem can be regarded as the form of (1.1) (1.2). In this case,

$$f_1(t, x, y) = \frac{1}{4}\sin x + \frac{1}{4}y + p(t),$$

$$f_2(t, x, y) = \frac{1}{4}x + \frac{1}{4}\cos y + q(t),$$

$$g_1(t, x, y) = \sin(\frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}x + \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}y),$$

$$g_2(t, x, y) = \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}\cos x + \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}\cos y.$$

We have $t_0 = \frac{1}{2}, \beta = \frac{1}{2}, n = 2$ and f_1, f_2 satisfy (2.1) with

$$a_{11} = b_{11} = \frac{1}{4}, \qquad a_1 = |p|_{C[0,\frac{1}{2}]},$$
$$a_{12} = b_{12} = \frac{1}{4}, \qquad b_1 = |p|_{C[\frac{1}{2},1]},$$
$$a_{21} = b_{21} = \frac{1}{4}, \qquad a_2 = |q|_{C[0,\frac{1}{2}]},$$
$$a_{22} = b_{22} = \frac{1}{4}, \qquad b_2 = |q|_{C[\frac{1}{2},1]}.$$

Also, g_1, g_2 satisfy (2.2) with

$$c_{11} = c_{12} = c_{21} = c_{22} = \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}.$$

In addition,

$$\alpha_{11}[1] = |\alpha_{11}| = 1,$$
$$\alpha_{12}[1] = |\alpha_{12}| = \frac{1}{2},$$
$$\alpha_{21}[1] = |\alpha_{21}| = \frac{1}{2},$$
$$\alpha_{22}[1] = |\alpha_{22}| = 1.$$

Then

$$\alpha[1] = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \qquad |\alpha| = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix},$$

$$|(I - \alpha[1])^{-1}||\alpha| + I = 2I, M_0 = \frac{\sqrt{2}}{2\Gamma(\frac{1}{2})}I, M_0 + C = \frac{\sqrt{2}}{\Gamma(\frac{1}{2})}I.$$

It is easy to see that $\rho(M_0 + C) = \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} < 1$. Therefore, according to Theorem 2.1 the problem (3.1) has at least one solution.

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