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Maximal restrained sets in graph

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Abstract

This paper is about restrainedness in graphs. A characterization of maximal restrained set has been given and also some results about maximal restrained set have been proved.

Keywords: restrained set, dominating set, maximal restrained set, vertex of minimality.

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1 Introduction

The concept of restrained dominating set was defined in [1]. In this paper, we define the concept of a restrained set and a maximal restrained set. For any graph G, a maximal restrained set with minimum cardinality is called an RE – set of G. $\mathrm{RE}(G)$ denotes the number of vertices in any RE – set of G. We observe some properties of maximal restrained set and changes in the RE – number of a graph when a vertex is removed from the graph.

2 Preliminaries and Notations

The vertex set of a graph G is denoted as $V(G)$ or V. For any subset S of the vertex set V, $G \setminus S$ is a subgraph of G obtained by removing the vertices of S and all edges incident to the vertices of S. If v is a vertex of G then $G \setminus v$ denotes the subgraph of G obtained by removing the vertex v and all edges incident to v. $\delta(G)$ denotes the minimum degree of the graph G. $N[v]$ denotes the set of vertices adjacent to v including v and $N(v)$ denotes the set of vertices which are adjacent to v.

We consider only simple, undirected graphs having finite vertex set.

Definition 2.1. A subset S of $V(G)$ is said to be a restrained set of G if every vertex outside S is adjacent to atleast one other vertex outside S.

The following example shows that a subset and a superset of a restrained set need not be restrained. The set { 1, 2, 3, 4 } is a restrained set but the subset { 1, 2, 4 } and the superset { 1, 2, 3, 4, 5 } are not restrained sets.

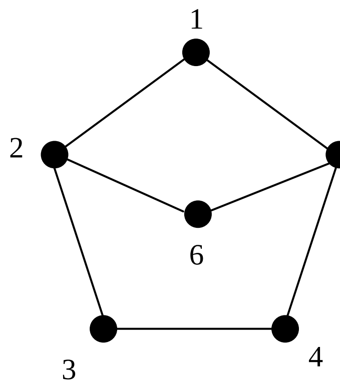

Figure 1

Definition 2.2. A subset S of $V(G)$ is said to be a maximal restrained set if the following conditions hold:

(1) S is a restrained set.

(2) $S \cup \{v\}$ is not a restrained set for every vertex v not in S.

3 Main Results

Theorem 3.1. A restrained set S is a maximal restrained set if and only if every vertex of $V \setminus S$ is adjacent to exactly one vertex of $V \setminus S$.

Proof: Let S be a maximal restrained set. Suppose that v is a vertex of $V \setminus S$ which is adjacent to more than one vertices of $V \setminus S$. Let $w \in N(v)$ such that $w \in V \setminus S$ and assume that the neighbors of w other than v are in S. Since v is adjacent to w and atleast one vertex of $V \setminus S$ other than w, $S \cup \{w\}$ is a restrained set. This is a contradiction since S is maximal. Thus every vertex of $V \setminus S$ is adjacent to exactly one vertex of $V \setminus S$.

Conversely, suppose that every vertex outside S is adjacent to exactly one vertex outside S. Let v be a vertex outside S, now there is a vertex x outside S which is adjacent to only one vertex outside S namely v then x is not adjacent to any vertex outside $S \cup \{v\}$. Thus $S \cup \{v\}$ is not a restrained set and hence S is maximal. ■

Corollary 3.2. If S is a maximal restrained set then $V \setminus S$ has even number of vertices.

Proof: By Theorem 3.1, if S is a maximal restrained set then every vertex of $V \setminus S$ is adjacent to exactly one vertex of $V \setminus S$. Thus, $V \setminus S$ has even number of vertices.

Corollary 3.3. For a graph G with $\delta(G) \geq 2$, a maximal restrained set is always a dominating set.

Proof: Suppose that S is a maximal restrained set of G. Let $v \in V \setminus S$. Since S is maximal, v is adjacent to exactly one vertex of $V \setminus S$. Since $\delta(G) \geq 2$, v is adjacent to some vertex of S. Thus S is a dominating set. ■

Definition 3.4. A vertex v of a restrained set S of a graph G is said to be a vertex of minimality (of restrainedness) of S if $S \setminus \{v\}$ is not a restrained set.

Theorem 3.5. If S is a restrained set of the graph G. A vertex v of S is a vertex of minimality of S if and only if $N[v]$ is a subset of S.

Proof: Let S be a restrained set of the graph G. Suppose that v is a vertex of minimality of S. Then $S \setminus \{v\}$ is not a restrained set. That is, there exists some vertex x outside $S \setminus \{v\}$ such that x is not adjacent to any vertex outside $S \setminus \{v\}$, that is, x is not adjacent to any vertex in $V \setminus (S \setminus \{v\})$. That is, x is not adjacent to any vertex outside S, which is not possible if $x \neq v$. Therefore $x = v$. Thus, v is not adjacent to any vertex outside S. Therefore, $N[v]$ is a subset of S.

Conversely, suppose that $N[v]$ is a subset of S. We have to prove that v is a vertex of minimality. Suppose not, then $S \setminus \{v\}$ is a restrained set. Since v is not in $S \setminus \{v\}$, v is adjacent to a vertex w outside $S \setminus \{v\}$. That is, there exists atleast one neighbor w of v in $V \setminus S$, which is a contradiction since $N[v]$ is a subset of S. Therefore, v is a vertex of minimality of the restrained set S. ■

Corollary 3.6. If S is a restrained set then every isolated vertex of a graph G in S is a vertex of minimality.

Proof: For an isolated vertex v of a graph G, $N[v]$ is an empty set, which is a subset of S. Therefore by Theorem 3.5, an isolated vertex v is a vertex of minimality. ■

Theorem 3.7. A restrained set S has no vertex of minimality if and only if $V \setminus S$ is a dominating set.

Proof: Let S be a restrained set of the graph G having no vertex of minimality. Consider a vertex v from S. Since v is not a vertex of minimality of S, by Theorem 3.5, $N[v]$ is not a subset of S. Hence, there exists atleast one neighbor w of v such that $w \in V \setminus S$. Thus for each $v \in S$, there exists atleast one neighbor w of v in $V \setminus S$. Thus $V \setminus S$ is a dominating set.

Conversely, suppose that $V \setminus S$ is a dominating set. Therefore, each vertex of S is adjacent to some vertex of $V \setminus S$. That is, there is no vertex v in S such that $N[v]$ is a subset of S. H the restrained set S has no vertex of minimality. ■

Theorem 3.8. A restrained subset S of G not containing v is a restrained set of $G \setminus v$ if and only if every neighbor of v which is outside S is also adjacent to some other vertex outside S.

Proof: Let S be a restrained set of the graph G not containing v. Suppose every neighbor of v, which is outside S is also adjacent to some other vertex outside S. We have to prove that S is a restrained set of $G \setminus v$. Consider the subgraph $G \setminus v$ and a vertex w not in S. If w is a neighbor of v then w is adjacent to some vertex x outside S (by the given condition). If w is not a neighbor of v, then since S is a restrained set of G, w is adjacent to some vertex x outside S. Hence, S is a restrained set of $G \setminus v$.

Conversely, suppose that S is a restrained set of $G \setminus v$. Suppose that w is a neighbor of v outside S. Since S is a restrained set of $G \setminus v$, w is adjacent to some vertex x outside S and hence the theorem is proved. ■

Definition 3.9. (devoted vertex) For a subset S of $V(G)$ and a vertex v of $V \setminus S$, v is said to be a devoted vertex to S if $N(v)$ is a subset of S.

Theorem 3.10. A restrained set of $G \setminus v$ is a restrained set of G if and only if neither v nor any of its neighbor outside S is devoted to S.

Proof: Let S be a restrained set of $G \setminus v$. Suppose neither v nor any of its neighbor outside S is devoted to S. Since v is not devoted to S, there is a neighbor w of v such that w is not in S.

Let x be any vertex of G outside S. If x is a neighbor of v then x is not devoted to S, hence there is a neighbor y of x outside S.

If x is not a neighbor of v then x is a vertex of the graph $G \setminus v$. Since S is a restrained set in $G \setminus v$, x must be adjacent to some vertex z outside S. Thus, S is a restrained set of G.

Conversely, suppose that S is a restrained set of G. Then, v is adjacent to some vertex w outside S. Thus v is not devoted to S.

Let x be any vertex of G outside S and x is a neighbor of v. Since S is a restrained set of G, x must be adjacent to some vertex outside S. Thus x is not devoted to S. ∎

Definition 3.11. (RE – set) A set S is said to be an RE – set if it is a maximal restrained set with minimum cardinality. The cardinality of an RE – set is denoted by RE(G).

Theorem 3.12. Let G be a graph and $v \in V(G)$, then $\text{RE}(G \setminus v) < \text{RE}(G)$ if and only if there is an RE – set S of G such that $v \in S$.

Proof: First suppose that $\text{RE}(G \setminus v) < \text{RE}(G)$.

Let S_1 be an RE – set of $G \setminus v$. Then S_1 cannot be a maximal restrained set in G because otherwise v would be adjacent to a unique vertex x outside S_1. Also since S_1 is a maximal restrained set in $G \setminus v$, x is adjacent to a unique vertex y in $G \setminus v$. Thus, x would be adjacent to two distinct vertices v and y outside S_1. This is a contradiction. Hence, S_1 cannot be a maximal restrained set in G.

Now consider the set $S = S_1 \cup \{v\}$. Obviously, S is a restrained set and infact a maximal restrained set in G. Since $\text{RE}(G \setminus v) < \text{RE}(G)$, S must be an RE – set of G. Obviously, S contains the vertex v.

Conversely, suppose there is an RE – set T of G such that $v \in T$. Consider the set $T_1 = T \setminus \{v\}$. We prove that T_1 is a maximal restrained set in $G \setminus v$. Let x be a vertex of $G \setminus v$ which is not in T_1. Then $x \notin T$. Since T is a maximal restrained set of G, x is adjacent to exactly one vertex y outside T. Obviously, $y \neq v$ and thus y is a vertex of $G \setminus v$. Then, T_1 is a maximal restrained set in $G \setminus v$ and hence $\text{RE}(G \setminus v) \leq |T_1| < |T| = \text{RE}(G)$. ∎

Remark 3.13. It may be noted that if $\text{RE}(G \setminus v) < \text{RE}(G)$, then $\text{RE}(G \setminus v) = \text{RE}(G) - 1$.

Theorem 3.14. Let G be a graph and $v \in V(G)$, then $\text{RE}(G \setminus v) > \text{RE}(G)$ if and only if for every RE – set S of G, $v \notin S$.

Proof: Suppose that $\text{RE}(G \setminus v) > \text{RE}(G)$.

Suppose that there is an RE – set S of G such that $v \in S$. Then by Theorem 3.12, it implies that $RE(G \setminus v) < RE(G)$, which is a contradiction. Thus, $v \notin S$ for every RE – set S of G.

Conversely, suppose that $v \notin S$ for every RE – set S of G. Let S be an RE – set of G. Now, v is adjacent to a unique vertex w outside S. Now consider the set $\underline{S}_1 = S \cup \{w\}$. We claim that S_1 is a maximal restrained set in $G \setminus v$. Let x be a vertex which is not in S_1. Therefore, $x \notin S$ and also $x \neq v$. Now there is a unique vertex y outside S such that x is adjacent to y. This vertex y cannot be equal to w, because otherwise w would be adjacent to two vertices x and v in G, which is a contradiction. Thus y is a vertex outside S_1 which is unique and it is adjacent to x in $G \setminus v$.

Suppose there is a maximal restrained set T of $G \setminus v$ such that $|T| < |S_1|$. Consider the set $T_1 = T \cup \{v\}$. Then $|T_1| \leq |S|$ and T_1 is a maximal restrained set in G. Hence $|T_1| = |S|$. Thus T_1 is an RE – set of G which contains the vertex v, which is a contradiction. Therefore, if T is a subset of $G \setminus v$ and $|T| < |S_1|$, then T cannot be a maximal restrained set in $G \setminus v$. Thus $S_1 = S \cup \{w\}$ is an RE – set of $G \setminus v$. Thus $RE(G) = |S| < |S \cup \{w\}| = RE(G \setminus v)$, and hence $RE(G \setminus v) > RE(G)$. ■

Remark 3.15.

(1) It may be observed that for any vertex v in a graph G, either $RE(G \setminus v) < RE(G)$ or $RE(G \setminus v) > RE(G)$. Thus it does not happen that $RE(G \setminus v) = RE(G)$. This is not like other parameters such as domination, total domination, independent domination, independence and others.

(2) From the proof of Theorem 3.14, it is clear that if $RE(G \setminus v) > RE(G)$, then $RE(G \setminus v) = RE(G) + 1$. This is also not like many other parameters related to graph.

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