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A new class of sets weaker than α-open sets

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Abstract

In this paper, we introduce a new class of sets, namely semi*α-open sets, using α-open sets and the generalized closure operator. We find characterizations of semi*α-open sets. We also define the semi*α-interior of a subset. Further, we study some fundamental properties of semi*α-open sets and semi*α-interior.

Keywords: **S**emi α-open set, semi α-interior, generalized closure, semi*α-open set, semi*α-interior.

AMS Subject Classification (**2010**)**:** 54A05.

1 Introduction

Norman Levine [7] introduced semi-open sets in topological spaces in 1963. Since the introduction of semi-open sets, many generalizations of various concepts in topology were made by considering semi-open sets instead of open sets. Njastad[13] introduced the concept of α-open sets in 1965. Levine [8] also defined and studied generalized closed sets in 1970. Dunham [4] introduced the concept of generalized closure using Levine's generalized closed sets and studied some of its properties. Govindappa Navalagi[12] defined the concept of semi α-open sets by considering α-open sets instead of open sets in the definition of semi-open sets. Hakeem A. Othman [5] introduced and studied various concepts concerning semi α-open sets. The authors have recently defined a new class of sets namely semi*-open sets [14] and investigated some of its properties.

In this paper, analogous to Navalagi's semi α-open sets, we define a new class of sets, namely semi*α-open sets, using the generalized closure operator due to Dunham instead of the closure operator in the definition of semi α-open sets. We further show that the concept of semi*α-open sets is weaker than the concept of α-open sets but stronger than the concept of semi α-open sets. We find characterizations of semi*α-open sets. We investigate fundamental properties of semi*α-open sets. We also define the semi*α-interior of a subset and study some of its basic properties.

2 Preliminaries

Throughout this paper (X, τ) will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of a space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively.

Definition 2.1. A subset A of a space X is generalized closed (briefly g-closed) [8] if $Cl(A)\subseteq U$ whenever U is an open set in X containing A.

Definition 2.2. If A is a subset of a space X, the generalized closure [4] of A is defined as the intersection of all g-closed sets in X containing A and is denoted by $Cl^*(A)$.

Definition 2.3. A subset A of a topological space (X, τ) is semi-open [7](respectively semi*-open [14]) if there is an open set U in X such that $U\subseteq A\subseteq Cl(U)$ (respectively $U\subseteq A\subseteq Cl^*(U)$) or equivalently if $A\subseteq Cl(Int(A))$ (respectively $A\subseteq Cl^*(Int(A))$).

Definition 2.4. A subset A of a topological space (X, τ) is pre-open [9] (respectively α-open [13], semi-preopen[2]= β - open [1], semi*-preopen) if $A\subseteq Int(Cl(A))$ (respectively $A\subseteq Int(Cl(Int(A)))$, $A\subseteq Cl(Int(Cl(A)))$, $A\subseteq Cl^*(pInt(A))$).

Definition 2.5. A subset A is semi α-open [12] if there is an α-open set U in X such that $U\subseteq A\subseteq Cl(U)$ or equivalently if $A\subseteq Cl(\alpha Int(A))$.

The class of all semi-open (respectively preopen, semi*-open, α-open, semi-preopen, semi*-preopen and semi α-open) sets in (X, τ) is denoted by SO(X) (respectively PO(X), S*O(X), αO(X) or τ^{α}, SPO(X), S*PO(X) and SαO(X)).

Definition 2.6. The semi-interior[3] (respectively semi*-interior[14], α-interior, pre-interior[11], semipre-interior[2], semi*-pre-interior and semi α-interior) of a subset A is defined to be the union of all semi-open (respectively semi*-open, α-open, preopen, semi-preopen, semi*-preopen and semi α-open) subsets of A. It is denoted by $sInt(A)$ (respectively $s^*Int(A)$, $\alpha Int(A)$, $pInt(A)$, $spInt(A)$, $s^*pInt(A)$ and $s\alpha Int(A)$).

Definition 2.7. A topological space X is $\mathbf{T_{1/2}}$ [8] if every g-closed set in X is closed.

Theorem 2.8. [4] Cl^* is a Kuratowski closure operator in X.

Definition 2.9. [4] If τ^* is the topology on X defined by the Kuratowski closure operator Cl^*, then (X, τ^*) is $T_{1/2}$.

Definition 2.10. [15] A space X is locally indiscrete if every open set in X is closed.

Definition 2.11. [15] A space X is extremally disconnected if the closure of every open set in X is open.

Theorem 2.12. A subset A is α-open if and only if there exists an open set G such that $G\subseteq A\subseteq Int(Cl(G))$.

Remark 2.13.

(i) In an extremally disconnected space, the semi-open sets and the α-open sets coincide.

(ii) In a locally indiscrete space, the open sets, the semi-open sets and the α-open sets coincide.

Definition 2.14. A collection $\Im$ of subsets of a set X is said to form a ***supra topology*** [10] on X if it satisfies (i) $\phi, X\in\Im$ (ii) $\Im$ is closed under arbitrary union.

Definition 2.15. The ***Khalimsky topology*** [6] or the ***digital topology*** is the topology **κ** on the set of integers generated by the collection of all triplets of the form {2n-1, 2n, 2n+1} as sub base. The digital line equipped with the Khalimsky topology is called the ***Khalimsky line***. The topological product of two copies of Khalimsky lines ($\mathbb{Z}$, κ) is called the ***Khalimsky plane or the digital plane***.

3 Semi*α-Open Sets

Definition 3.1: A subset A of a topological space (X, τ) is called a ***semi*α-open set*** if there exists an α-open set U in X such that $U\subseteq A\subseteq Cl^*(U)$.

The class of all semi*α-open sets in (X, τ) is denoted by $S^*\alpha O(X, \tau)$ or simply $S^*\alpha O(X)$.

Theorem 3.2. For a subset A of a topological space (X, τ) the following statements are equivalent:

(i) A is semi*α-open.

(ii) $A\subseteq Cl^*(\alpha Int(A))$.

(iii) $Cl^*(\alpha Int(A))=Cl^*(A)$.

(iv) $Cl^*(A\cap Int(Cl(Int(A))))=Cl^*(A)$.

Proof: (i)⟹(ii): If A is semi*α-open, then there is an α-open set U such that $U\subseteq A\subseteq Cl^*(U)$.
Now $U\subseteq A \Longrightarrow U=\alpha Int(U)\subseteq\alpha Int(A) \Longrightarrow A\subseteq Cl^*(U)\subseteq Cl^*(\alpha Int(A))$.
(ii)⟹(iii): By assumption, $A\subseteq Cl^*(\alpha Int(A))$. Since Cl^* is a Kuratowski operator, we have $Cl^*(A)\subseteq Cl^*(Cl^*(\alpha Int(A)))=Cl^*(\alpha Int(A))$. Now $\alpha Int(A)\subseteq A$ implies that $Cl^*(\alpha Int(A))\subseteq Cl^*(A)$. Therefore, $Cl^*(\alpha Int(A))=Cl^*(A)$.
(iii)⟹(i): Take $U=\alpha Int(A)$. Then U is an α-open set in X such that $U\subseteq A\subseteq Cl^*(A)=Cl^*(\alpha Int(A))=Cl^*(U)$. Therefore by Definition 3.1, A is semi*α-open.

(iii)⟺(iv): Follows from the fact that for any subset A, $\alpha Int(A)=A\cap Int(Cl(Int(A)))$.

Theorem 3.3. If a subset A of a topological space (X, τ) is semi*α-open, then the following statements hold:

(i) There exists an open set G such that $G\subseteq A\subseteq Cl^*(Int(Cl(G)))$.

(ii) There exists an open set G such that $G\subseteq A\subseteq Cl^*(sCl(G))$.

(iii) There exists an open set G such that $G\subseteq A\subseteq Cl^*(G\cup Int(Cl(G)))$.

(iv) $A\subseteq Cl^*(Int(Cl(Int(A))))$.

Proof: (i) Since A is a semi*α-open set, by Definition 3.1, there exists an α–open set U such that $U\subseteq A\subseteq Cl^*(U)$. Since U is an α-open set, by Theorem 2.12, there is an open set G such that $G\subseteq U\subseteq Int(Cl(G))$. This implies $G\subseteq A\subseteq Cl^*(Int(Cl(G)))$.

(ii) Follows from (i) and the fact that for any open set G, $Int(Cl(G))=sCl(G)$.

(iii) Follows from (ii) and the fact that for any subset G, $sCl(G)=G\cup Int(Cl(G))$.

(iv) From (i), $G\subseteq A$ implies that $Cl^*(Int(Cl(G)))\subseteq Cl^*(Int(Cl(Int(A))))$. Hence, $A\subseteq Cl^*(Int(Cl(Int(A))))$.

Remark 3.4. (i) In any topological space (X, τ), ϕ and X are semi*α-open sets.
(ii) From Theorem 3.3(i), every nonempty semi*α-open set must contain a nonempty open set and therefore cannot be nowhere dense.
(iii) In any topological space, a singleton set is semi*α-open if and only if it is open.

Theorem 3.5. Arbitrary union of semi*α-open sets in X is also semi*α-open in X.

Proof: Let $\{A_i\}$ be a collection of semi*α-open sets in X. Since each A_i is semi*α-open, there is an α-open set U_i in X such that $U_i\subseteq A_i\subseteq Cl^*(U_i)$. Then $\cup U_i\subseteq \cup A_i\subseteq \cup Cl^*(U_i)\subseteq Cl^*(\cup U_i)$. Since $\cup U_i$ is α-open, by Definition 3.1, $\cup A_i$ is semi*α-open.

Remark 3.6. The intersection of two semi*α-open sets need not be semi*α-open as seen from the following examples. However the intersection of a semi*α-open set and an open set is semi*α-open as shown in Theorem 3.9.

Example 3.7: Let $X = \{a, b, c, d\}$ and $\tau =\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. In the space (X, τ), the subsets $\{a, d\}$ and $\{b, d\}$ are semi*α-open but their intersection $\{d\}$ is not semi*α-open.

Example 3.8. Consider the subspace X of the digital plane where $X= \{1, 2\}\times\{1, 2, 3\}$.
In X, the subsets $A=\{(1,1), (2,2)\}$ and $B=\{(1,3), (2,2)\}$ are semi*α-open but $A\cap B=\{(2,2)\}$ is not semi*α-open.

Theorem 3.9. If A is semi*α-open in X and B is open in X, then $A\cap B$ is semi*α-open in X.

Proof: Since A is semi*α-open in X, there is an α-open set U such that $U\subseteq A\subseteq Cl^*(U)$.
Since B is open, we have $U\cap B\subseteq A\cap B\subseteq Cl^*(U)\cap B\subseteq Cl^*(U\cap B)$. Since $U\cap B$ is α-open, by Definition 3.1, $A\cap B$ is semi*α-open in X.

Remark 3.10. From Remark 3.4(i) and Theorem 3.5, in general, $S^*\alpha O(X, \tau)$ forms a supra-topology on X. However $S^*\alpha O(X, \tau)$ forms a topology on X if and only if it is closed under finite intersection. Also in any topological space(X, τ), $S^*\alpha O(X, \tau)$ generates a topology on X that is finer than the original topology τ as seen from Theorem 3.16.

Theorem 3.11. In any topological space,
(i) Every α-open set is semi*α-open.
(ii) Every open set is semi*α-open.
(iii) Every semi*-open set is semi*α-open.
(iv) Every semi*α-open set is semi α-open.
(v) Every semi*α-open set is semi*-preopen.
(vi) Every semi*α-open set is semi-preopen.
(vii) Every semi*α-open set is semi-open.

Proof:(i) Let U be α-open in X. Then by Definition 3.1, U is semi*α-open. (ii) follows from (i) since every open set is α-open. If A is semi*-open, then there is an open set U such that $U\subseteq A\subseteq Cl^*(U)$. Since every open set is α-open, by Definition 3.1, A is semi*α-open. This proves (iii). Let A be a semi*α-open set. Then there is an α-open set U in X such that $U\subseteq A\subseteq Cl^*(U)$. Since $Cl^*(U)\subseteq Cl(U)$, we have $U\subseteq A\subseteq Cl(U)$. Hence A is semi α-open. Thus (iv) is proved. Let A be a semi*α-open set. Then there is an α-open set U in X such that $U\subseteq A\subseteq Cl^*(U)$. Since every α-open set is preopen, by Definition 2.4, A is semi*-preopen. This proves (v). The statement (vi) follows from (v) and the fact that every semi*-preopen set is semi-preopen. Suppose A is a semi*α-open set. Then from Theorem 3.2, $A\subseteq Cl^*(\alpha Int(A))$. Since $Cl^*(\alpha Int(A)) \subseteq Cl(\alpha Int(A))$ and $\alpha Int(A)=A\cap Int(Cl(Int(A)))$, we have $A\subseteq Cl(Int(A))$. Hence, A is semi-open. This proves (vii).

Remark 3.12. The converse of each of the statements in Theorem 3.11 is not true as shown in the following examples.

Example 3.13. Consider the topological space (X, τ) in Example 3.7. The subset $\{a, d\}$ is semi*α-open in X but it is neither open nor α-open.

Example 3.14. Consider the subspace X of the digital plane given in Example 3.8. In X, the subsets $\{(1,1),(1,3),(2,2)\}$, $\{(1,1),(1,3),(2,1),(2,2)\}$ and $\{(1,1),(1,2),(1,3),(2,2),(2,3)\}$ are semi*α-open but not open.

Example 3.15. In the space (X, τ) where $X=\{a, b, c, d, e\}$ and $\tau=\{\phi, \{a\}, \{b\}, \{a, b\},\{a, b, c, d\}, X\}$, the subset $\{a, c\}$ is semi-open but not semi*α-open.

Theorem 3.16. If (X, τ) is a topological space, let $\tau_{s^*\alpha}= \{U\in S^*\alpha O(X, \tau) : U\cap A\in S^*\alpha O(X, \tau) \text{ for all } A\in S^*\alpha O(X, \tau)\}$. Then $\tau_{s^*\alpha}$ is a topology on X that is finer than τ.

Proof: Clearly from definition, $\phi, X\in\tau_{s^*\alpha}$. Let $U_\beta\in \tau_{s^*\alpha}$ and $U=\cup U_\beta$. Since each $U_\beta\in S^*\alpha O(X,\tau)$, by Theorem 3.5, $U=\cup U_\beta\in S^*\alpha O(X, \tau)$. Let $A\in S^*\alpha O(X, \tau)$. Then $U_\beta\cap A\in S^*\alpha O(X, \tau)$, for each β and hence by Theorem 3.5, $U\cap A=(\cup U_\beta)\cap A =\cup (U_\beta\cap A)\in S^*\alpha O(X, \tau)$. Therefore $U\in\tau_{s^*\alpha}$. Now let $U_1, U_2\in\tau_{s^*\alpha}$. Then $U_1,U_2\in S^*\alpha O(X, \tau)$ and from definition of $\tau_{s^*\alpha}$, we get $U_1\cap U_2\in S^*\alpha O(X, \tau)$. If $A\in S^*\alpha O(X,\tau)$, then by the definition of $\tau_{s^*\alpha}$, we have $(U_1\cap U_2)\cap A\in S^*\alpha O(X, \tau)$. Hence $U_1\cap U_2\in\tau_{s^*\alpha}$. Inductively, it can be shown that $\tau_{s^*\alpha}$ is closed under finite intersection. This shows that $\tau_{s^*\alpha}$ is a topology on X. Let $V\in\tau$. By Theorem 3.11(ii), $V\in S^*\alpha O(X, \tau)$. Also by Theorem 3.9, $V\cap A \in S^*\alpha O(X, \tau)$ for all $A\in S^*\alpha O(X, \tau)$. Hence $V\in\tau_{s^*\alpha}$. Thus $\tau_{s^*\alpha}$ is finer than τ.

Example 3.17. Consider the topological space(X, τ) where, $X = \{a, b, c, d\}$ and $\tau =\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Here $S^*\alpha O(X, \tau)=\{\phi, \{a\} ,\{b\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\tau_{s^*\alpha}=\{\phi, \{a\} ,\{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Here $\tau\subsetneq\tau_{s^*\alpha}$.

Example 3.18. Consider the topological space(X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi,\{a\},\{a, b, c\}, X\}$. Here $S^*\alpha O(X,\tau)=\{\phi, \{a\},\{a ,b\},\{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$ and $\tau_{s^*\alpha}=S^*\alpha O(X, \tau)$.

Example 3.19. Consider the topological space(X, τ) where, $X=\{a, b, c\}$ and $\tau=\{\phi,\{a\}, \{b\}, \{a, b\}, X\}$. Here $S^*\alpha O(X)= \{\phi,\{a\}, \{b\}, \{a, b\}, \{a, c\}. \{b, c\}, X\}$ and $\tau_{s^*\alpha}= \tau$.

Theorem 3.20. In any topological space (X,τ),

(i) $\tau\subseteq \alpha O(X, \tau)\subseteq S^*\alpha O(X, \tau)\subseteq S\alpha O(X, \tau) \subseteq SPO(X, \tau)$

(ii) $\tau\subseteq \alpha O(X, \tau)\subseteq S^*\alpha O(X, \tau)\subseteq S^*PO(X, \tau) \subseteq SPO(X, \tau)$

(iii) $\tau \subseteq S^*O(X, \tau) \subseteq S^*\alpha O(X, \tau) \subseteq S\alpha O(X, \tau)\subseteq SPO(X, \tau)$ and

(iv) $\tau \subseteq S^*O(X, \tau) \subseteq S^*\alpha O(X, \tau) \subseteq SO(X, \tau)\subseteq SPO(X, \tau)$.

Proof: Follows from the facts that every open set is α-open, every semi α-open set is semi-preopen and every α-open set is preopen and from Theorem 3.11.

Corollary 3.21. If (X, τ) is a locally indiscrete space,
$\tau =\alpha O(X, \tau)= S^*\alpha O(X, \tau) = S\alpha O(X, \tau)=S^*O(X,\tau)=SO(X, \tau)$.

Proof: Let (X, τ) be a locally indiscrete space. From Remark 2.13(ii), the semi-open sets, the open sets and the α-open sets in X coincide. Hence by Theorem 3.18, we have
$\tau =\alpha O(X, \tau) \subseteq S^*O(X, \tau) \subseteq S^*\alpha O(X, \tau)= S\alpha O(X,\tau) =SO(X,\tau)=\tau$. This implies that
$\tau =\alpha O(X, \tau)=S^*O(X, \tau) = S^*\alpha O(X, \tau)= S\alpha O(X,\tau)=SO(X,\tau)$.

Remark 3.22. (i) In the Sierpinski space (X, τ), where $X = \{a, b\}$ and $\tau =\{\phi, \{a\}, X\}$,
$\tau=\alpha O(X, \tau)=S^*O(X, \tau)=SO(X, \tau)=S^*\alpha O(X, \tau)=S\alpha O(X, \tau)=PO(X, \tau)=S^*PO(X, \tau)=SPO(X, \tau)$.
(ii) In a $T_{1/2}$ space, the g-closed sets and the closed sets coincide and hence $Cl^*(U) = Cl(U)$. Therefore, the family of semi*α-open sets equals the family of semi α-open sets. In particular, in the Khalimsky line and in the real line with usual topology, the semi*α-open sets and the semi α-open sets coincide.
(iii) The inclusions in Theorem 3.20 involving semi*α-open sets may be strict and equality may also hold. This can be seen from the following examples.

Example 3.23. In the topological space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{b, c, d\}, X\}$
$\alpha O(X, \tau)=S^*\alpha O(X, \tau)=S\alpha O(X, \tau)$.

Example 3.24. In the topological space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$, $\alpha O(X, \tau) \subsetneq S^*\alpha O(X, \tau) = S\alpha O(X, \tau)$.

Example 3.25. In the topological space (X, τ) where $X= \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\alpha O(X, \tau)= S^*\alpha O(X, \tau) \subsetneq S\alpha O(X, \tau)$.

Example 3.26. Consider the topological space (X, τ) where $X=\{a, b, c, d, e\}$ and
$\tau =\{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b\ c, d\}, X\}$.
Here, $\alpha O(X, \tau) \subsetneq S^*\alpha O(X, \tau) \subsetneq S\alpha O(X, \tau)$.

Example 3.27. Consider the subspace (X, τ) of the digital plane where $X=\{1, 2, 3\} \times \{0, 1\}$.
If a, b, c, d, e and f denote the points (1, 0), (1, 1), (2, 0), (2, 1), (3, 0) and (3,1) respectively,
then $\tau =\{\phi, \{b\}, \{f\}, \{a, b\}, \{b, f\}, \{e, f\}, \{a, b, f\}, \{b, d, f\}, \{b, e, f\}, \{a, b, d, f\},\{a, b, e, f\}, b, d, e, f \}, \{a, b, d, e, f\}, X\}$. Here $\alpha O(X, \tau)\subsetneq S^*\alpha O(X, \tau)=S^*PO(X, \tau) \subsetneq S\alpha O(X, \tau)=SPO(X, \tau)$. Also, $S^*O(X, \tau) = S^*\alpha O(X, \tau)$.

Theorem 3.28. Let A be semi*α-open and $B\subseteq X$ such that $\alpha Int(A)\subseteq B\subseteq Cl^*(A)$. Then B is semi*α-open.

Proof: Since A is semi*α-open, by Theorem 3.2, we have $Cl^*(A)=Cl^*(\alpha Int(A))$. Since $\alpha Int(A)\subseteq B$, $\alpha Int(A)\subseteq\alpha Int(B)$ and hence $Cl^*(\alpha Int(A))\subseteq Cl^*(\alpha Int(B))$. Therefore by the assumption, we have $B\subseteq Cl^*(A)=Cl^*(\alpha Int(A))\subseteq Cl^*(\alpha Int(B))$. Hence $B\subseteq Cl^*(\alpha Int(B))$. Again by invoking Theorem 3.2, B is semi*α-open.

Theorem 3.29. Let $\mathcal{B}$ be a collection of subsets in (X, τ) satisfying (i) $\alpha O(X, \tau)\subseteq \mathcal{B}$ (ii) If $B\in\mathcal{B}$ and if D is a subset of X satisfying $\alpha Int(B)\subseteq D\subseteq Cl^*(B)$ then $D\in\mathcal{B}$. Then $S^*\alpha O(X, \tau)\subseteq \mathcal{B}$. Thus $S^*\alpha O(X, \tau)$ is the smallest collection satisfying the conditions (i) and (ii).

Proof: By Theorem 3.20(i), $S^*\alpha O(X, \tau)$ satisfies (i) and by Theorem 3.28, $S^*\alpha O(X, \tau)$ satisfies the condition (ii). Let $\mathcal{B}$ be a collection of subsets satisfying the conditions (i) and (ii). If $A\in S^*\alpha O(X, \tau)$, then there is an α-open set U in X such that $U\subseteq A\subseteq Cl^*(U)$. By condition (i), $U\in \mathcal{B}$. Since $\alpha Int(U)=U$, by condition (ii), $A\in \mathcal{B}$. Thus, $S^*\alpha O(X, \tau)\subseteq \mathcal{B}$. Thus $S^*\alpha O(X, \tau)$ is the smallest collection satisfying both (i) and (ii).

Remark 3.30. The concept of semi*α-open sets are independent of each of the concepts of g-open sets and pre-open as seen from the following example:

Example 3.31. In the topological space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, the subset $\{a, d\}$ is semi*α-open but not g-open and $\{b, c\}$ is g-open but not semi*α-open.

Example 3.32. In the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{b\}, \{a, b\},\{a, b, c\}, X\}$, the subset $\{a, d\}$ is semi*α-open but not pre-open.

Example 3.33. In the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi,\{a, b\},\{a, b, c\}, X\}$, the subset $\{b, c\}$ is pre-open but not semi*α-open.

From the above discussions we have the following diagram:

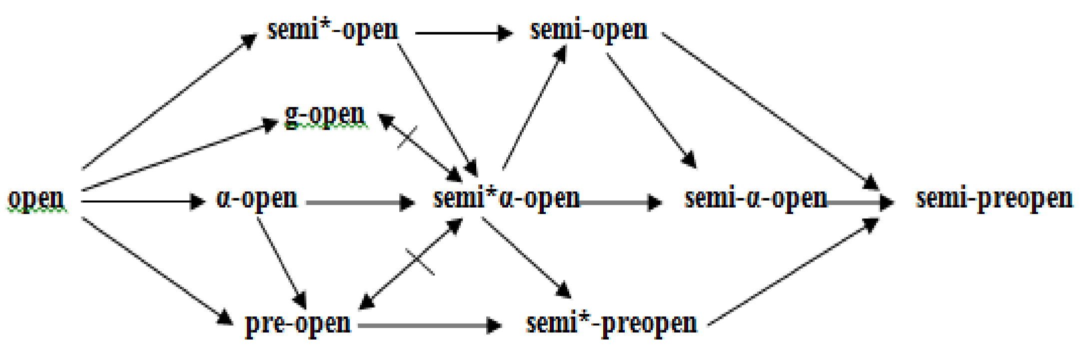

Figure 1.

4 Semi*α-Interior of a Set

Definition 4.1. The ***semi*α-interior*** of A is defined as the union of all semi*α-open sets of X contained in A. It is denoted by $s^*\alpha Int(A)$.

Definition 4.2. Let A be a subset of a topological space (X, τ). A point x in X is called a ***semi*α-interior point*** of A if there is a semi*α-open subset of A that contains x.

Theorem 4.3. If A is any subset of a topological space (X, τ), then

(i) $s^*\alpha Int(A)$ is the largest semi*α-open set contained in A.

(ii) A is semi*α-open if and only if $s^*\alpha Int(A)=A$.

(iii) $s^*\alpha Int(A)$ is the set of all semi*α-interior points of A.

(iv) A is semi*α-open if and only if every point of A is a semi*α-interior point of A.

Proof:(i) Being the union of all semi*α-open subsets of A, by Theorem 3.5, $s^*\alpha Int(A)$ is semi*α-open and contains every semi*α-open subset of A. This proves (i).

(ii) A is semi*α-open implies $s^*\alpha Int(A)=A$ is obvious from Definition 4.1. On the other hand, suppose $s^*\alpha Int(A)=A$. By (i), $s^*\alpha Int(A)$ is semi*α-open and hence A is semi*α-open.

(iii) By Definition 4.1, $x\in s^*\alpha Int(A)$ if and only if x belongs to some semi*α-open subset U of A. That is, if and only if x is a semi*α-interior point of A.

(iv) follows from (ii) and (iii).

Theorem 4.4. (Properties of Semi*α-Interior)

In any topological space (X, τ) the following statements hold:

(i) $s^*\alpha Int(\phi)=\phi$.

(ii) $s^*\alpha Int(X)=X$.

If A and B are subsets of X,

(iii) $s^*\alpha Int(A)\subseteq A$.

(iv) $A\subseteq B \Longrightarrow s^*\alpha Int(A)\subseteq s^*\alpha Int(B)$.

(v) $s^*\alpha Int(s^*\alpha Int(A))=s^*\alpha Int(A)$.

(vi) $Int(A)\subseteq \alpha Int(A)\subseteq s^*\alpha Int(A)\subseteq s\alpha Int(A)\subseteq spInt(A)\subseteq A$.

(vii) $Int(A)\subseteq s^*Int(A)\subseteq s^*\alpha Int(A)\subseteq s\alpha Int(A)\subseteq spInt(A)\subseteq A$.

(viii) $Int(A)\subseteq \alpha Int(A)\subseteq s^*\alpha Int(A)\subseteq s^*pInt(A)\subseteq spInt(A)\subseteq A$.

(ix) $s^*\alpha Int(A\cup B)\supseteq s^*\alpha Int(A)\cup s^*\alpha Int(B)$.

(x) $s^*\alpha Int(A\cap B)\subseteq s^*\alpha Int(A) \cap s^*\alpha Int(B)$.

(xi) $Int(s^*\alpha Int(A))=Int(A)$.

(xii) $s^*\alpha Int(Int(A))=Int(A)$.

Proof: (i), (ii), (iii) and (iv) follow from Definition 4.1. By Theorem 4.3(i), $s^*\alpha Int(A)$ is semi*α-open and by Theorem 4.3(ii), $s^*\alpha Int(s^*\alpha Int(A))=s^*\alpha Int(A)$. Thus (v) is proved. The statements (vi), (vii) and (viii) follow from Theorem 3.20 and the facts that every open set is α-open, every semi-α-open set is semi-peopen and that every semi*-preopen set is semi-preopen. Since $A\subseteq A\cup B$, from statement (iv) we have $s^*\alpha Int(A)\subseteq s^*\alpha Int(A\cup B)$. Similarly, $s^*\alpha Int(B)\subseteq s^*\alpha Int(A\cup B)$. This proves (ix). The proof for (x) is similar. Since $s^*\alpha Int(A)\subseteq A$, $Int(s^*\alpha Int(A))\subseteq Int(A)$. From (vi), $Int(A)\subseteq s^*\alpha Int(A)$ and so $Int(A)\subseteq Int(s^*\alpha Int(A))$.Therefore $Int(s^*\alpha Int(A))=Int(A)$. This proves (xi). Since $Int(A)$ is open, by Theorem 3.11(ii), $Int(A)$ is semi*α-open and hence by invoking Theorem 4.3(ii), we have $s^*\alpha Int(Int(A))=Int(A)$.

Remark 4.5. In (vi), (vii) and (viii) of Theorem 4.4, the inclusions involving semi*α-interior may be strict and equality may also hold. This can be seen from the following examples:

Example 4.6. Consider the topological space (X, τ) where $X=\{a, b, c, d, e\}$ and $\tau=\{\phi,\{a\},\{b\},\{a, b\}, \{b, c\},\{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}, X\}$. Let $A=\{b, c, d, e\}$.
Then $\alpha Int(A)=\{b, c\}$, $s^*\alpha Int(A)=\{b, c, e\}$, $s\alpha Int(A)=\{b, c, d, e\}$. Here $\alpha Int(A)\subsetneq s^*\alpha Int(A)\subsetneq s\alpha Int(A)$.
Let $B=\{a, e\}$. Then $\alpha Int(B)=\{a\}$, $s^*\alpha Int(B)= s\alpha Int(B)=\{a, e\}$. Here $\alpha Int(B)\subsetneq s^*\alpha Int(B)=s\alpha Int(B)$.
Let $C=\{a, b, d\}$. Then $Int(C)=s^*Int(C)=sInt(C)=C$.

Example 4.7. Consider the topological space (X, τ) where $X=\{a, b, c, d, e\}$ and $\tau=\{\phi,\{a\},\{ b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}, X\}$. Let $A=\{a, b, d, e\}$. Then $\alpha Int(A)=\{a\}$, $s^*\alpha Int(A)=\{a, e\}$, $s^*pInt(A)=\{a, b, d, e\}$. Here $\alpha Int(A)\subsetneq s^*\alpha Int(A)\subsetneq s^*pInt(A)$. Let $B=\{b, c, d, e\}$. Then $\alpha Int(B)=\{b, c, d\}$, $s^*\alpha Int(B)= s^*pInt(B)=\{b, c, d, e\}$. Here, $\alpha Int(B)\subsetneq s^*\alpha Int(B)=s^*pInt(B)$.

Example 4.8. Consider the topological space (X, τ) where $X=\{a, b, c, d, e\}$ and $\tau=\{\phi,\{a\},\{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, X\}$. Let $A=\{a, d, e\}$.
Then $s^*\alpha Int(A)=s\alpha Int(A)=s^*pInt(A)=spInt(A)=\{a, d, e\}$, $s^*Int(A)=\{a, e\}$
Here $s^*Int(A)\subsetneq s^*\alpha Int(A)=s\alpha Int(A)=s^*pInt(A)$.

Remark 4.9. The inclusions in (ix) and (x) of Theorem 4.4 may be strict and equality may also hold. This can be seen from the following examples.

Example 4.10. Consider the topological space (X, τ) in Example 4.6. Let $A=\{a, d\}$ and $B=\{ b, d\}$. Then $A\cup B=\{a, b, d\}$ and $A\cap B=\phi$; $s^*\alpha Int(A)=\{a\}$; $s^*\alpha Int(B)=\{b\}$; $s^*\alpha Int(A\cup B)=\{a, b, d\}$; $s^*\alpha Int(A\cap B)=\phi$. Here $s^*\alpha Int(A)\cup s^*\alpha Int(B)\subsetneq s^*\alpha Int(A\cup B)$ and $s^*\alpha Int(A\cap B)\subsetneq s^*\alpha Int(A)\cap s^*\alpha Int(B)$. Let $C=\{a, b, d\}$ and $D=\{b, c, e\}$ then $C\cap D=\{b\}$, $C\cup D=X$. $s^*\alpha Int(C)=\{a, b, d\}$; $s^*\alpha Int(D)=\{ b, c, e\}$; $s^*\alpha Int(C\cap D)= s^*\alpha Int(C) \cap s^*\alpha Int(D)=\{b\}$. Here $s^*\alpha Int(C\cup D)=s^*\alpha Int(C) \cap s^*\alpha Int(D)=X$.

References

[1] M.. Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud, *β - open sets and β - continuous mappings*, Bull. Fac. Sci. Assiut Univ., 12 (1983), 77-90.

[2] D. Andrijevic, *Semi-Preopen Se*ts, Mat. Vesnik, 38(1986), 24-32.

[3] P. Das, *Note on some applications of semiopen sets*, Progre. Math. Soc., Allahabad.7 (1973), 33-44.

[4] W. Dunham, *A new closure operator for Non-T_1 topologies*, Kyungpook Math. J. 22(1) (1982), 55-60.

[5] Hakeem A. Othman, *New types of α-continuous mapping*, Master Degree Thesis, Al-Mustansiriya University, IRAQ (2004).

[6] E.D. Khalimsky, *Applications of Connected Ordered Topological spaces in Topology*, Conference of Math. Departments of Povolsia, 1970.

[7] N. Levine, *Semi-open sets and Semi-continuity in topological spaces*, American Mathematical Monthly, 70 (1)(1963), 36-41.

[8] N. Levine, *Generalized Closed Sets in Topology*, Rend. Circ. Mat. Palermo, 19(2) (1970), 89-96.

[9] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, *On Precontinuous and Weak Precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, 53(1982), 47-53.

[10] S. Mashhour, A.A. Allam, F.S. Mahmoud and F.H. Khedr, *On Supra topological spaces*, Indian J. Pure and Appl. Math., 4(14) (1983), 502-510.

[11] G.B. Navalagi, Pre-neighbourhoods, The Mathematics Education, Vol.XXXII,No.4, Dec.(1998),201-206

[12] G. B. Navalagi, *Definition Bank in General Topology*, Topology Atlas (2000).

[13] O. Njastad, *Some Classes of Nearly Open Sets*, Pacific J. Math., 15 (3) (1965), 961-970.

[14] A. Robert, and S. Pious Missier, *A New Class of Nearly Open Sets*, International Journal of Mathematical Archive, 3(7), 2012, 1-8.

[15] S. Willard, *General Topology*, Addison–Wesley Publishing Company, Inc, 1970.