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Some graceful graphs

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Abstract

In this paper, we have investigate some general classes of connected graceful graphs. We prove that cycles C_n, C_m joined by an arbitrary path P_k, where $m, n \equiv 0 \pmod 4$, a cycle C_n, ($n \equiv 2 \pmod 4$) with twin chords and a path union of cycles C_n, ($n \equiv 0 \pmod 4$) are graceful graphs.

Keywords: Cycle, path, path union, graceful graph, cycle with twin chords.
AMS Subject Classification(2010): 05C78.

1 Introduction

The graceful labeling was introduced by A. Rosa [7] during 1967. He proved that the cycle C_n is graceful, when $n \equiv 0, 3 \pmod 4$. Golomb [1] proved that the complete graph K_n is graceful when $n \leq 4$. Hoede and Kuiper [5], Frucht [3] proved that all wheels $W_n = C_n + K_1$ are graceful.

The cordial labeling was introduced by Cahit [2] in 1987 as a weaker version of graceful labeling. Vaidya et al. [10] proved that the graph G obtained by joining two copies of cycle C_n by a path of arbitrary length is cordial. Shee and Ho [8] proved that the path union of cycles is cordial. Vaidya et al. [9] proved that cycles C_n with twin chords are cordial as well as $3-$equitable. For a detailed survey of graph labeling one can refer Gallian [4].

We begin with a simple, undirected and finite graph $G = (V, E)$. For all standard terminology and notations we follow Harary [6]. We give a brief summary of definitions which are useful for this paper.

Definition 1.1. If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

Definition 1.2. A function f is called *graceful labeling* of a graph $G=(V,E)$ if $f: V \longrightarrow \{0,1,\ldots,q\}$ is injective and the induce function $f^*: E \longrightarrow \{1,2,\ldots,q\}$ defined as $f^*(e=uv)=|f(u)-f(v)|$ is bijective, $\forall\ e=uv \in E$. A graph G, which admits a graceful labeling is called *graceful graph*.

Definition 1.3. Let $G=(V,E)$ be a graph. A function $f: V \longrightarrow \{0,1\}$ is called *binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex* v of G under f.

For an edge $e=uv$, the induced function $f^*: E \rightarrow \{0,1\}$ is given as $f^*(e=uv)=|f(u)-f(v)|$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^*.

A binary vertex labeling f of a graph G is called *cordial labeling* if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph which admits a cordial labeling is called *cordial graph*.

Definition 1.4. Let G be a graph and $G_1, G_2, \ldots, G_n$, $n \geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} (for $i=1,2,\ldots,n-1$) is called *path union* of G.

Definition 1.5. Let $G=(V,E)$ be a graph. A function $f: V \rightarrow$\{0,1,2\} is called *ternary vertex* labeling of G and $f(v)$ is called the *label* of the vertex v of G under f.

For an edge $e=uv$, the induced edge labeling $f^*: E \rightarrow$\{0,1,2\} is given by $f^*(e)=|f(u)-f(v)|$. Let $v_f(0)$, $v_f(1)$, $v_f(2)$ be the number of vertices of G having labels 0,1 and 2 respectively under f and let $e_f(0)$, $e_f(1)$, $e_f(2)$ be the number of edges having labels 0, 1 and 2 respectively under f^*.

A ternary vertex labeling of a graph G is called $3-$*equitable labeling* if $|v_f(i)-v_f(j)| \leq 1$ and $|e_f(i)-e_f(j)| \leq 1$, $0 \leq i,j \leq 2$. A graph G is called $3-$*equitable* if it admits a $3-$equitable labeling.

Definition 1.6. Two chords of a cycle are said to be *twin chords* if they form a triangle with an edge of the cycle C_n.

In the present paper we prove some cycle related graphs with graceful labeling.

2 Main Results

Theorem 2.1. The graph obtained by joining two cycles C_n, C_m by a path of arbitrary length is graceful, where $n, m \equiv 0 \pmod 4$.

Proof: Let $G=(V,E)$ be the graph obtained by joining two cycles $C_n(n \equiv 0 \pmod 4)$, $C_m(m \equiv 0 \pmod 4)$ by a path P_k of length k. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle C_n; $w_1, w_2, \ldots, w_m$ be the vertices of the cycle C_m and $v_0, v_1, \ldots, v_k$ be the vertices of the path P_k with $v_0=u_n$ and $v_k=w_1$. We consider the following two cases.

Case 1: k is odd.

Define $f: V \longrightarrow \{0,1,\ldots,q\}$, where $q=n+m+k$ as follows:

$$\begin{aligned} f(u_j) &= n+m+k-\tfrac{j-1}{2}, && \forall\ j=1,3,\ldots,n-1 \\ &= \tfrac{j-2}{2}, && \forall\ j=2,4,\ldots,\tfrac{n}{2} \\ &= \tfrac{j}{2}, && \forall\ j=\tfrac{n}{2}+2,\tfrac{n}{2}+4,\ldots,n \end{aligned}$$

$$
\begin{aligned}
f(v_i) &= \tfrac{n}{2} + \tfrac{i}{2}, && \forall\, i = 0, 2, , \ldots, k-1\\
&= m + \tfrac{n}{2} + k - \tfrac{i-1}{2}, && \forall\, j = 1, 3, \ldots, k\\
f(w_j) &= m + \tfrac{n}{2} + \tfrac{k-j+2}{2}, && \forall\, j = 1, 3, \ldots, m-1\\
&= \tfrac{n+k+1}{2} + \tfrac{j-2}{2}, && \forall\, j = 2, 4, \ldots, \tfrac{m}{2}\\
&= \tfrac{n+k+1}{2} + \tfrac{j}{2}, && \forall\, j = \tfrac{m}{2}+2, \tfrac{m}{2}+4, \ldots, m.
\end{aligned}
$$

Case 2: k is even.

Define $f : V \longrightarrow \{0, 1, \ldots, q\}$, where $q = n + m + k$ as follows:

$$
\begin{aligned}
f(u_j) &= n + m + k - \tfrac{j-1}{2}, && \forall\, j = 1, 3, \ldots, n-1\\
&= \tfrac{j-2}{2}, && \forall\, j = 2, 4, \ldots, \tfrac{n}{2}\\
&= \tfrac{j}{2}, && \forall\, j = \tfrac{n}{2}+2, \tfrac{n}{2}+4, \ldots, n\\
f(v_i) &= \tfrac{n}{2} + \tfrac{i}{2}, && \forall\, i = 0, 2, , \ldots, k\\
&= m + \tfrac{n}{2} + k - \tfrac{i-1}{2}, && \forall\, j = 1, 3, \ldots, k-1\\
f(w_j) &= \tfrac{n+k}{2} + \tfrac{j-1}{2}, && \forall\, j = 3, 5, \ldots, m-1\\
&= m + \tfrac{n+k}{2} - \tfrac{j}{2}, && \forall\, j = 2, 4, \ldots, \tfrac{m}{2}\\
&= m + \tfrac{n+k}{2} - \tfrac{j+2}{2}, && \forall\, j = \tfrac{m}{2}+2, \tfrac{m}{2}+4, \ldots, m.
\end{aligned}
$$

In both the cases, we can verify that f is a graceful labeling of the graph G. ∎

Illustration 2.2. A graceful labeling of the graph obtained by joining C_8 and C_{12} by a path P_5 is shown in Figure 1.

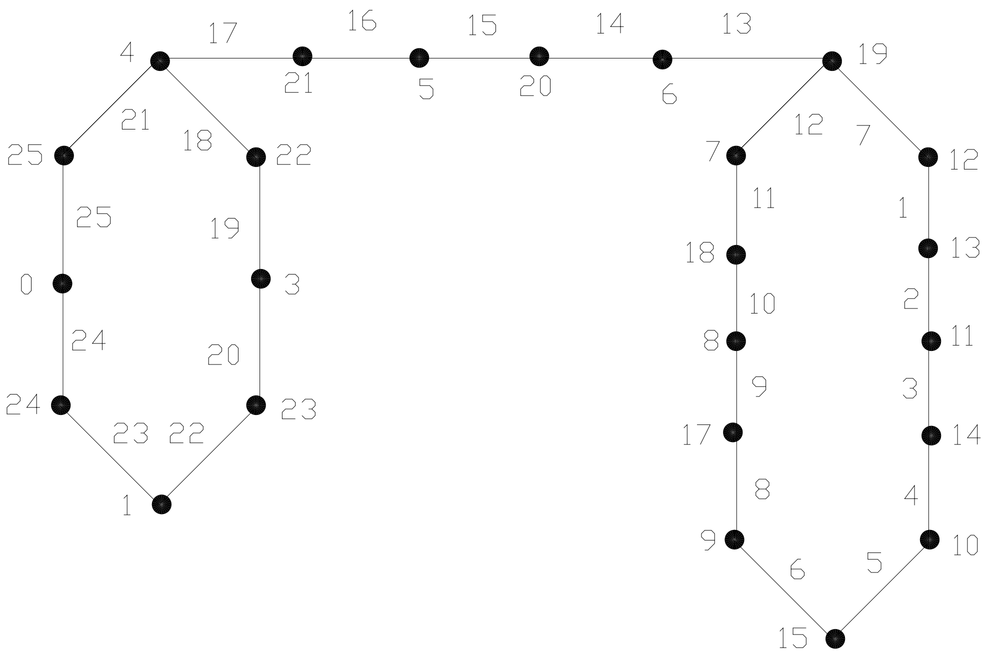

Figure 1: A graceful labeling of the graph obtained by joining C_8 and C_{12} by P_5.

Theorem 2.3. The path union of a finite number of copies of cycle C_n ($n \equiv 0 \pmod 4$) is a graceful graph.

Proof: Let $G = (V, E)$ be the graph obtained by the path union of k copies of cycle C_n where $n \equiv 0 \pmod 4$. Let $C_n^{(i)}$, $i = 1, 2, \ldots, k$ be k copies of cycle C_n ($n \equiv 0 \pmod 4$) in which $v_{i,j}$; $j = 1, 2, \ldots, n$ be the vertices of cycle $C_n^{(i)}$; $i = 1, 2, \ldots, k$. Form the path union of $C_n^{(i)}$ and $C_n^{(i+1)}$ by joining $v_{i,n}$ with $v_{i+1,1}$ by an edge for $1 \leq i \leq n-1$. Hence, G has nk vertices and $nk + (k-1)$ edges.

Define $f : V \longrightarrow \{0, 1, \ldots, q\}$, where $q = nk + (k-1)$ as follows:

$$\begin{aligned} f(v_{i,j}) &= (2k - i + 1)\tfrac{n}{2} + k - \tfrac{j+1}{2}, && \forall\, j = 1, 3, \ldots, n-1 \\ &= (\tfrac{n+2}{2})(i-1) + \tfrac{j-2}{2}, && \forall\, j = 2, 4, \ldots, \tfrac{n}{2} \\ &= (\tfrac{n+2}{2})(i-1) + \tfrac{j}{2}, && \forall\, j = \tfrac{n}{2}+2, \tfrac{n}{2}+4, \ldots, n; \\ & \qquad \forall\, i = 1, 2, \ldots, k. \end{aligned}$$

Then, f is a graceful labeling of the graph G. ∎

Illustration 2.4. A graceful labeling of the path union of three copies of C_8 is given in Figure 2.

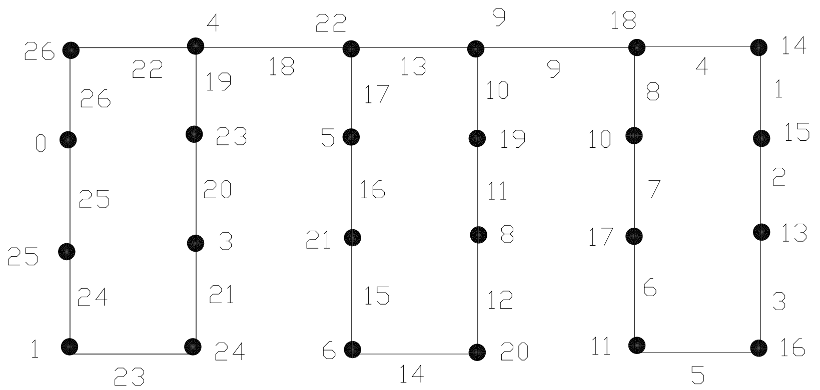

Figure 2: A graceeful labeling of the path union of three copies of C_8.

Theorem 2.5. The path union of a finite number of copies of cycle C_n ($n \equiv 0 \pmod 4$) obtained by joining any vertex of one cycle to any vertex of next cycle is graceful.

Proof: Let $G = (V, E)$ be the path union of finite number of copies of cycle C_n ($n \equiv 0 \pmod 4$) obtained by joining any vertex of one cycle to any vertex of next cycle. Let $C_n^{(i)}$, $i = 1, 2, \ldots, k$ be the k copies of the cycle C_n ($n \equiv 0 \pmod 4$). G has nk vertices and $nk + (k-1)$ edges.

We consider the following two cases.

Case 1: k is odd.

Define $f : V \longrightarrow \{0, 1, \ldots, q\}$, where $q = nk + (k-1)$ as follows:

$$f(u_{1,j}) = k(n+1) - \tfrac{j+1}{2}, \qquad \forall\, j = 1, 3, \ldots, n-1$$

$$
\begin{aligned}
&= \tfrac{j-2}{2}, && \forall\, j = 2,4,\ldots,\tfrac{n}{2}\\
&= \tfrac{j}{2}, && \forall\, j = \tfrac{n}{2}+2, \tfrac{n}{2}+4,\ldots,n\\
f(u_{l,j}) &= f(u_{(l-1),j}) - \tfrac{n}{2}, && \forall\, j = 1,3,\ldots,n-1\\
&= f(u_{(l-1),j}) + \tfrac{n}{2}+1, && \forall\, j = 2,4,\ldots,n;\\
& \forall\, l = 2,3,\ldots,\tfrac{k+1}{2}\\
f(u_{(\frac{k+3}{2}),j}) &= f(u_{(\frac{k+1}{2}),n}) + \tfrac{j+1}{2}, && \forall\, j = 1,3,\ldots,n-1\\
&= f(u_{(\frac{k+1}{2}),(n-1)}) - \tfrac{j}{2}, && \forall\, j = 2,4,\ldots,\tfrac{n}{2}\\
&= f(u_{(\frac{k+1}{2}),(n-1)}) - \tfrac{j+2}{2}, && \forall\, j = \tfrac{n}{2}+2, \tfrac{n}{2}+4,\ldots,n\\
f(u_{l,j}) &= f(u_{(l-1),j}) + \tfrac{n}{2}, && \forall\, j = 1,3,\ldots,n-1\\
&= f(u_{(l-1),j}) - \tfrac{n+2}{2}, && \forall\, j = 2,4,\ldots,n;\\
& \forall\, l = \tfrac{k+3}{2}, \tfrac{k+5}{2},\ldots,k.
\end{aligned}
$$

Case 2: k is even.

Define $f : V \longrightarrow \{0,1,\ldots,q\}$ as follows:

$$
\begin{aligned}
f(u_{1,j}) &= k(n+1) - \tfrac{j+1}{2}, && \forall\, j = 1,3,\ldots,n-1\\
&= \tfrac{j-2}{2}, && \forall\, j = 2,4,\ldots,\tfrac{n}{2}\\
&= \tfrac{j}{2}, && \forall\, j = \tfrac{n}{2}+2, \tfrac{n}{2}+4,\ldots,n\\
f(u_{l,j}) &= f(u_{(l-1),j}) - \tfrac{n}{2}, && \forall\, j = 1,3,\ldots,n-1\\
&= f(u_{(l-1),j}) + \tfrac{n}{2}+1, && \forall\, j = 2,4,\ldots,n;\\
& \forall\, l = 2,3,\ldots,\tfrac{k}{2}\\
f(u_{(\frac{k+2}{2}),j}) &= f(u_{(\frac{k}{2}),n}) + \tfrac{j+1}{2}, && \forall\, j = 1,3,\ldots,n-1\\
&= f(u_{(\frac{k}{2}),(n-1)}) - \tfrac{j}{2}, && \forall\, j = 2,4,\ldots,\tfrac{n}{2}\\
&= f(u_{(\frac{k}{2}),(n-1)}) - \tfrac{j+2}{2}, && \forall\, j = \tfrac{n}{2}+2, \tfrac{n}{2}+4,\ldots,n\\
f(u_{l,j}) &= f(u_{(l-1),j}) + \tfrac{n}{2}, && \forall\, j = 1,3,\ldots,n-1\\
&= f(u_{(l-1),j}) - \tfrac{n+2}{2}, && \forall\, j = 2,4,\ldots,n;\\
& \forall\, l = \tfrac{k+2}{2}, \tfrac{k+4}{2},\ldots,k.
\end{aligned}
$$

Then, f induces the labels $(i-1)(n+1)+1, (i-1)(n+1)+2, \ldots, i(n+1)-1$ for the edges of the cycles $C_n^{(i)}$, for $i = 1,2,\ldots,k$.
For the function f to be graceful on G, we should have labels $n+1$, $2(n+1), \ldots, (k-1)(n+1)$ for the edges of the path in G joining a vertex of one cycle to a vertex of the next cycle.

For this, we rearrange the cycles as $C_1, C_3, C_2, C_5, C_4, \ldots, C_k, C_{k-1}$, when k is odd and $C_1, C_3, C_2, C_5, C_4, \ldots, C_{k-1}, C_{k-2}, C_k$, when k is even.

Note that C_1 contains one vertex with label 0, C_3 contains one vertex with label $(k-1)(n+1)$, C_2 contains one vertex with label $n+1$, C_5 contains one vertex with label $(k-2)(n+1)$, C_4 contains one vertex with label $2(n+1)$ and so on. Hence, f defines a graceful labeling for G. ■

Illustration 2.6. A graceful labeling of the path union of five copies of C_8 obtained by joining a vertex of one cycle to a vertex of next cycle are shown in Figure 3.

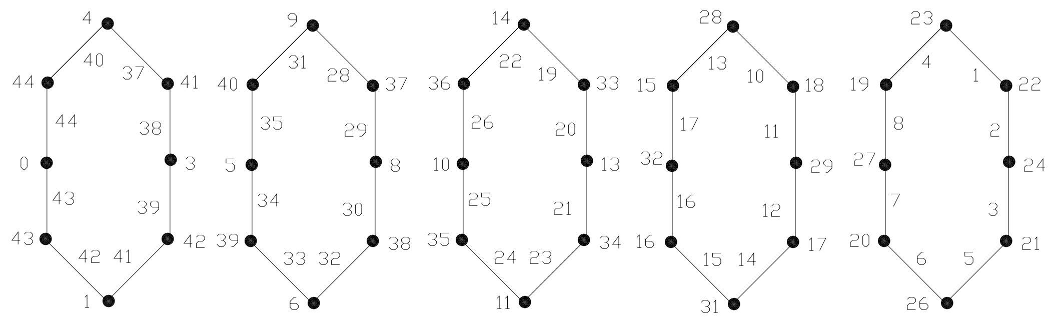

Figure 3(a): Labeling of 8 copies of C_8.

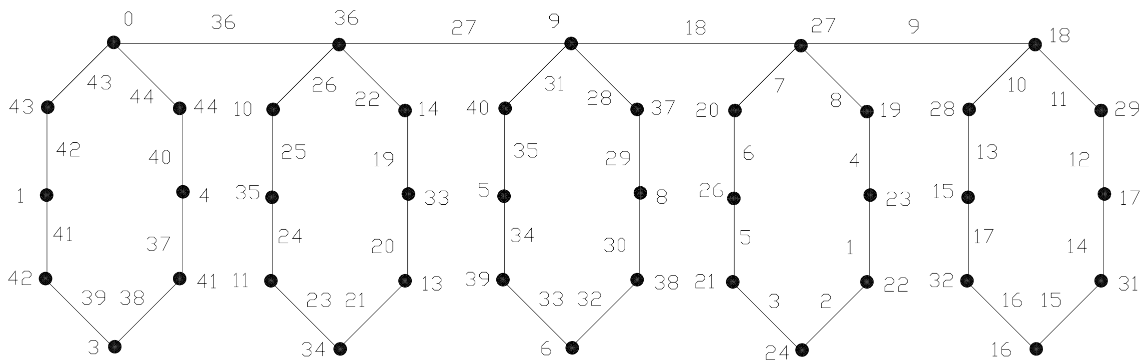

Figure 3(b): Graceful labeling of the path union of five copies of C_8.

Theorem 2.7. Any cycle C_n ($n \equiv 2 \pmod 4$) with twin chords is graceful.

Proof: Let $G=(V,E)$ be a cycle C_n, where $n \equiv 2 \pmod 4$ with twin chords. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and v_2v_n, v_2v_{n-1} be the twin chords of C_n.

Define $f : V \longrightarrow \{0,1,\ldots,q\}$, where $q = n+2$ as follows:

$f(v_n) = q = n+2$

$f(v_{n-1}) = q-2 = n$

$f(v_1) = q-1 = n+1$

$f(v_i) = \frac{i-2}{2}, \qquad \forall\, i = 2,4,\ldots,\frac{n}{2}-1$

$$= \frac{i}{2}, \qquad \forall\, i = \frac{n}{2}+1, \frac{n}{2}+3, \ldots, n-2$$
$$= n - \frac{i-1}{2}, \qquad \forall\, i = 3, 5, \ldots, n-3.$$

Then, f is a graceful labeling of G and hence G is a graceful graph. ■

Illustration 2.8. A graceful labeling of a cycle C_{10} with twin chords is shown in Figure 5.

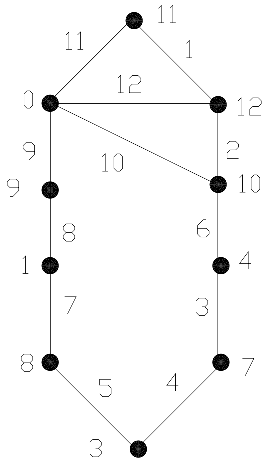

Figure 4: A graceful labeling of C_{10} with twin chords.

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