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Minimum covering energy of binary labeled graph

Pradeep G. Bhat, Sabitha D'Souza
Department of Mathematics
Manipal Institute of Technology
Manipal University, Manipal-576 104, India.
E-mail: pg_bhat@rediff.com, sabithachetan@gmail.com

Abstract

Let G be a graph with vertex set $V(G)$ and edge set $X(G)$ and consider the set $A = \{0, 1\}$. A mapping $l : V(G) \longrightarrow A$ is called a binary vertex labeling of G and $l(v)$ is called the label of the vertex v under l. In this paper we introduce a new kind of graph energy for the binary labeled graph, the minimum covering label energy $E_{cl}(G)$. Depending on the underlying graph G and its binary labeling, the upper and lower bounds for $E_{cl}(G)$ are established. The minimum covering label energies of complete and star graphs are computed. The characteristic polynomial of complete bipartite graph is also obtained.

Keywords: Minimum covering label matrix, minimum covering label eigenvalues, minimum covering label energy.
AMS Subject Classification(2010): 05C50.

1 Introduction

Let G be a graph of order n. The energy of the graph G was defined by Gutman [7] in 1978 as the sum of the absolute eigenvalues of G. It represents a proper generalization of a formula valid for the total π-electron energy of a conjugated hydrocarbon as calculated by the Huckel Molecular Orbital (HMO) method in quantum chemistry. For recent mathematical work on the energy of a graph see [1–3, 9, 14–16]. In connection with graph energy, energy -like quantities were also considered for other matrices: Laplacian [8], distance [10], minimum covering [4], maximum degree [5] and the like.

In 2012, the minimum covering energy of a graph was introduced by Chandrashekar Adiga et.al [4]. Also, the present authors of this paper have defined label adjacency matrix [13] and label energy for a binary labeled graph. This paper is the extension of minimum covering energy and the label energy for the binary labeled graph. For all terminologies and notations, we refer [6]. All graphs considered in this paper are finite, simple and undirected.

Let $G = (V, X)$ be a simple binary labeled graph [12] with n vertices $\{v_1, v_2, \ldots, v_n\}$. A subset C of V is called a covering set of G if every edge of G is incident to at least one vertex of C. Any covering set with minimum cardinality is called a *minimum covering set*.

Let C be a minimum covering set of a graph G. The *minimum covering label matrix* of G is the $n \times n$

matrix $A_{cl}(G) = (l_{ij})$, where

$$l_{ij} = \begin{cases} a, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = l(v_j) = 0, \\ 1, & \text{if } i = j \text{ and } v_i \in C, \\ b, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = l(v_j) = 1 \\ c, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = 0, l(v_j) = 1 \text{ or vice-versa,} \\ 0, & \text{otherwise.} \end{cases}$$

where, a, b and c are nonzero distinct real numbers.
The characteristic polynomial of the label matrix $A_{cl}(G)$ is defined by

$$\begin{aligned} \phi(A_{cl}(G), \lambda) &= \det(\lambda I - A_{cl}(G)) \\ &= c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + ... + c_n \end{aligned}$$

where I is the unit matrix of order n. The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ assumed in non-increasing order of $\phi(A_{cl}(G), \lambda) = 0$ are called *minimum covering label eigenvalues* of binary labeled graph G. The *minimum covering label energy* of a graph G is defined as $E_{cl}(G) = \sum_{i=1}^{n} |\lambda_i|$. Since $A_{cl}(G)$ is a real symmetric matrix, these eigenvalues of binary labeled graph are real and we label them in non-increasing order. Thus $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

This paper deals only the mathematical aspects of minimum covering label energy of a graph and it is worth saying that it means totally a new concept in the literature.

In this paper, we obtain the explicit expression for the coefficient c_i of characteristic polynomial of minimum covering label matrix, bounds for $E_{cl}(G)$. We also derive the minimum covering label energy for complete graph , star graph and derive the characteristic polynomial of complete bipartite graph. In particular, we consider $a = 1, b = 2, c = 3$ in minimum covering label matrix.

2 Basic properties and bounds for the minimum covering label energy

Example 1. Consider the graph G in Figure 1.

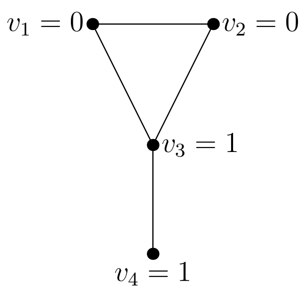

Figure 1: Binary graph G.

For the graph G, $C = \{v_1, v_3\}$ is the minimum covering set.
The minimum covering label matrix of G is

$$A_{cl}(G) = \begin{bmatrix} 1 & a & c & 0 \\ a & 0 & c & 0 \\ c & c & 1 & b \\ 0 & 0 & b & 0 \end{bmatrix}$$

In particular, for $a = 1, b = 2 \; and \; c = 3$, the characteristic polynomial of $A_{cl}(G)$ is $\lambda^4 - \lambda^3 - 22\lambda^2 - 4\lambda + 4$. The minimum covering label eigenvalues of G are $\lambda_1 = -3.6479$, $\lambda_2 = -0.5483$, $\lambda_3 = 0.3416$, $\lambda_4 = 5.8545$ and the Minimum covering label energy of the given graph G is $E_{cl}(G) = 10.3923$.

We now give the explicit expression for the coefficient c_i of λ^{n-i}, $i = 0, 1, 2, 3$ in the *characteristic polynomial* of $A_{cl}(G)$.

Theorem 2.1. Let G be a labeled graph with vertex set V and edge set X with a minimum covering set C. Let $\phi(A_{cl}(G), \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} \cdots + c_n$ be the characteristic polynomial of $A_{cl}(G)$. Then,

(i) $c_0 = 1$,

(ii) $c_1 = -|C|$,

(iii) $c_2 = \binom{|C|}{2} - \left[n_1(1)^2 + n_2(2)^2 + n_3(3)^2\right]$, where n_1, n_2, n_3 denote the number of edges of the graph G with end vertices are labeled as $(0, 0), (1, 1)$ and $(0, 1)$ respectively.

(iv) $c_3 = \sum_{v_i \in C} \left[(n_1 - n_{i1}) \times 1^2 + (n_2 - n_{i2}) \times 2^2 + (n_3 - n_{i3}) \times 3^2\right] - \binom{|C|}{3} - 2 \sum_{\substack{\triangle v_i v_j v_k \\ i<j<k}} S(i)$, where, n_{i1}, n_{i2}, n_{i3} denote the number of edges of G incident on v_i having end vertices as $(0, 0), (1, 1)$ and $(0, 1)$ respectively, $l_i = l(v_i)$ and $S(i) = \begin{cases} (l_i + 1)^3, & \text{if } \; l_i = l_j = l_k = 0 \; or \; 1 \\ (l_i + l_j + 2)^2(l_i + 1), & \text{if } \; l_i \text{ is the repeated label} \end{cases}$

Proof: (i) Directly from the definition of $\phi(A_{cl}(G), \lambda)$, it follows that $c_0 = 1$.
(ii) Since the sum of the diagonal elements of $A_{cl}(G)$ is equal to the cardinality of the set C, we have $(-1)c_1 = trace(A_{cl}(G)) = |C|$.
(iii)

$$\begin{aligned} c_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} l_{ii} & l_{ij} \\ l_{ji} & l_{jj} \end{vmatrix} \\ &= \sum_{1 \le i < j \le n} (l_{ii}l_{jj} - l_{ij}^2) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le i<j\le n} l_{ii}l_{jj} - \sum_{1\le i<j\le n} l_{ij}^2 \\
&= \binom{|C|}{2} - [n_1(1)^2 + n_2(2)^2 + n_3(3)^2].
\end{aligned}
$$

(iv) We have

$$
\begin{aligned}
c_3 &= (-1)^3 \sum_{1\le i<j<k\le n} \begin{vmatrix} l_{ii} & l_{ij} & l_{ik} \\ l_{ji} & l_{jj} & l_{jk} \\ l_{ki} & l_{kj} & l_{kk} \end{vmatrix} \\
&= -\sum_{1\le i<j<k\le n} \left(l_{ii}\left[l_{jj}l_{kk} - l_{kj}l_{jk}\right] - l_{ij}\left[l_{ji}l_{kk} - l_{ki}l_{jk}\right] + l_{ik}\left[l_{ji}l_{kj} - l_{ki}l_{jj}\right]\right) \\
&= -\sum_{1\le i<j<k\le n} l_{ii}l_{jj}l_{kk} + \sum_{1\le i<j<k\le n} (l_{ii}l_{jk}^2 + l_{jj}l_{ik}^2 + l_{kk}l_{ij}^2) - 2\sum_{1\le i<j<k\le n} l_{ij}l_{jk}l_{ki} \\
&= -\binom{|C|}{3} + \sum_{v_i\in C} l_{ii} \sum_{\substack{i\ne j\ne k \\ 1\le i<j<k\le n}} l_{jk}^2 - 2\sum_{\substack{i<j<k \\ \triangle v_iv_jv_k}} S(i) \\
&= -\binom{|C|}{3} + \sum_{v_i\in C} [(n_1 - n_{i1})\times 1^2 + (n_2 - n_{i2})\times 2^2 + (n_3 - n_{i3})\times 3^2] - 2\sum_{\substack{i<j<k \\ \triangle v_iv_jv_k}} S(i)
\end{aligned}
$$

where, n_{i1}, n_{i2}, n_{i3} denote the number of edges of G incident on v_i having end vertices as $(0,0), (1,1)$ and $(0,1)$ respectively. Consider a triangle of vertices v_i, v_j, v_k, when $i < j < k$. There are only four possible non-isomorphic labellings of a triangle i.e. $(0,0,0), (1,1,1), (0,1,0), (1,0,1)$. Then,

$$
S(i) = \begin{cases} (l_i+1)^3, & \text{if } l_i = l_j = l_k = 0 \textit{ or } 1 \\ (l_i + l_j + 2)^2(l_i+1), & \text{if } l_i \text{ is the repeated label.} \end{cases}
$$

∎

Theorem 2.2. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A_{cl}(G)$, then $\sum_{i=1}^{n} \lambda_i^2 = 2[n_1(1^2) + n_2(2^2) + n_3(3^2)] + |C|$.

Proof: We have

$$
\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= trace\ (A_{cl}(G))^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}l_{ji}
\end{aligned}
$$

$$\begin{aligned} &= 2\sum_{i<j} l_{ij}^2 + \sum_{i=1}^{n} l_{ii}^2 \\ &= 2[n_1(1)^2 + n_2(2)^2 + n_3(3)^2] + |C|. \end{aligned}$$

∎

Bounds for $E_{cl}(G)$, similar to McClelland's inequalities [11] for graph energy are given in the following theorem.

Theorem 2.3. Let G be a labeled graph of order n and C be a minimum covering set of G. Then,

$$\sqrt{2[n_1(1)^2 + n_2(2)^2 + n_3(3)^2] + |C| + n(n-1)D^{\frac{2}{n}}} \leq E_{cl}(G) \leq \sqrt{n[2(n_1(1)^2 + n_2(2)^2 + n_3(3)^2) + |C|]}.$$

Proof: We have

$$\begin{aligned} E_{cl}^2(G) &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i\neq j} |\lambda_i||\lambda_j| \\ &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= 2[n_1(1)^2 + n_2(2)^2 + n_3(3)^2] + |C| + n(n-1)\left|\prod_{i=1}^{n} \lambda_i\right|^{\frac{2}{n}} \\ &= 2[n_1(1)^2 + n_2(2)^2 + n_3(3)^2] + |C| + n(n-1)D^{\frac{2}{n}} \end{aligned}$$

by using Arithmetic and Geometric means inequality and Theorem 2.2 .

Hence, $[E_{cl}(G)]^2 \geq \sqrt{2(n_1(1)^2 + n_2(2)^2 + n_3(3)^2) + |C| + n(n-1)D^{\frac{2}{n}}}$ where, $D = \left|\prod_{i=1}^{n} \lambda_i\right|$.

$$\begin{aligned} E_{cl}(G)^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \\ &\leq n\sum_{i=1}^{n} |\lambda_i|^2 \text{ Employing Cauchy Schwarz inequality.} \\ &= n\sum_{i=1}^{n} \lambda_i^2 \end{aligned}$$

$$= \quad n\left(2\left[n_1(1)^2+n_2(2)^2+n_3(3)^2\right]+|C|\right).$$

Hence,

$$\sqrt{2[n_1(1)^2+n_2(2)^2+n_3(3)^2]+|C|+n(n-1)D^{\frac{2}{n}}} \quad \leq \quad E_{cl}(G) \leq \sqrt{n[2(n_1(1)^2+n_2(2)^2+n_3(3)^2)+|C|]}.$$

∎

3 Minimum covering label energies of complete graph, star graph and complete bipartite graph

Theorem 3.1. Let $m \geq 1$. Let $\{v_1, v_2, \ldots, v_m\}$ be the vertices of complete graph K_n labeled with 0's and remaining vertices be labeled with 1's. Let $C = \{v_2, v_3, \ldots, v_n\}$ be the minimum covering set. Then,
$\phi(A_{cl}(K_n), \lambda) = \lambda^{m-2}(\lambda+1)^{n-(m+1)}[\lambda^3 - (2(n-1)-m)\lambda^2 - (7m(n-m)+2(n-1))\lambda - (7(n-m)-1)(m-1)]$, for $n \geq 2$ *and* $m \leq n$.

Proof: Let $v_1, v_2, \cdots, v_m$ vertices of K_n be labeled with 0's and $v_{m+1}, v_{m+2}, \cdots, v_n$ be labeled with 1's. Then

$$A_{cl}(K_n) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 3 & 3 & 3 & \cdots & 3 & 3 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 3 & 3 & 3 & \cdots & 3 & 3 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 3 & 3 & 3 & \cdots & 3 & 3 \\ \vdots & & & \ddots & & & & & & \ddots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 & 3 & 3 & 3 & \ldots & 3 & 3 \\ 3 & 3 & 3 & \cdots & 3 & 3 & 1 & 2 & 2 & \cdots & 2 & 2 \\ 3 & 3 & 3 & \cdots & 3 & 3 & 2 & 1 & 2 & \cdots & 2 & 2 \\ \vdots & & & \ddots & & & & & & \ddots & & \vdots \\ 3 & 3 & 3 & \cdots & 3 & 3 & 2 & 2 & 2 & \cdots & 2 & 1 \end{bmatrix}$$

The *characteristic polynomial* of $A_{cl}(K_n)$ is

$$\phi(A_{cl}(K_n),\lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 & -3 & -3 & -\cdots & -3 & -3 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -3 & -3 & \cdots & -3 & -3 \\ \vdots & & & \ddots & & & & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & \lambda-1 & -3 & -3 & \ldots & -3 & -3 \\ -3 & -3 & -3 & \cdots & -3 & \lambda-1 & -2 & \cdots & -2 & -2 \\ -3 & -3 & -3 & \cdots & -3 & -2 & \lambda-1 & \cdots & -2 & -2 \\ \vdots & & & \ddots & & & & \ddots & & \vdots \\ -3 & -3 & -3 & \cdots & -3 & -2 & -2 & \cdots & -2 & \lambda-1 \end{vmatrix}$$

Step 1: Replace R_i by $R_i = R_{i+1} - R_i$ for $i = 1, 2, \ldots, m-1, m+1, \ldots, n-1$. Then we see that

$\phi(A_{cl}(K_n),\lambda) = \lambda^{m-2}(\lambda+1)^{n-(m+1)}det(B)$.

Step 2: In $det(B)$, replacing C_i by $C_i' = C_i + C_{i+1} + \ldots + C_{m-1} + C_{m-2} + \ldots + C_n$, for $i = 2, 3, \ldots, m-1, m+2, \ldots, n$ and simplifying ,we get, $\phi(A_{cl}(K_n),\lambda) = \lambda^{m-2}(\lambda+1)^{n-(m+1)}det(C)$ where,

$$det(C) = \begin{vmatrix} \lambda & -(m-1) & -(m-2) & \ldots & -1 & -3(n-m) \\ -(\lambda+1) & \lambda & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ -3m & -3(m-1) & -3(m-2) & \ldots & -3 & (\lambda+1)-2(n-m) \end{vmatrix}$$

Expanding the $det(C)$ over the last column, we get,

$$det(C) = -9(n-m)[m\lambda + m - 1] + [(\lambda+1) - 2(n-m)][\lambda^2 - (\lambda+1)(m-1)].$$

Substituting $det(C)$ in $\phi(A_{cl}(K_n),\lambda)$ obtained in step 2, we get

$$\phi(A_{cl}(K_n),\lambda) = \lambda^{m-2}(\lambda+1)^{n-(m+1)}\times \left[\lambda^3 - (2(n-1)-m)\lambda^2 - (7m(n-m)+2(n-1))\lambda - (7(n-m)-1)(m-1)\right]. \quad \blacksquare$$

Corollary 3.2. For $m = 1$, the minimum covering label energy of complete graph K_n is $E_{cl}(K_n) = n - 2 + \sqrt{4n^2 + 24n - 27}$ if the vertex with 0 label is not in the covering set C .

Proof: Put $m = 1$ in the characteristic polynomial of $A_{cl}(K_n)$ in Theorem 3.1, we get

$$\phi(A_{cl}(K_n),\lambda) = (\lambda+1)^{n-2}[\lambda^2 - (2n-3)\lambda - 9(n-1)].$$

The spectrum of $A_{cl}(K_n)$ is

$$\begin{pmatrix} -1 & \frac{(2n-3+\sqrt{4n^2+24n-27})}{2} & \frac{(2n-3-\sqrt{4n^2+24n-27})}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Hence, $E_{cl}(K_n) = n - 2 + \sqrt{4n^2 + 24n - 27}$. $\blacksquare$

Corollary 3.3. For $m = 1$, the characteristic polynomial of minimum covering label matrix of K_n is $\phi(A_{cl}(K_n),\lambda) = (\lambda+1)^{n-3}[\lambda^3 - (2n-4)\lambda^2 - (11n-12)\lambda - (14n-19)]$ if the vertex with 0 label is in the covering set C.

Remark 3.4. From corollaries 3.2 and 3.3, it is evident that $E_{cl}(K_n)$ not only depends on the selection of minimum covering set but also on the combination of labels to the vertices.

Theorem 3.5. For $n \geq 2$, the minimum covering label energy of star graph $K_{1,n-1}$ is equal to $\sqrt{36n - 32m - 3}$ when m vertices including the central vertex are labeled zero and remaining vertices are labeled 1.

Proof: Let the vertices $v_1, v_2, \cdots, v_m$ be labeled with 0's and the remaining vertices be labeled with 1's. Let v_1 be the central vertex. The minimum covering set is $C = \{v_1\}$. Then

$$A_{cl}(K_{1,n-1}) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 3 & 3 & \cdots & 3 & 3 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & & & \ddots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & & & \ddots & & \vdots \\ 3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The *characteristic* polynomial is

$$|\lambda I_n - A_{cl}(K_{1,n-1})| = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 & -3 & -3 & \cdots & -3 & -3 \\ -1 & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & \lambda & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & & & \ddots & & \vdots \\ -1 & 0 & 0 & \cdots & \lambda & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ -3 & 0 & 0 & \cdots & 0 & 0 & \lambda & 0 & \cdots & 0 & 0 \\ -3 & 0 & 0 & \cdots & 0 & 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & & & \ddots & & \vdots \\ -3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}$$

Step 1: Replace R_i by $R_i = R_{i+1} - R_i$ for $i = 3, 4, \dots, m-1, m+1, \dots, n-1$. Then we see that $\phi(A_{cl}(K_{n,1}), \lambda) = \lambda^{n-3} det(B)$.

Step 2: In $det(B)$, replacing C_i by $C_i' = C_i + C_{i+1} + \dots + C_{m-1} + C_{m-2} + \dots + C_n$, for $i = 2, 3, \dots, m-1, m+2, \dots, n$ and simplifying, we get, $\phi(A_{cl}(K_{n,1}), \lambda) = \lambda^{n-3} det(C)$ where

$$det(C) = \begin{vmatrix} \lambda-1 & -(m-1) & -(m-2) & \dots & -m & -3(n-m) \\ -1 & \lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ -3 & 0 & 0 & \dots & 0 & \lambda \end{vmatrix}$$

Expanding the determinant over the last row, we get,

$$\begin{aligned}\det(C) &= (-1)^{m+3}9(n-m)(-1)^{m+2}\lambda + \lambda[\lambda(\lambda-1)-(m-1)] \\ &= \lambda[\lambda^2 - \lambda - (9n-8m-1)].\end{aligned}$$

Substituting $det(C)$ in $\phi(A_{cl}(K_n), \lambda)$ obtained in step 2, we get

$$|\lambda I_n - A_{cl}(K_{1,n-1})| = \lambda^{n-2}[\lambda^2 - \lambda - (9n-8m-1)].$$

The spectrum of $A_{cl}(K_{1,n-1})$ is

$$\begin{pmatrix} 0 & \frac{1+\sqrt{36n-32m-3}}{2} & \frac{1+\sqrt{36n-32m-3}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Hence, $E_{cl}(K_{1,n-1}) = \sqrt{36n-32m-3}$. ■

Corollary 3.6. For $n \geq 2$, the minimum covering label energy of star graph $K_{1,n-1}$ is equal to $\sqrt{36n-20m-15}$ when m vertices including the central vertex are labeled with 1's and remaining vertices are labeled with 0's, $0 \leq m \leq n$.

Proof: The proof follows by replacing the element 1 by 2 in matrix $A_{cl}(K_{1,n-1})$ of Theorem 3.5. ■

Corollary 3.7. Each positive integer $(2p-1) \geq 3$ is the minimum covering label energy of a star graph.

Proof: We have from Corollary 3.6, $E_{cl}(K_{1,n-1}) = \sqrt{36n-20m-15} = \sqrt{4(9n-5m-4)+1}$.
By substituting $9n-5m-4 = p^2 - p$, we get $E_{cl}(K_{1,n-1}) = 2p-1$. ■

Theorem 3.8. The characteristic polynomial of minimum covering label matrix of complete bipartite graph $K_{r,s}$ with $m_1 \leq r, m_2 \leq s$, the number of zeros in the vertex set of order r, s respectively, is given by, $\phi(A_{cl}(K_{r,s}), \lambda) = \lambda^{s-2}(\lambda-1)^{r-2}[\lambda^2(\lambda-1)^2 - \lambda(\lambda-1)\{(4r+5m_1)s+(5r-13m_1)m_2\} + 49m_1m_2(r-m_1)(s-m_2)]$, for $r \leq s$

Proof: Let the labels of the $r+s$ vertices of $K_{r,s}$ be $\underbrace{000\ldots0}_{m_1}\underbrace{111\ldots1}_{r-m_1}$ and $\underbrace{000\ldots0}_{m_2}\underbrace{111\ldots1}_{s-m_2}$.
As $r \leq s$, $C = \{v_1, v_2, \ldots, v_r\}$ is the minimum covering set for $K_{r,s}$.

$$A_{cl}(K_{r,s}) = \left[\begin{array}{c|c} I_r & -B_{r\times s} \\ \hline -B^T_{s\times r} & O_s \end{array}\right]_{(r+s)\text{x}(r+s)}$$

where $B = \left(\begin{array}{c|c} J_{m_1\times m_2} & 3J_{m_1\times s-m_2} \\ \hline 3J_{r-m_1\times m_2} & 2J_{r-m_1\times s-m_2} \end{array}\right)$, J is the matrix of all 1s.

The characteristic polynomial of $A_{cl}(K_{r,s})$ is,
$|\lambda I - A_{cl}(K_{r,s})| = \begin{vmatrix} (\lambda-1)I_r & B \\ B^T & \lambda I_s \end{vmatrix}$ where
$B^T B = \left[\begin{array}{c|c} [(9r-8m_1)]J_{m_2\times m_2} & [6r-3m_1]J_{m_2\times(s-m_2)} \\ \hline [6r-3m_1]J_{(s-m_2)\times m_2} & [4r+5m_1]J_{(s-m_2)\times(s-m_2)} \end{array} \right].$

Therefore,

$$\begin{aligned} |\lambda I - A_{cl}(K_{r,s})| &= |(\lambda-1)I_r| \left| \lambda I_s - (B^T) \times \frac{I_r}{(\lambda-1)} \times (B) \right| \\ &= (\lambda-1)^r \left| \frac{\lambda(\lambda-1)I_s - B^T B}{(\lambda-1)} \right| \end{aligned}$$

$$|\lambda I - A_{cl}(K_{r,s})| = (\lambda-1)^{r-s} \left|\lambda(\lambda-1)I_s - B^T B\right| \qquad (3.1)$$

Using elementary row and column operations successively, we get

$$B^T B = \begin{bmatrix} m_2[9r-8m_1] & 0 & \cdots & 0 & (s-m_2)[6r-3m_1] & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ m_2[6r-3m_1] & 0 & \cdots & 0 & (s-m_2)[4r+5m_1] & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & \ddots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Let $X = \lambda(\lambda-1)$.
$|XI_s - B^T B|$

$$= \left| \begin{array}{cccc|cccc} X - m_2[9r-8m_1] & 0 & \cdots & 0 & -(s-m_2)[6r-3m_1] & 0 & \cdots & 0 \\ 0 & X & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & & \ddots \\ 0 & 0 & \cdots & X & 0 & 0 & \cdots & 0 \\ \hline -m_2[6r-3m_1] & 0 & \cdots & 0 & X-(s-m_2)[4r+5m_1] & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & X & \cdots & 0 \\ \vdots & & & \ddots & & & & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & X \end{array} \right|$$

$= X^{m_2-1}[\lambda(\lambda-1) - m_2(9r-8m_1)] \left| \text{diag}[X - (s-m_2)(4r+5m_1)] - \frac{m_2[6r-3m_1]^2(s-m_2)}{X - m_2[9r-8m_1]}, X, \cdots, X] \right|$

$= X^{m_2-1}[X - m_2(9r-8m_1)]X^{(s-m_2-1)}\ [X - (s-m_2)(4r+5m_1) - \frac{m_2(6r-3m_1)^2(s-m_2)}{X-m_2(9r-8m_1)}].$

Substituting $X = \lambda(\lambda-1)$, we obtain

$$\left|\lambda(\lambda-1)I_s - B^TB\right| = \lambda^{s-2}(\lambda-1)^{s-2}\{[(\lambda(\lambda-1))^2 - (\lambda(\lambda-1))\,[m_2(5r-13m_1)+s(4r+5m_1)] + 49m_1m_2(r-m_1)(s-m_2)\}.$$

By substituting in equation 3.1 we obtain,

$\phi(A_{cl}(K_{r,s}),\lambda) = \lambda^{s-2}(\lambda-1)^{r-2}[\lambda^2(\lambda-1)^2 - \lambda(\lambda-1)\{(4r+5m_1)s+(5r-13m_1)m_2\}+49m_1m_2(r-m_1)(s-m_2)]$.

■

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