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Packing Chromatic Number of Circular Fans and Mesh of Trees

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k for which there exists a mapping $\pi : V(G) \longrightarrow \{1, 2, ..., k\}$ such that any two vertices of color i are at distance at least $i + 1$. It is a frequency assignment problem used in wireless networks, which is also called broadcast coloring. It is proved that packing coloring is NP-complete for general graphs and even for trees. In this paper, we compute the packing chromatic number for circular fans with two and four chords and mesh of trees.

Keywords: Packing chromatic number, circular fans with two and four chords, mesh of trees.
AMS Subject Classification(2010): 05C15.

1 Introduction

Let G be a connected graph and k be an integer, $k \geq 1$. A packing k-coloring of a graph G is a mapping $\pi : V(G) \longrightarrow \{1, 2, ..., k\}$ such that any two vertices of color i are at distance at least $i + 1$. The packing chromatic number $\chi_\rho(G)$ of G is the smallest integer k for which G has packing k-coloring. The concept of packing coloring comes from the area of frequency assignment in wireless networks and was introduced by Goddard et al. [8] under the name Broadcast coloring. It also has several applications, such as, resource replacement and biological diversity. The term packing chromatic number was introduced by Brešar [3].

Goddard et al. [8] proved that the packing coloring problem is NP-complete for general graphs and Fiala and Golovach [5] proved that it is NP-complete even for trees. Sloper [13] studied a special type of packing coloring, called eccentric coloring and proved that the infinite 3-regular tree has packing chromatic number 7. For the infinite planar square lattice $\mathbb{Z}^2$, 10 $\leq \chi_\rho(\mathbb{Z}^2) \leq 17$ [6, 9]. The packing coloring of distance graphs were studied in [4, 14]. For the infinite hexagonal lattice $\mathbb{H}$, $\chi_\rho(\mathbb{H}) = 7$ [3, 11].

Argiroffo et al. [1, 2] proved that the packing coloring is solvable in polynomial time for the class of $(q, q-4)$ graphs, partner limited graphs and for an infinite subclass of lobsters, including caterpillars. It is proved in [7, 12] that the infinite, planar triangular lattice and the three dimensional square lattice have unbounded packing chromatic number. In this paper, we study the packing chromatic number of circular fans with two and four chords and mesh of trees.

2 Preliminaries

Proposition 2.1. [8] Let H be a subgraph of G. Then $\chi_\rho(H) \leq \chi_\rho(G)$.

Proposition 2.2. [8] $\chi_\rho(C_n) = \begin{cases} 3 & \text{when } n \text{ is a multiple of } 4 \\ 4 & \text{when } n \text{ is not a multiple of } 4 \end{cases}$

We introduce the following notations: $\alpha_0(G)$ for the vertex cover number and $\beta_0(G)$ for the independence number. By Gallai's theorem $n - \beta_0(G) = \alpha_0(G)$[8].

Proposition 2.3. [8] For every graph G, $\chi_\rho(G) = \alpha_0(G) + 1$ if G has diameter 2.

Proposition 2.4. [8] Let T be a tree of diameter 4 with *central vertex* v. For $i = 1, 2, 3$, let n_i denote the number of neighbours of v of degree i, and L denote the *large* (vertex with degree 4 or more) neighbours of v. If $L = 0$ then $\chi_\rho(T) = 4$ if $n_3 \geq 2$ and $n_1 + n_2 + n_3 \geq 3$

3 Circular fan with four and two chords

Definition 3.1. [10] Let $C : x_1x_2...x_mx_1$ be a cycle on m vertices. Let u be a new vertex. The graph obtained by adding edges $(u, x_i), i = 1, 2, ..., m - 8$ to C and chords (x_{m-2}, x_m), (x_{m-4}, x_{m-1}), (x_{m-6}, x_{m-3}) and (x_{m-5}, x_{m-7}) is called a circular fan with four chords and is denoted by $F(m, 4)$. See Figure 1(a).

Theorem 3.2. $\chi_\rho(F(m, 4)) = (\lceil \frac{m-7}{2} \rceil) + 2$ for $m \geq 16$.

Proof: Let G denote $F(m, 4)$, $m \geq 16$. Then, consider the subgraph H of $F(m, 4)$ induced by u and its neighbours. The graph H is of diameter 2 with $m - 7$ vertices. Therefore, the independence number is $\lfloor \frac{m-7}{2} \rfloor$ and the vertex cover number is $(m - 7) - \lfloor \frac{m-7}{2} \rfloor = \lceil \frac{m-7}{2} \rceil$. By Proposition 2.3, $\chi_\rho(H) = (\lceil \frac{m-7}{2} \rceil) + 1$. Since $m \geq 16$, at most four vertices of H are colored with 2,3,4 and 5.

Consider the subgraph H' induced by $V(G) \setminus V(H)$. By Definition 3.1, H' consists of only eight vertices constituting four edges (x_{m-2}, x_m),(x_{m-4}, x_{m-1}),(x_{m-6}, x_{m-3}), (x_{m-5}, x_{m-7}). Since diameter of $F(m, 4)$ is 5, no vertex of H' can be colored 5.

Since $d(\{x_{m-7}, x_{m-6}, x_{m-5}, x_{m-2}, x_{m-1}, x_m\}, x_i) \leq 4$ for $1 \leq i \leq m - 8$, vertices x_m, x_{m-1}, x_{m-2}, x_{m-5}, x_{m-6}, x_{m-7} can not be colored with 4. Thus, either x_{m-3} or x_{m-4} can be colored with 4. Since $d(\{x_m, x_{m-7}\}, x_i) \leq 3$ for $1 \leq i \leq m - 8$, vertices x_m and x_{m-7} can not be colored 3 and giving color 3 to any other vertex of H' , no other vertices receive color 3 because except vertices x_m and x_{m-7}, all other vertices are at distance less than 4 to each other. Thus at most one vertex can be colored 4 and one vertex can be colored 3. Since diameter of H' is 4, at most two vertices with color 2 and three vertices with color 1 can be colored and the remaining one vertex should receive distinct color greater than $(\lceil\frac{m-7}{2}\rceil)+1$. Hence $\chi_\rho(F(m,4)) \geq (\lceil\frac{m-7}{2}\rceil)+2$. We give an algorithm to color the circular fan $F(m,4)$ and prove that the bound is sharp.

Procedure PACKING COLORING $(F(m,4)), m \geq 16$

Algorithm

Step 1: Color the vertices $x_{2i-1}, 1 \leq i \leq \lceil\frac{m-8}{2}\rceil$ with color 1.

Step 2: Color the vertices x_2,x_4 and x_6 with 2,3 and 4.

Step 3: Color the vertices x_{m-1}, x_{m-2}, x_{m-3}, x_{m-4}, x_{m-5}, x_{m-6}, x_{m-7} sequentially with 2,1,4,1,3,2,1 when m is even and with 2,1,4,1,3,1,2 sequentially when m is odd.

Step 4: Color the vertices x_{2i}, u and x_m, $4 \leq i \leq \lfloor\frac{m-8}{2}\rfloor$ consecutively beginning with 5.

Output: $\chi_\rho(F(m,4)) = \lceil\frac{m-7}{2}\rceil + 2$.

Proof of Correctness: Since $d(x_6, x_{m-3}) = 5$, vertices x_6 and x_{m-3} are colored with 4 for all m.

Since $d(x_4, x_{m-5}) = 4$, vertices x_4 and x_{m-5} are colored with 3 for all m.

Since $d(x_2, x_{m-1}) = 3$, $d(x_2, x_{m-6}) = 3$ and $d(x_{m-1}, x_{m-6}) = 3$, vertices x_2, x_{m-1} and x_{m-6} are colored with 2 when m is even. Similarly, since $d(x_2, x_{m-1}) = 3$, $d(x_2, x_{m-7}) = 3$ and $d(x_{m-1}, x_{m-7}) = 3$, vertices x_2, x_{m-1} and x_{m-7} are colored with 2 when m is odd.

Since $d(x_{m-2}, x_{m-4}) = 2$, $d(x_{m-2}, x_{m-7}) = 3$, $d(x_{m-4}, x_{m-7}) = 2$ and $d(\{x_{m-2}, x_{m-4}, x_{m-7}\}, x_{2i-1}) \geq 2$ for $1 \leq i \leq \lceil\frac{m-8}{2}\rceil$, vertices x_{m-2}, x_{m-4}, x_{m-7} and x_{2i-1} for $1 \leq i \leq \lceil\frac{m-8}{2}\rceil$ are colored with 1 when m is even.

Since $d(x_{m-2}, x_{m-4}) = 2$, $d(x_{m-2}, x_{m-6}) = 3$, $d(x_{m-4}, x_{m-6}) = 2$ and $d(\{x_{m-2}, x_{m-4}, x_{m-6}\}, x_{2i-1}) \geq 2$, $1 \leq i \leq \lceil\frac{m-8}{2}\rceil$, vertices x_{m-2}, x_{m-4}, x_{m-6} and x_{2i-1} for $1 \leq i \leq \lceil\frac{m-8}{2}\rceil$ are colored with 1 when m is odd.

The remaining $(\lceil\frac{m-7}{2}\rceil - 2)$ vertices are labeled consecutively beginning with 5.

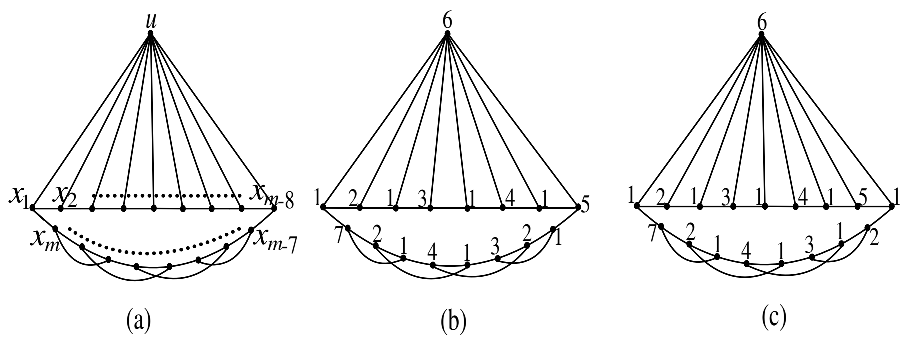

Figure 1: (a) $F(m,4)$, (b) $\chi_\rho(F(16,4)) = 7$ and (c) $\chi_\rho(F(17,4)) = 7$.

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The proof of Theorems 3.4, 3.6 and 3.8 are similar to that of Theorem 3.2.

Definition 3.3. [10] Let $C : x_1x_2...x_mx_1$ be a cycle on m vertices. Let u be a new vertex. The graph obtained by adding edges $(u, x_i), i = 1, 2, ..., m-4$ to C and chords (x_{m-2}, x_m) and (x_{m-1}, x_{m-3}) is called a circular fan with two chords and is denoted by $F(m, 2)$. See Figure 2(a).

Theorem 3.4. $\chi_\rho(F(m, 2)) = \lceil \frac{m-4}{2} \rceil + 3$ for $m \geq 10$.

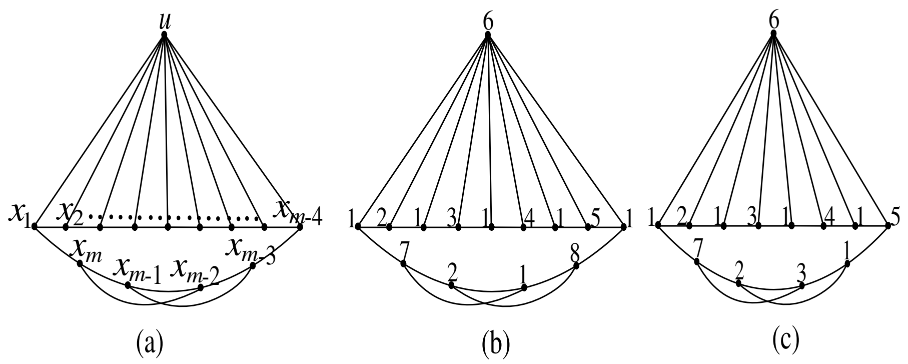

Figure 2: (a) $F(m, 2)$, (b) $\chi_\rho(F(13, 2)) = 8$ and (c) $\chi_\rho(F(12, 2)) = 7$.

Definition 3.5. [10] Let $C : x_1x_2...x_mx_1$ be a cycle on m vertices. Let u and v be two new vertices. The graph obtained by adding edges $(u, x_i), i = 1, 2, ..., m-3$ and $(v, x_i), i = m-2, m-1, m$ to C is called a double headed circular fan and is denoted by $DF(m)$. See Figure 3(a).

Theorem 3.6. $\chi_\rho(DF(m)) = \lceil \frac{m-3}{2} \rceil + 3$ for $m \geq 10$.

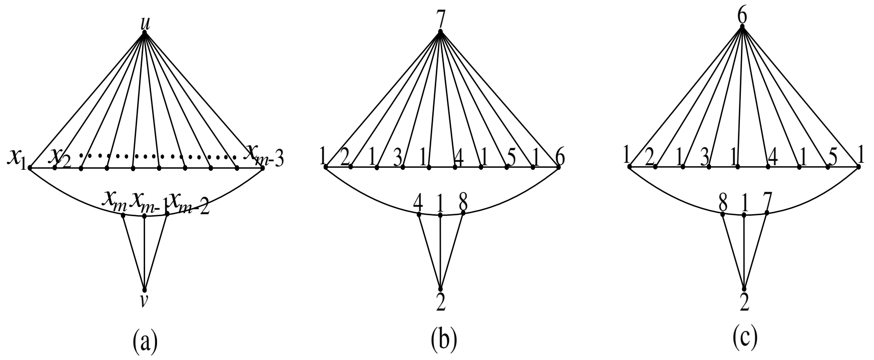

Figure 3: (a) $DF(m)$, (b) $\chi_\rho(DF(13)) = 8$ and (c) $\chi_\rho(F(12)) = 8$.

Definition 3.7. [10] Let $C : x_1x_2...x_mx_1$ be a cycle on m vertices. Let u and v be two new vertices. The graph obtained by adding edges $(u, x_i), i = 1, 2, ..., m-7$ and $(v, x_i), i = m-3, m-4, m-5$ to C and chords (x_{m-2}, x_m) and (x_{m-1}, x_{m-6}) is called a double headed circular fan with two chords and is denoted by $DF(m, 2)$. See Figure 4(a).

Theorem 3.8. $\chi_\rho(DF(m,2)) = \lceil \frac{m-7}{2} \rceil + 3$ for $m \geq 16$.

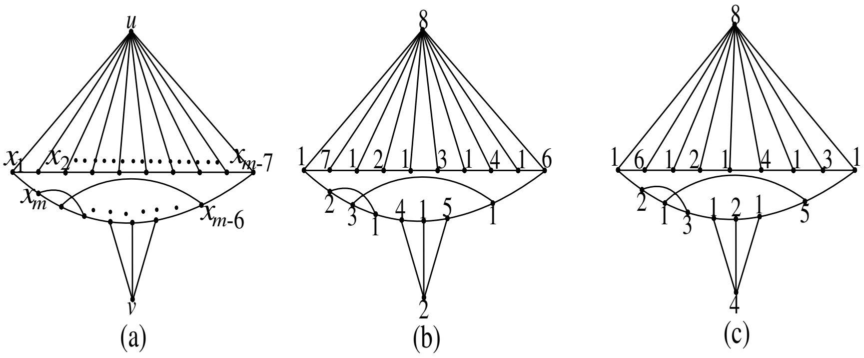

Figure 4: (a) $DF(m,2)$, (b) $\chi_\rho(DF(17,2)) = 8$ and (c) $\chi_\rho(DF(16,2)) = 8$.

4 Mesh of Trees

Let $N = 2^n$. The $N \times N$ mesh of trees denoted by $MT(n)$ is constructed from an $N \times N$ grid of vertices by adding vertices and edges to form a complete binary tree in each row and each column and is said to be of dimension n. The leaves of the tree are precisely the original vertices of the grid, and the added vertices are precisely the internal vertices of trees. Overall, the network has $3N^2 - 2N$ vertices. The leaf and root vertices have degree 2 and all other vertices have degree 3.

Since $MT(1)$ is a cycle on 8 vertices, by Propositions 2.1 and 2.2, $\chi_\rho(MT(1)) = 3$. hence, we have the following theorem.

Theorem 4.1. $\chi_\rho(MT(1)) = 3$.

Theorem 4.2. $\chi_\rho(MT(2)) = 3$.

Proof: Since C_8 is a subgraph of $MT(2)$, by Propositions 2.1 and 2.2, $\chi_\rho(MT(2)) \geq 3$. The coloring shown in Figure 5 shows that $\chi_\rho(MT(2)) = 3$. ■

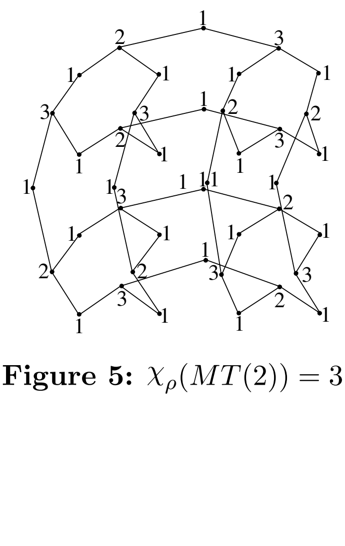

Figure 5: $\chi_\rho(MT(2)) = 3$.

Theorem 4.3. $\chi_\rho(MT(3) = 4$.

Proof: Since a tree of diameter 4 with 4 colors is a subgraph of $MT(3)$, by Propositions 2.1 and 2.4, $\chi_\rho(MT(3)) \geq 4$. The coloring of $MT(3)$ in Figure 6 shows that $\chi_\rho(MT(3)) = 4$. ■

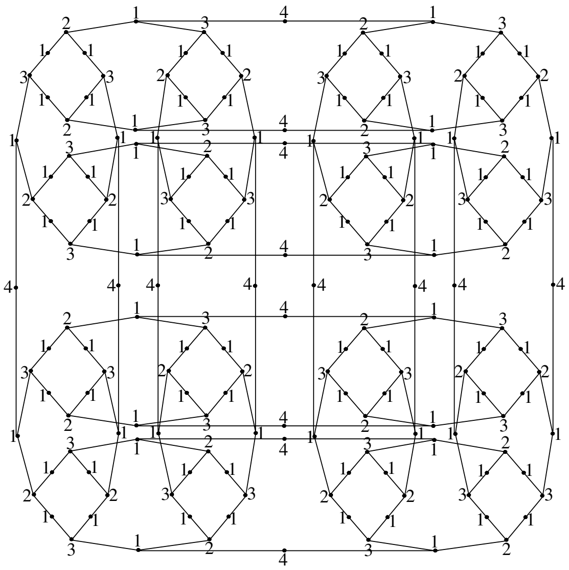

Figure 6: $\chi_\rho(MT(3)) = 4$.

Theorem 4.4. For $MT(4)$, $4 \leq \chi_\rho(MT(4)) \leq 5$.

Proof: Since $MT(3)$ is a subgraph of $MT(4)$, $\chi_\rho(MT(4)) \geq 4$.

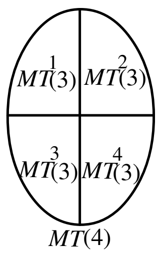

Figure 7: Four copies of $MT(3)$ in $MT(4)$.

We give an algorithm to show that the upper bound for $MT(4)$ is 5. See Figure 8.

Procedure PACKING COLORING $MT(4)$

The Mesh of Tree $MT(4)$ has 4 copies of $MT(3)$. We call them as $MT^1(3)$, $MT^2(3)$, $MT^3(3)$ and $MT^4(3)$. See Figure 7.

Algorithm

Step 1: Copy the coloring of $MT(3)$ given in the proof of Theorem 4.3 to $MT^1(3)$ and $MT^4(3)$.

Step 2: Copy the coloring of $MT(3)$ to $MT^2(3)$ and $MT^3(3)$ and replace the color 4 by 5.

Step 3: Color the remaining vertices with color 1.

Output: $\chi_\rho(MT(4)) \leq 5$

Proof of Correctness: Coloring $MT^1(3)$, $MT^2(3)$, $MT^3(3)$ and $MT^4(3)$ as in $MT(3)$ implies that the vertices receive color i are at distance $i+1$ apart from each other for $1 \leq i \leq 3$. Since the distance between any vertex of $MT^1(3)$ to $MT^4(3)$ is at most 9, the vertices in $MT^1(3)$ and $MT^4(3)$ can be given color 4. Similarly, since the distance between any vertex of $MT^2(3)$ to $MT^3(3)$ is at most 9, the vertices in $MT^2(3)$ and $MT^3(3)$ can be given color 5. Since the root vertices of degree two connecting four copies of $MT^1(3)$, $MT^2(3)$, $MT^3(3)$ and $MT^4(3)$ are at distance at most 2 from the vertices which received color 1, these vertices of degree 2 are colored with 1. Thus the number of colors used is 5. Therefore $\chi_\rho(MT(4)) \leq 5$.

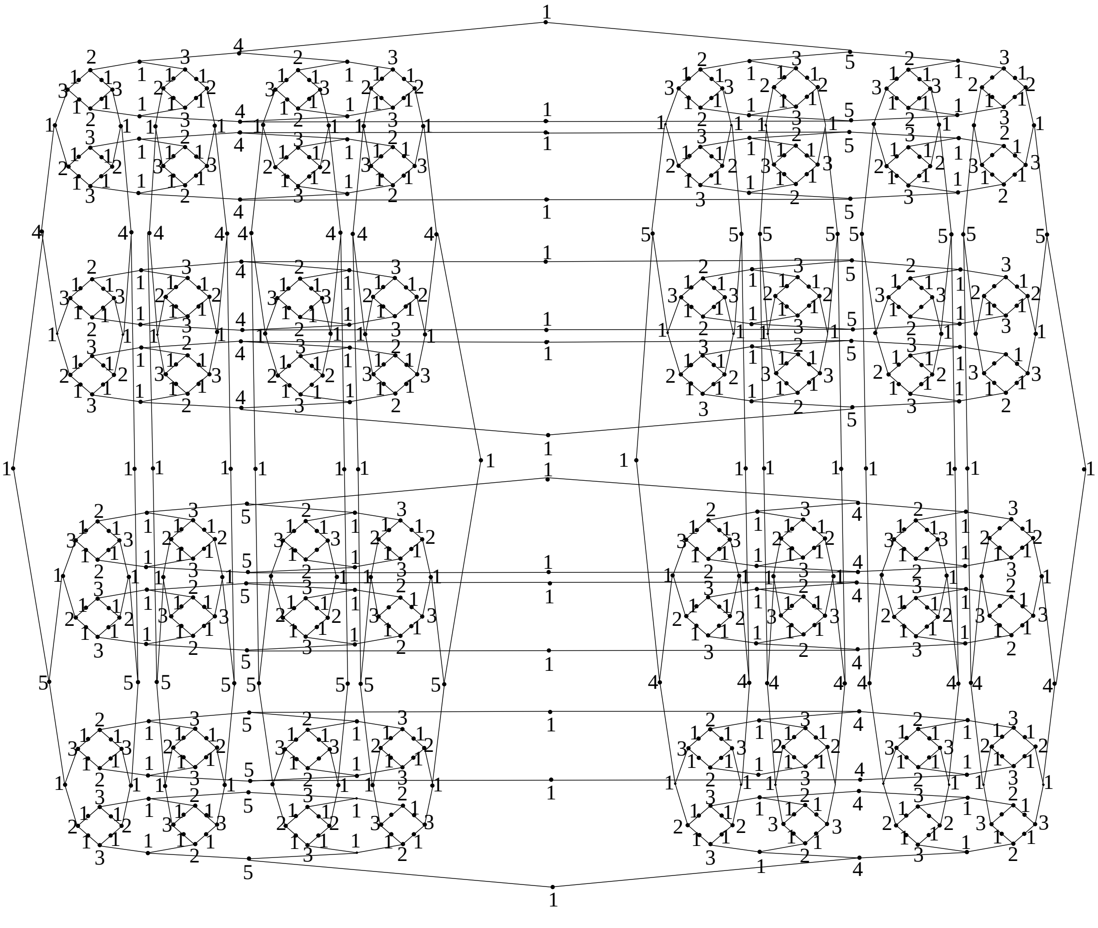

Figure 8: $\chi_\rho(MT(4)) = 5$.

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