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Some cycle-supermagic graphs

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Abstract

A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs to a subgraph of G isomorphic to H. G is H-magic if there is a total labeling $f : V \cup E \to \{1, 2, 3, \cdots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of G isomorphic to H, $\sum\limits_{v \in V_1} f(v) + \sum\limits_{e \in E_1} f(e) = s$ is constant. When $f(V) = \{1, 2, \cdots, |V|\}$, then G is said to be H-supermagic. In this paper, we show that $P_{m,n}$ and the splitting graph of a cycle C_n are cycle-supermagic.

Keywords: Total labeling, H-magic, H-covering, H- supermagic covering.
AMS Subject Classification(2010): 05C78.

1 Introduction

The concept of H-magic graphs was introduced in [2]. An edge-covering of a graph G is a family of different subgraphs $H_1, H_2, \ldots, H_k$ such that each edge of E belongs to at least one of the subgraphs $H_i, 1 \le i \le k$. Then, it is said that G admits an $(H_1, H_2, \ldots, H_k)$ - edge covering. If every H_i is isomorphic to a given graph H, then we say that G admits an H-covering.

If all subgraphs in the covering are edge-disjoint, the covering is also called an H-decomposition of G.

Let $G = (V, E)$ admit an H-covering. We say that a bijective function $f : V \cup E \to \{1, 2, 3, \cdots, |V| + |E|\}$ is an H-magic labeling of G if there is a positive integer $m(f)$, which we call magic sum, such that for each subgraph $H' = (V', E')$ of G isomorphic to H, we have, $f(H') = \sum\limits_{v \in V'} f(v) + \sum\limits_{e \in E'} f(e) = m(f)$. In this case we say that the graph G is H-magic. If $f(V) = \{1, 2, \cdots, |V|\}$, we say that f is an H-supermagic labeling. An H-covering of G is said to be an H-(super)magic covering of G if G admits an H-(super)magic labeling and the supermagic sum is denoted by $s(f)$.

We use the following notations. For any two integers $n < m$, we denote by $[n, m]$, the set of all consecutive integers from n to m. For any set $\mathbb{I} \subset \mathbb{N}$ we write, $\sum \mathbb{I} = \sum\limits_{x \in \mathbb{I}} x$ and for any integers k,

$\mathbb{I}+k=\{x+k : x \in \mathbb{I}\}$. Thus $k+[n,m]$ is the set of consecutive integers from $k+n$ to $k+m$. It can be easily verified that $\sum(\mathbb{I}+k)=\sum\mathbb{I}+k|\mathbb{I}|$.

If $\mathbb{P}=\{X_1,X_2,\cdots,X_n\}$ is a partition of a set X of integers with the same cardinality then we say $\mathbb{P}$ is an n-equipartition of X. Also we denote the set of subsets sums of the parts of $\mathbb{P}$ by $\sum\mathbb{P}=\{\sum X_1,\sum X_2,\cdots,\sum X_n\}$. Finally, given a graph $G=(V,E)$ and a total labeling f on it we denote by $f(G)=\sum f(V)+\sum f(E)$.

2 Preliminary Results

Lemma 2.1. [4] Let h and k be two positive integers and h is odd. Then there exists a k-equipartition $\mathbb{P}=\{X_1,X_2,\cdots,X_k\}$ of $X=[1,hk]$ such that $\sum X_r=\frac{(h-1)(hk+k+1)}{2}+r$ for $1\le r\le k$. Thus, $\sum\mathbb{P}$ is a set of consecutive integers given by $\sum\mathbb{P}=\frac{(h-1)(hk+k+1)}{2}+[1,k]$.

Lemma 2.2. [4] Let h and k be two positive integers such that h is even and $k\ge 3$ is odd. Then there exists a k-equipartition $\mathbb{P}=\{X_1,X_2,\cdots,X_k\}$ of $X=[1,hk]$ such that $\sum X_r=\frac{(h-1)(hk+k+1)}{2}+r$ for $1\le r\le k$. Thus, $\sum\mathbb{P}$ is a set of consecutive integers given by $\sum\mathbb{P}=\frac{(h-1)(hk+k+1)}{2}+[1,k]$.

Lemma 2.3. [4] If h is even, then there exists a k-equipartition $\mathbb{P}=\{X_1,X_2,\cdots,X_k\}$ of $X=[1,hk]$ such that $\sum X_r=\frac{h(hk+1)}{2}$ for $1\le r\le k$.

Lemma 2.4. [4] Let h and k be two even positive integers. If $X=[1,hk+2]-\{1,\frac{k}{2}+2\}$, there exists a k-equipartition $\mathbb{P}=\{X_1,X_2,\cdots,X_k\}$ of X such that $\sum X_r=\frac{h^2k+5h-k-2}{2}+r$ for $1\le r\le k$. Thus, $\sum\mathbb{P}$ is a set of consecutive integers $\frac{h^2k+5h-k-2}{2}+[1,k]$.

Lemma 2.5. [4] Let $h\ge 3$ be an odd integer. If k is odd then there exists a k-equipartition $\mathbb{P}=\{X_1,X_2,\cdots,X_k\}$ of $X=[1,hk]$ such that $\sum X_r=\frac{(h(hk+1)}{2}$ for $1\le r\le k$.

Lemma 2.6. [4] Let $h\ge 3$ be an odd integer. If k is even then there exists a k-equipartition $\mathbb{P}=\{X_1,X_2,\cdots,X_k\}$ of $X=[1,hk+1]-\{\frac{k}{2}+1\}$ such that $\sum X_r=\frac{h^2k+3h-1}{2}$ for $1\le r\le k$.

Lemma 2.7. [4] Let h and k be two even positive integers. If $X=[1,hk+1]-\{\frac{k}{2}+1\}$ there exists a k-equipartition $\mathbb{P}=\{X_1,X_2,\cdots,X_k\}$ of X such that $\sum X_r=\frac{h^2k+3h-k-2}{2}+r$ for $1\le r\le k$. Thus, $\sum\mathbb{P}$ is a set of consecutive integers $\frac{h^2k+3h-k-2}{2}+[1,k]$.

Lemma 2.8. [4] If h is even then there exists a k-equipartition $\mathbb{P}=\{X_1,X_2,\cdots,X_k\}$ of $X=[1,hk]$ such that $\sum X_r=\frac{k(h^2-2)+h-2}{2}+2r$ for $1\le r\le k$.

3 Main Results

Definition 3.1. [3] Let u and v be two fixed vertices. We connect u and v by means of $n\ge 2$ internally disjoint paths of length $m\ge 2$ each. The resulting graph embedded in a plane is denoted by $P_{m,n}$.

$P_{m,n}$ has $(m-1)n+2$ vertices and mn edges.

Theorem 3.2. The graph $P_{m,n}$ is C_{2m}-supermagic for all $m, n \geq 2$.

Proof: Let u and v be two fixed vertices. We join u and v by means of n internally disjoint paths of length m.
For $1 \leq i \leq n$, let $P_i = uv_{i,1}v_{i,2}\cdots v_{i,m-1}v$ be the i^{th} path between u and v. Let $C_{i,j} = P_iP_j^{-1}$. Then $C_{i,j} = uv_{i,1}v_{i,2}\cdots v_{i,m-1}vv_{j,m-1}v_{j,m-2},\cdots v_2v_1u$ is a cycle of length $2m$.

Then clearly $\{C_{i,j} : i, j = 1, 2, \cdots, n \text{ and } i \neq j\}$ is a covering for $P_{m,n}$. Also, $C_{i,j} \cong C_{2m}$ for $i, j = 1, 2, \cdots, n$ and $i \neq j$. Therefore, $\{C_{i,j} : i, j = 1, 2, \cdots, n \text{ and } i \neq j\}$ is a C_{2m}-covering for $P_{m,n}$.

Now, we prove that there exists a C_{2m}-supermagic covering for $P_{m,n}$.

Let V be the vertex set and E be the edge set of $P_{m,n}$. Then $|V| = (m-1)n + 2$ and $|E| = mn$. Let V_i be the vertex set and E_i be the edge set of the path P_i for $i = 1, 2, \cdots, n$. Let $V_i' = V_i - \{u, v\}$.

Case 1: m is even and n is odd.

Since $m-1$ is odd, by Lemma 2.1 there exists an n-equipartition $\mathbb{Q}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[1, (m-1)n]$ such that

$$\begin{aligned}\sum X_i &= \frac{(m-2)[(m-1)n + n + 1]}{2} + i \text{ for } 1 \leq i \leq n \\ &= \frac{(m-2)(mn+1)}{2} + i \text{ for } 1 \leq i \leq n.\end{aligned}$$

Since m is even and n is odd, by Lemma 2.2 there exists an n-equipartition $\mathbb{Q}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1, mn]$ such that

$$\sum Y_i' = \frac{(m-1)(mn+n+1)}{2} + i \text{ for } 1 \leq i \leq n.$$

Add $(m-1)n+2$ to each element of the set $[1, mn]$. We get an n-equipartition $\mathbb{Q}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[(m-1)n + 3, 2mn - n + 2]$ such that

$$\sum Y_i = [(m-1)n + 2]m + \frac{(m-1)(mn+n+1)}{2} + i \quad \text{for } 1 \leq i \leq n.$$

Define a total labeling $f : V \cup E \to [1, 2mn - n + 2]$ as follows:

$$\begin{aligned} f(u) &= (m-1)n + 1 \qquad \text{and } f(v) = (m-1)n + 2. \\ f(V_i') &= X_i \qquad \text{for } 1 \leq i \leq n. \\ f(E_i) &= Y_{n-i+1} \qquad \text{for } 1 \leq i \leq n. \end{aligned}$$

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(P_i) &= f(u) + f(v) + \sum f(V_i') + \sum f(E_i) \\ &= f(u) + f(v) + \sum X_i + \sum Y_{n-i+1} \end{aligned}$$

$$= constant.$$

Thus, $f(P_i)$ is constant for $1 \leq i \leq n$. Now, $f(C_{i,j}) = f(P_i) + f(P_j) - f(u) - f(v)$ which is a constant. Since $C_{i,j} \cong C_{2m}$, $P_{m,n}$ is C_{2m}-supermagic.

Case 2: m is even and n is even.

Since $m-1$ is odd and n is even, by Lemma 2.6 there exists an n-equipartition $\mathbb{Q}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[1, (m-1)n+1] - \left\{\frac{n}{2}+1\right\}$ such that

$$\sum X_r = \frac{(m-1)^2 n + 3(m-1) - 1}{2} \text{ for } 1 \leq i \leq n.$$

Since m is even, by Lemma 2.3 there exists an n-equipartition $\mathbb{Q}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1, mn]$ such that

$$\sum Y_i' = \frac{m(mn+1)}{2} \text{ for } 1 \leq i \leq n.$$

Add $(m-1)n+2$ to each element of the set $[1, mn]$. We get an n-equipartition $\mathbb{Q}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[(m-1)n+3, 2mn-n+2]$ such that

$$\sum Y_i = [(m-1)n+2]m + \frac{m(mn+1)}{2} \text{ for } 1 \leq i \leq n.$$

Define a total labeling $f : V \cup E \to [1, 2mn-n+2]$ as follows:

$$\begin{aligned} f(u) &= \frac{n}{2}+1 \qquad \text{and } f(v) = (m-1)n+2. \\ f(V_i') &= X_i \qquad \text{for } 1 \leq i \leq n. \\ f(E_i) &= Y_i \qquad \text{for } 1 \leq i \leq n. \end{aligned}$$

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(P_i) &= f(u) + f(v) + \sum f(V_i') + \sum f(E_i) \\ &= f(u) + f(v) + \sum X_i + \sum Y_i \\ &= constant. \end{aligned}$$

Thus, $f(P_i)$ is constant for $1 \leq i \leq n$. Now, $f(C_{i,j}) = f(P_i) + f(P_j) - f(u) - f(v)$ which is a constant. Since $C_{i,j} \cong C_{2m}$, $P_{m,n}$ is C_{2m}-supermagic.

Case 3: m is odd and n is odd.

Since $m-1$ is even and n is odd, by Lemma 2.2 there exists an n-equipartition $\mathbb{Q}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[1, (m-1)n]$ such that

$$\sum X_i = \frac{(m-2)[(m-1)n+n+1]}{2} + i \text{ for } 1 \leq i \leq n$$

$$= \frac{(m-2)(mn+1)}{2} + i \text{ for } 1 \leq i \leq n.$$

Since m is odd, by Lemma 2.1 there exists an n-equipartition $\mathbb{Q}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1, mn]$ such that

$$\sum X_i = \frac{(m-1)(mn+n+1)}{2} + i \text{ for } 1 \leq i \leq n.$$

Add $(m-1)n+2$ to each element of the set $[1, mn]$. We get an n-equipartition $\mathbb{Q}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[(m-1)n+3, 2mn-n+2]$ such that

$$\sum Y_i = [(m-1)n+2]m + \frac{(m-1)(mn+n+1)}{2} + i \text{for } 1 \leq i \leq n.$$

Define a total labeling $f : V \cup E \to [1, 2mn-n+2]$ as follows:

$$\begin{aligned} f(u) &= (m-1)n+1 \qquad \text{and } f(v) = (m-1)n+2. \\ f(V_i') &= X_i \qquad \text{for } 1 \leq i \leq n. \\ f(E_i) &= Y_{n-i+1} \qquad \text{for } 1 \leq i \leq n. \end{aligned}$$

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(P_i) &= f(u) + f(v) + \sum f(V_i') + \sum f(E_i) \\ &= f(u) + f(v) + \sum X_i + \sum Y_{n-i+1} \\ &= constant. \end{aligned}$$

Hence, $f(C_{i,j})$ is a constant and consequently $P_{m,n}$ is C_{2m}-supermagic.

Case 4: m is odd and n is even.
Since $m-1$ and n are even, by Lemma 2.4 there exists an n-equipartition $\mathbb{Q}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[1, (m-1)n+2] - \{1, \frac{n}{2}+2\}$ such that

$$\sum X_i = \frac{(m-1)^2 n + 5(m-1) - n - 2}{2} + i \text{ for } 1 \leq i \leq n.$$

Since m is odd, by Lemma 2.1 there exists an n-equipartition $\mathbb{Q}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1, mn]$ such that

$$\sum Y_i' = \frac{(m-1)(mn+n+1)}{2} + i \quad \text{for } 1 \leq i \leq n.$$

Add $(m-1)n+2$ to each element of the set $[1, mn]$. We get an n-equipartition $\mathbb{Q}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[(m-1)n+3, 2mn-n+2]$ such that

$$\sum Y_i = [(m-1)n+2]m + \frac{(m-1)(mn+n+1)}{2} + i \quad \text{for } 1 \leq i \leq n.$$

Define a total labeling $f : V \cup E \to [1, 2mn - n + 2]$ as follows:

$$\begin{aligned} f(u) &= 1 \qquad \text{and } f(v) = \frac{n}{2} + 2 \\ f(V_i') &= X_i \qquad \text{for } 1 \le i \le n \\ f(E_i) &= Y_{n-i+1} \qquad \text{for } 1 \le i \le n. \end{aligned}$$

Then for $1 \le i \le n$,

$$\begin{aligned} f(P_i) &= f(u) + f(v) + \sum f(V_i') + \sum f(E_i) \\ &= f(u) + f(v) + \sum X_i + \sum Y_{n-i+1} \\ &= constant. \end{aligned}$$

Thus, $f(P_i)$ is constant for $1 \le i \le n$. Now, $f(C_{i,j}) = f(P_i) + f(P_j) - f(u) - f(v)$ which is a constant. Since $C_{i,j} \cong C_{2m}$, $P_{m,n}$ is C_{2m}-supermagic. Hence, $P_{m,n}$ is C_{2m}-supermagic for all $m, n \ge 2$. ∎

Illustration 3.3. C_{10}-supermagic labeling of $P_{5,5}$ is given in Figure 1.

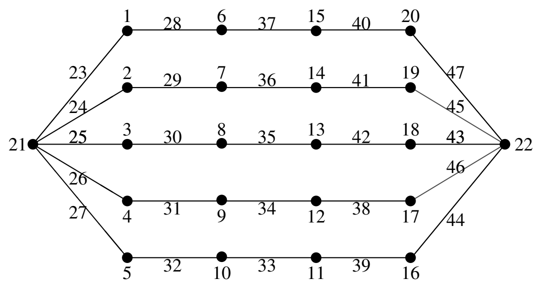

Figure 1: C_{10}-supermagic labeling of $P_{5,5}$.

Illustration 3.4. C_{10}-supermagic labeling of $P_{5,4}$ is given in Figure 2.

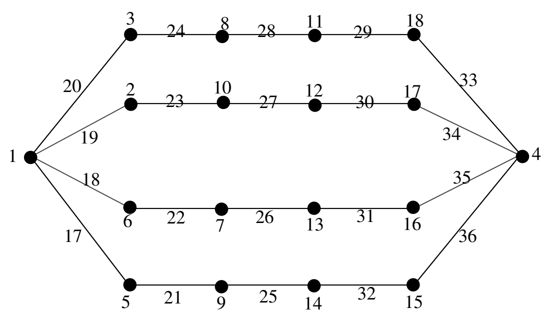

Figure 2: C_{10}-supermagic labeling of $P_{5,4}$.

Definition 3.5. [1] For a graph G, the splitting graph of G; $S(G)$, is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v.

Theorem 3.6. The splitting graph of a cycle C_n is C_4-supermagic for $n \neq 4$.

Proof: Let $C_n = v_1 v_2 \cdots v_n$ be a cycle of length n and $S(C_n)$ be its splitting graph. Let v'_1, v'_2, $\cdots v'_n$ be the added vertices corresponding to v_1, v_2, $\cdots v_n$. Let V be the vertex set and E be the edge set of the splitting graph $S(C_n)$. Then,
$V = \{v_1, v_2, \cdots, v_n, v'_1, v'_2, \cdots, v'_n\}$ and
$E = \{v_1v_2, v_2v_3, \cdots v_{n-1}v_n, v_nv_1\} \cup \{v'_1v_2, v'_2v_3, \cdots v'_{n-1}v_n, v'_nv_1\} \cup \{v'_1v_n, v'_2v_1, , v'_3v_2, \cdots v'_nv_{n-1}\}$.

Let $C_4^i = v_iv_{i+1}v'_iv_{i-1}$ for $2 \leq i \leq n-1$, $C_4^1 = v_1v_2v'_1v_n$ and $C_4^n = v_nv_1v'_nv_{n-1}$. Then, clearly $\{C_4^i : 1 \leq i \leq n\}$ is a covering for $S'(C_n)$. Since $C_4^i \cong C_4$ for $1 \leq i \leq n$ we have $\{C_4^i : 1 \leq i \leq n\}$ is a C_4-covering and we prove that it is a C_4-supermagic covering.
Define a total labeling $f : V \cup E \leftarrow \{1, 2, 3, \cdots, |V \cup E|\}$ by

$$
\begin{aligned}
f(v_i) &= i \text{ for } 1 \leq i \leq n \\
f(v'_i) &= 2n - i \text{ for } 1 \leq i \leq n-1 \\
f(v'_n) &= 2n \\
f(v_iv_{i+1}) &= 3n - i + 1 \text{ for } 1 \leq i \leq n-1 \\
f(v_nv_1) &= 2n + 1 \\
f(v'_iv_{i+1}) &= 3n + i \text{ for } 1 \leq i \leq n-1 \\
f(v'_nv_1) &= 4n \\
f(v'_iv_{i-1}) &= 5n - i + 1 \text{ for } 2 \leq i \leq n \\
f(v'_1v_n) &= 5n
\end{aligned}
$$

For $2 \leq i \leq n-1$,

$$
\begin{aligned}
f(C_4^i) &= f(v_i) + f(v_{i+1}) + f(v'_i) + f(v_{i-1}) + f(v'_iv_{i+1}) + f(v_{i+1}v'_i) + f(v'_iv_{i-1}) + f(v_{i-1}v_i) \\
&= i + i + 1 + 2n - i + i - 1 + 3n - i + 1 + 3n - i + 5n - i + 1 + 3n - i + 1 + 1 \\
&= 16n + 4. \\
f(C_4^1) &= f(v_1) + f(v_2) + f(v'_1) + f(v_n) + f(v_1v_2) + f(v_2v'_1) + f(v'_1v_n) + f(v_1v_n) \\
&= 1 + 2 + 2n - 1 + n + 3n + 3n + 1 + 5n + 2n + 1 \\
&= 16n + 4. \\
f(C_4^n) &= f(v_n) + f(v_1) + f(v'_n) + f(v_{n-1}) + f(v_nv_1) + f(v_1v'_n) + f(v'_nv_{n-1}) + f(v_{n-1}v_n) \\
&= n + 1 + 2n + n - 1 + 2n + 1 + 4n + 4n + 1 + 2n + 2 \\
&= 16n + 4.
\end{aligned}
$$

Hence $f(C_4^i) = 16n+4$ for $1 \leq i \leq n$. Since $C_4^i \cong C_4$ for $1 \leq i \leq n$ we have $\{C_4^i\}$ is a C_4-supermagic covering for the splitting graph $S'(C_n)$. Hence, $S'(C_n)$ is C_4-supermagic.

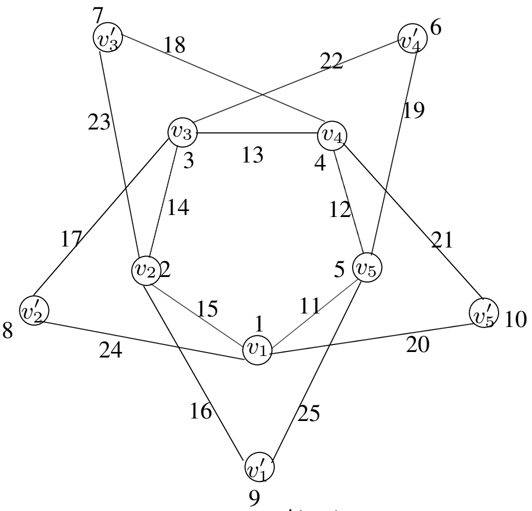

Figure 3: A C_4-supermagic labeling of $S'(C_n)$ with supermagic strength 84.

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