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b-Chromatic number of some cycle related graphs

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Abstract

A b-coloring by k-colors is a proper coloring of the vertices of graph G such that in each color classes there exists a vertex that has neighbours in all the other $k-1$ color classes. The b-chromatic number $\varphi(G)$ is the largest integer k for which G admits a b-coloring with k-colors. If $\chi(G)$ is the chromatic number of G then G is said to be b-continuous if b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$. We investigate the b-chromatic number of some cycle related graphs and also study their b-continuity.

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1 Introduction

We begin with a simple, finite, connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A coloring of the vertices of G is a mapping $f : V(G) \rightarrow \mathbb{N}$. For every vertex $v \in V(G)$, $f(v)$ is called the color of v. If any two adjacent vertices have different colors then f is called proper coloring. The chromatic number $\chi(G)$ is the smallest integer k such that G admits a proper coloring using k colors. The set of vertices with a particular color is called a color class.

A *b-coloring* by k colors is a proper coloring of the vertices of G such that in each color class there exists a vertex that has neighbours in all the other *k-1* color classes. In other words each color class contains a vertex which has at least one neighbour in all the other color classes. Such vertex is called a color dominating vertex. If v is a color dominating vertex of a color class c then we write $cdv(c) = v$. It is obvious that every coloring of a graph G by $\chi(G)$ colors is a b-coloring of G. The b-chromatic number $\varphi(G)$ is the largest integer k such that G admits a b-coloring with k colors. The concept of b-coloring was introduced by Irving and Manlove [6] and showed that the problem of determining $\varphi(G)$ is NP-hard for general graphs but it is polynomial-time solvable for trees. According to Faik [5] the graph is b-continuous if b-coloring exists for every integer k satisfying $\chi(G) \leq k \leq \varphi(G)$. The bounds for the b-chromatic number for various graphs are established by Kouider and Zaker in [7] while the

discussion on b-coloring of central graph of some graphs is reported in Thilagavathi *et al.* [9]. The b-chromatic number of cartesian product of some families of graphs is studied by Balakrishnan *et al.* [3] while b-coloring for square of cartesian product of two cycles is investigated by Chandrakumar and Nicholas [4]. Let G be a graph and A be a P_4 in G,a partner of A is a vertex v in $G - A$ such that $A \cup v$ induces at least two P_4 in G. A graph G is P_4-tidy if every induced P_4 has at most one partner. It has been proved by Velasquez *et al.* that all P_4-tidy graphs are b-continuous. Some results on b-coloring and b-continuity are reported in Alkhateeb [1]. The b-chromatic number of C_n is well known but we initiate the study of b-coloring for the graphs obtained from C_n by means of various graph operations. For any undefined term we refer to West [10].
We present some results which are useful for the present investigations.

Propostion 1.1. [2] For any graph G, $\varphi(G) \leq \Delta(G) + 1$.

Propostion 1.2. [6] If a graph G admits a b-coloring with m-colors,then G must have at least m vertices with degree at least $m - 1$ (Since each color class has a b-vertex).

2 Main Results

Definition 2.1. A vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every vertex which are not adjacent to v in G.

Lemma 2.2. If G_v is the graph obtained by switching of an arbitrary vertex v in cycle C_n then

$$\chi(G_v) = \begin{cases} 2, & for\ the\ cycle\ C_n;\ n = 4 \\ 3, & for\ the\ cycle\ C_n;\ n \geq 5\,. \end{cases}$$

Proof: Let $G = C_n$ and $v_1, v_2, ..., v_n$ be the vertices of C_n. Without loss of generality we switch the vertex v_1 of C_n and denote the resultant graph by G_v. We consider the following two cases.
Case 1: For $n = 4$.

As the graph obtained by switching of v_1 is a bipartite graph, we have $\chi(G_v) = 2$.

Case 2: For $n \geq 5$.

In this case G_v contains odd cycles which implies that $\chi(G_v) \geq 3$. Now for proper coloring of G_v, we need to assign color to only vertex v_1, as $G_v - v_1$ is 2-colorable. Hence $\chi(G_v) = 3$. ■

Remark 2.3. We avoid the case when $n = 3$ as C_3 is a complete graph and switching of any vertex in C_3 will yield a disconnected graph.

Theorem 2.4. Let G_v be the graph obtained by switching of an arbitrary vertex v in cycle C_n then

$$\varphi(G_v) = \begin{cases} 2, & for\,the\,cycle\,C_n; n = 4 \\ 3, & for\,the\,cycle\,C_n; n = 5 \\ 4, & for\,the\,cycle\,C_n; n \geq 6. \end{cases}$$

Proof: We continue with the terminology and notations used in Lemma 2.2 and consider the following cases. Here $|V(G_v)| = n$.

Case 1: For $n = 4$.

We have $|V(G_v)| = 4$ and $V((G_v)) = \{v_1, v_2, v_3, v_4\}$. The graph G_v is isomorphic to $K_{1,3}$ and $\varphi(K_{1,3}) = 2$ as proved by Kratochvil *et al.* [8]. Hence $\varphi(G_v) = 2$.

Case 2: For $n = 5$.

We have $|V(G_v)| = 5$ and $V((G_v)) = \{v_1, v_2, v_3, v_4, v_5\}$. Also G_v has two vertices of degree three, a vertex of degree two and two vertices of degree one. As $\Delta(G_v) = 3$, then by Proposition 1.1, $\varphi(G_v) \leq 4$.

If $\varphi(G_v) = 4$ then according to Proposition 1.2 we need minimum four vertices of degree at least three, which is not possible as there are only two vertices of degree three, one vertex of degree two and two vertices of degree one.

For G_v, the graph $G_v - v_1$ is a path which is a bipartite graph. Hence it is 2-colorable. For b-coloring the third color must be assigned to v_1 . As $d(v_1) = 2$ and v_1 is adjacent to the vertices of both the color classes. Now consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 2, f(v_5) = 3$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_4, cdv(3) = v_3$.

Hence $\varphi(G_v) = 3$.

Case 3: For $n = 6$.

Here $|V(G_v)| = 6$ and let $V(G_v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Also G_v has two vertices of degree one and four vertices of degree three. As $\Delta(G_v) = 3$, by Proposition 1.1 we have $\varphi(G_v) \leq 4$.

If $\varphi(G_v) = 4$ then according to Proposition 1.2, the graph must have four vertices of degree at least three, which is possible for the graph G_v as there are exactly four vertices of degree three and the remaining two vertices are of degree one.

Now consider the set of colors $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1$, $f(v_2) = 4$, $f(v_3) = 2$, $f(v_4) = 3$, $f(v_5) = 4$, $f(v_6) = 2$.

This proper coloring gives $cdv(1) = v_1, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5$. Hence, $\varphi(G_v) = 4$.

Case 4: For $n > 6$.

We color the vertices $v_1, v_2, ..., v_6$ as in G_v and for the remaining vertices assign the colors as

$$f(v_i) = \begin{cases} 3, & when\ i\ is\ odd \\ 2, & when\ i\ is\ even \end{cases}$$

The color dominating vertices are same as in the case when $n = 6$. Thus $\varphi(G_v) = 4$, for all $n > 6$. ■

Corollary 2.5. G_v is b-continuous.

Proof: To prove the result we consider the following cases.

Case 1: For $n = 4$.

In this case the graph G_v is b-continuous as $\chi(G_v) = \varphi(G_v) = 2$.

Case 2: For $n = 5$.

In this case the graph is P_4-tidy and as mentioned earlier it is b-continuous.

Case 3: For $n \geq 6$.

By Lemma 2.2, $\chi(G_v) = 3$ and by Theorem 2.4, $\varphi(G_v) = 4$. Hence b-coloring exists for every integer k satisfying $\chi(G_v) \leq k \leq \varphi(G_v)$ (Here $k = 3, 4$).

Hence G_v is b-continuous for all n . ■

Definition 2.6. The splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' respectively in $S'(G)$.

Lemma 2.7.

$$\chi\left(S'\left(C_n\right)\right) = \begin{cases} 2, & when\ n\ is\ even \\ 3, & when\ n\ is\ odd. \end{cases}$$

Proof: Let $v_1, v_2, ..., v_n$ be the vertices and $e_1, e_2, .., e_n$ be the edges of cycle C_n. Let $v'_1, v'_2, .., v'_n$ be the newly added vertices corresponding to the vertices $v_1, v_2, ..., v_n$ to construct the graph $S'(C_n)$.

Here $V(S'(C_n)) = V(C_n) \cup \{v'_i / 1 \leq i \leq n\}$ and $E(S'(C_n)) = E(C_n) \cup \{v'_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v'_n v_1\} \cup \{v_i v'_{i+1} / 1 \leq i \leq n-1\} \cup \{v_n v'_1\}$. Also $|V(S'(C_n))| = 2n$ and $|E(S'(C_n))| = 3n$.

We consider the following two cases.

Case 1: When n is even.

$S'(C_n)$ is a bipartite graph $\Rightarrow \chi(S'(C_n)) = 2$.

Case 2: When n is odd.

In this case $\chi(S'(C_n)) \geq 3$ as the graph $S'(C_n)$ contains odd cycles. Here the graph $S'(C_n) - \{v_1, v'_1\}$ is a bipartite graph and hence it is 2-colorable. As mutually non-adjacent vertices v_1 and v'_1 are adjacent to the vertices of both the color classes and for proper coloring the third color must be assigned to v_1 and v'_1. Hence $\chi(S'(C_n)) = 3$. ■

Theorem 2.8. $$\varphi(S'(C_n)) = \begin{cases} 3, & n = 3 \\ 2, & n = 4 \\ 4, & n = 5, 6, 8 \\ 5, & n = 7 \\ 5, & n \geq 9. \end{cases}.$$

Proof: We continue with the terminology and notations used in Lemma 2.7 and consider the following five cases.

Case 1: For $n = 3$.

We have $|V(S'(C_3))| = 6$ and $V(S'(C_3)) = \{v_1, v_2, v_3, v'_1, v'_2, v'_3\}$. Also $S'(C_3)$ has three vertices of degree two and three vertices of degree four. Then by Proposition 1.1, $\varphi(S'(C_3)) \leq 5$ as

$\Delta(S'(C_3)) = 4$.
If $\varphi(S'(C_3)) = 5$, then according to Proposition 1.2, the graph $S'(C_3)$ must have five vertices of degree four, which is not possible as there are only three vertices of degree four and the remaining three vertices are of degree two.
If $\varphi(S'(C_3)) = 4$ then according to Proposition 1.2 we need minimum four vertices of degree at least three, which is not possible as there are only three vertices of degree four and the remaining three vertices are of degree two.
If $\varphi(S'(C_3)) = 3$, then according to Proposition 1.2, the graph must have three vertices of degree two. This is possible for $S'(C_3)$. For the b-coloring consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = f(v_1') = 1, f(v_2) = f(v_2') = 2, f(v_3) = f(v_3') = 3$. This proper coloring gives $cdv(1) = v_1', cdv(2) = v_2', cdv(3) = v_3'$. Thus $\varphi\,(S'\,(C_3)) = 3$.
Case 2: For $n = 4$.

For $S'(C_4)$ we have $|V(S'(C_4))| = 8$ and $V\,(S'(C_4)) = \{v_1, v_2, v_3, v_4, v_1', v_2', v_3', v_4'\}$. Also $S'\,(C_4)$ has four vertices of degree two and four vertices of degree four. Then by Proposition 1.1, we have $\varphi\,(S'(C_4)) \leq 5$ as $\Delta\,(S'(C_4)) = 4$.
If $\varphi\,(S'\,(C_4)) = 5$ then according to Proposition 1.2 the graph $S'(C_4)$ must have five vertices of degree four, which is not possible as there are only four vertices of degree four and the remaining vertices are of degree two.
Due to nature of the graph $S'(C_4)$ any proper coloring with four or three colors have at least one color class which does not have any color dominating vertex. Hence such coloring will not be a b-coloring for the graph $S'(C_4)$.
For b-coloring we color the vertices with two colors. Define the color function $f : V \to \{1, 2\}$ as

$$\begin{aligned} f\,(v_i) = \quad & f(v_i') = \quad 1, \quad i = 1, 3 \\ f(v_i) = \quad & f\,(v_i') = \quad 2, \quad i = 2, 4 \end{aligned}$$

This proper coloring gives $cdv(1) = v_1', cdv(2) = v_2'$. Hence, $\varphi\,(S'\,(C_4)) = 2$.

Case 3: For $n = 5, 6, 8$.

Subcase 1: For $n = 5$.

For $S'(C_5)$ we have $|V(S'\,(C_5))| = 10$ and $V(S'(C_5)) = \{v_1, v_2, v_3, v_4, v_5, v_1',\ v_2', v_3',\ v_4', v_5'\}$. Also in the graph $S'(C_5)$ we have five vertices of degree two and five vertices of degree four. Since $\Delta\,(S'(C_5)) = 4$, by Proposition 1.1, $\varphi\,(S'(C_5)) \leq 5$.
If $\varphi\,(S'(C_5)) = 5$, then the graph $S'(C_5)$ must have five vertices of degree four which is possible. But due to nature of the graph $S'(C_5)$, any proper coloring with five colors has at least one color class which does not have any color dominating vertex. Hence, such coloring will not be a b-coloring for the graph $S'(C_5)$. Thus we can color the graph by four colors.

For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_1') = 3, f(v_2) = 2,$

$f(v_2') = 3, f(v_3) = 4, f(v_3') = 1, f(v_4) = 3, f(v_4') = 1, f(v_5) = 2, f(v_5') = 4$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_4, cdv(4) = v_3$. Hence, $\varphi(S'(C_5)) = 4$.

Subcase 2: For $n = 6$.

We have $|V(S'(C_6))| = 12$ and $V(S'(C_6)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_1', v_2', v_3' v_4', v_5', v_6'\}$. Also in the graph $S'(C_6)$ we have six vertices of degree two and six vertices of degree four. Then by Proposition 1.1, $\varphi(S'(C_6)) \leq 5$ as $\Delta(S'(C_6))$= 4.

If $\varphi(S'(C_6)) = 5$, $S'(C_6)$ must have five vertices of degree four which is possible. But due to nature of the graph $S'(C_6)$ any proper coloring with five colors has at least one color class which does not have any color dominating vertex. Hence such coloring will not be b-coloring for the graph $S'(C_6)$. Thus we color the graph by four colors.

For b-coloring consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_1') = 3, f(v_2) = 2, f(v'_2) = 3, f(v_3) = 4, f(v_3') = 1, f(v_4) = 3, f(v_4') = 1, f(v_5) = 2, f(v_5') = 2, f(v_6) = 4, f(v_6') = 4$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_4, cdv(4) = v_3$. Hence, $\varphi(S'(C_6)) = 4$.

Subcase 3: For $n = 8$.

We have $|V(S'(C_8))| = 16$ and $V(S'(C_8)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1' v_2', v_3', v_4', v_5', v_6', v_7', v_8'\}$. Also in the graph $S'(C_8)$ we have eight vertices of degree two and eight vertices of degree four. Then by Proposition 1.1, $\varphi(S'(C_8)) \leq 5$ as $\Delta(S'(C_8)) = 4$.

If $\varphi(S'(C_8)) = 5$, $S'(C_8)$ must have five vertices of degree four which is possible. But due to the nature of the graph $S'(C_8)$ any proper coloring with five colors has at least one color class which does not have any color dominating vertex. Hence such coloring will not be b-coloring for the graph $S'(C_8)$. For b-coloring with four colors consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_1') = 3$, $f(v_2) = 2, f(v'_2) = 3, f(v_3) = 4, f(v_3') = 1, f(v_4) = 3, f(v_4') = 1, f(v_5) = 2, f(v_5') = 2$, $f(v_6) = 3, f(v_6') = 3, f(v_7) = 1, f(v_7') = 1, f(v_8) = 4, f(v_8') = 4$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_4, cdv(4) = v_3$. Hence, $\varphi(S'(C_8)) = 4$.

Case 4: For $n = 7$.

We have $|V(S'(C_7)| = 14$ and $V(S'(C_7)) = \{v_1, v_2, v_3\ v_4, v_5, v_6, v_7, v_1', v_2', v_3', v_4', v_5', v_6', v_7'\}$. Also the graph $S'(C_7)$ has seven vertices of degree two and seven vertices of degree four. Then by Proposition 1.1, $\varphi(S'(C_7)) \leq 5$ as $\Delta(S'(C_7)) = 4$. If $\varphi(S'(C_7)) = 5$ then according to Proposition 1.2, we need minimum five vertices of degree at least four which is possible as there are seven vertices of degree two and seven vertices of degree four.

For b-coloring consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v_1) = 1, f(v_1') = 3, f(v_2) = 2, f(v'_2) = 3, f(v_3) = 4, f(v_3') = 5, f(v_4) = 1, f(v_4') = 5$, $f(v_5) = 3, f(v_5') = 3, f(v_6) = 2, f(v_6') = 4, f(v_7) = 5, f(v_7') = 4$. This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_5, cdv(4) = v_3, cdv(5) = v_7$. Hence, $\varphi(S'(C_7)) = 5$.

Case 5: For $n \geq 9$.

When $n = 9$ $|V(S'(C_9))| = 18$ and let $V(S'(C_9)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6\ v'_7, v'_8, v'_9\}$. Also the graph $S'(C_9)$ has nine vertices of degree two and nine vertices of degree four. Then by Proposition 1.1, $\varphi\,(S'(C_9)) \leq 5$ as $\Delta\,(S'(C_9)) = 4$. According to Proposition 1.2, if $\varphi\,(S'(C_9)) = 5$ then we need minimum five vertices of degree at least four, which is possible as there are nine vertices of degree two and nine vertices of degree four. For b-coloring consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v_1) = 4, f(v'_1) = 3, f(v_2) = 1, f(v'_2) = 3, f(v_3) = 2, f(v'_3) = 5, f(v_4) = 4, f(v'_4) = 5, f(v_5) = 3, f(v'_5) = 3, f(v_6) = 1, f(v'_6) = 2, f(v_7) = 4, f(v'_7) = 3, f(v_8) = 5, f(v'_8) = 3, f(v_9) = 1, f(v'_9) = 2$. This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_5, cdv(4) = v_7, cdv(5) = v_8$.
Thus $\varphi\,(S'(C_9)) = 5$.

When $n > 9$, we repeat the colors as in the graph $S'(C_9)$ for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\ , v_7\ v_8, v_9, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v'_9\}$ and for the remaining vertices give the colors as

$$\begin{aligned} f(v_{2i}) = \quad & f(v'_{2i}) = \quad 5; \quad i = 5, 6, 7, ... \\ f(v_{2i+1}) = \quad & f(v'_{2i+1}) = \quad 1; \quad i = 5, 6, 7, ... \end{aligned}$$

The color dominating vertices are the same as the color dominating vertices in the case of $S'(C_9)$. Therefore $\varphi\,(S'(C_n)) = 5$, for all $n > 9$. ∎

Corollary 2.9. $S'(C_n)$ is b-continuous.

Proof: To prove the result we consider the following cases.
Case 1: For $n = 3$.

In this case the graph $S'(C_3)$ is b-continuous as $\chi(S'(C_3)) = \varphi\,(S'(C_3)) = 3$.

Case 2: For $n = 4$.

In this case the graph $S'(C_4)$ is b-continuous as $\chi(S'(C_4)) = \varphi\,(S'(C_4)) = 2$.

Case 3: For $n = 5$.

By Lemma 2.7, $\chi\,(S'(C_5)) = 3$ and by Theorem 2.8, $\varphi\,(S'(C_5)) = 4$. Hence b-coloring exists for every integer k satisfying $\chi\,(S'\,(C_5)) \leq k \leq \varphi\,(S'\,(C_5))$ (Here $k = 3, 4$). Hence $S'(C_5)$ is b-continuous.

Case 4: For $n = 6$.

By Lemma 2.7, $\chi\,(S'(C_6)) = 2$ and by Theorem 2.8, $\varphi\,(S'(C_6)) = 4$. It is obvious that b-coloring for the graph $S'(C_6)$ is possible using the number of colors $k = 2, 4$. Now for $k = 3$ the b-coloring for the graph $S'(C_6)$ as follows. Consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = 1, f(v'_1) = 3, f(v_2) = 2, f(v'_2) = 3, f(v_3) = 1, f(v'_3) = 1, f(v_4) = 3, f(v'_4) = 2, f(v_5) = 2, f(v'_5) = 2, f(v_6) = 3, f(v'_6) = 3$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_4$.

Thus $S'(C_6)$ is three colorable. Hence b-coloring exists for every integer k satisfying $\chi\left(S'\left(C_6\right)\right) \leq k \leq \varphi\left(S'\left(C_6\right)\right)$ (Here $k = 2, 3, 4$). Thus $S'(C_6)$ is b-continuous.

Case 5: For $n = 7$.

By Lemma 2.7, $\chi\left(S'(C_7)\right) = 3$ and by Theorem 2.8, $\varphi\left(S'(C_7)\right) = 5$. It is obvious that b-coloring for the graph $S'(C_7)$ is possible using the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring for the graph $S'(C_7)$ is as follows.

Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_1') = 3, f(v_2) = 2, f(v_2') = 3, f(v_3) = 4, f(v_3') = 4, f(v_4) = 1, f(v_4') = 1, f(v_5) = 3, f(v_5') = 3, f(v_6) = 2, f(v_6') = 4, f(v_7) = 3, f(v_7') = 4$.

This proper coloring gives the color dominating vertices $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_5, cdv(4) = v_3$. Thus $S'(C_7)$ is four colorable. Hence b-coloring exists for every integer k satisfying $\chi\left(S'\left(C_7\right)\right) \leq k \leq \varphi\left(S'\left(C_7\right)\right)$ (Here $k = 3, 4, 5$). Thus $S'(C_7)$ is b-continuous.

Case 6: For $n = 8$.

By Lemma 2.7, $\chi\left(S'(C_8)\right) = 2$ and by Theorem 2.8, $\varphi\left(S'(C_8)\right) = 4$. It is obvious that b-coloring for the graph $S'(C_8)$ is possible with the number of colors $k = 2, 4$. Now for $k = 3$ the b-coloring for the graph $S'(C_8)$ is as follows.

Consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = 1, f(v_1') = 3, f(v_2) = 2, f(v_2') = 3, f(v_3) = 1, f(v_3') = 1, f(v_4) = 3, f(v_4') = 3, f(v_5) = 2, f(v_5') = 2, f(v_6) = 3, f(v_6') = 3, f(v_7) = 2, f(v_7') = 2, f(v_8) = 3, f(v_8') = 3$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_4$. Thus $S'(C_8)$ is three colorable. Hence b-coloring exists for every integer k satisfying $\chi\left(S'\left(C_8\right)\right) \leq k \leq \varphi\left(S'\left(C_8\right)\right)$ (Here $k = 2, 3, 4$). Thus $S'(C_8)$ is b-continuous.

Case 7: For $n \geq 9$.

By Lemma 2.7, $\chi\left(S'(C_9)\right) = 3$ and by Theorem 2.8, $\varphi\left(S'(C_9)\right) = 5$. It is obvious that b-coloring for the graph $S'(C_9)$ is possible with the number of colors $k = 3, 5$. Now for $k = 4$ the b-coloring for the graph $S'(C_9)$ is as follows.

Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 2, f(v_1') = 4, f(v_2) = 1, f(v_2') = 3, f(v_3) = 2, f(v_3') = 3, f(v_4) = 4, f(v_4') = 4, f(v_5) = 1, f(v_5') = 2, f(v_6) = 3, f(v_6') = 3, f(v_7) = 4, f(v_7') = 4, f(v_8) = 3, f(v_8') = 3, f(v_9) = 1, f(v_9') = 1$.

This proper coloring gives the color dominating vertices as $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_6, cdv(4) = v_4$. Thus $S'(C_9)$ is four colorable. Hence b-coloring exists for every integer k satisfying $\chi\left(S'\left(C_9\right)\right) \leq k \leq \varphi\left(S'\left(C_9\right)\right)$ (Here $k = 3, 4, 5$).

When $n > 9$, repeat the colors as in $S'(C_9)$ for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_1', v_2', v_3', v_4', v_5', v_6', v_7', v_8', v_9'\}$ and for the remaining vertices give the colors as

When $k = 4$:

$$f(v_i) = \begin{cases} 3, & i \quad even \\ 1, & i \quad odd \end{cases}$$

Therefore, $S'(C_n)$, for all $n > 9$ is b-continuous. ■

Definition 2.10. The edge splitting $S'_e(G)$ of a graph G is obtained by adding to each edge $e_i = v_i v_j$ a new edge $e'_i = v'_i v'_j$ such that $N(v'_i) = N(v_i) \cup \{v'_{i+1}\} - \{v_{i+1}\}$ and $N(v'_{i+1}) = N(v_{i+1}) \cup \{v'_i\} - \{v_i\}$ in $S'_e(G)$.

Illustration 2.11. For better understanding of the above definition the edge splitting graph of C_5 is shown in Figure 1 in which the light lines are of C_5 while the dark lines are newly added in order to obtain $S'_e(C_5)$.

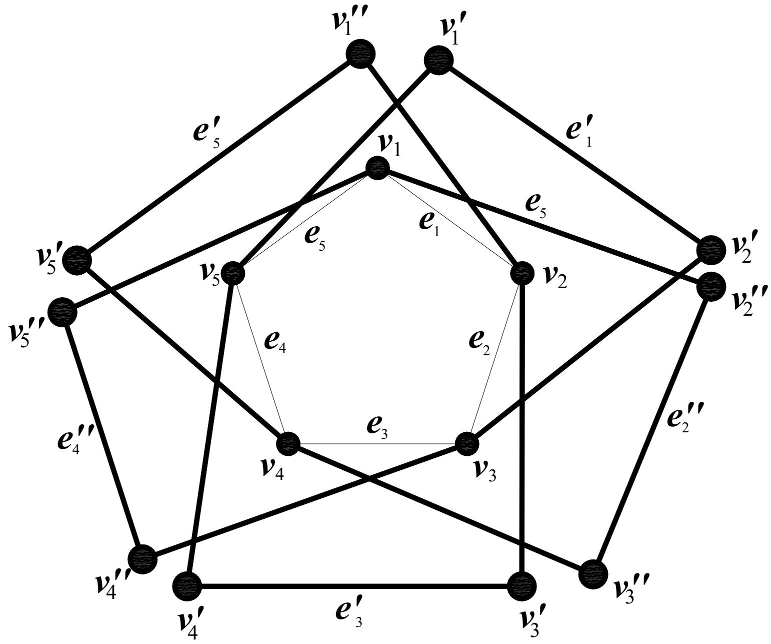

Figure 1: Edge Splitting Graph of C_5.

Lemma 2.12.

$$\chi\left(S'_e(C_n)\right) = \begin{cases} 2 & n \ even \\ 3 & n \ odd. \end{cases}$$

Proof: Let $v_1, v_2, ..., v_n$ be the vertices and $e_1, e_2, .., e_n$ be the edges of cycle C_n such that $e_i = v_i v_{i+1}, for \ 1 \le i \le n-1$. Let $v'_1, v'_2, ..., v'_n$ and $v_1'', v_2'', ..., v_n''$ be the vertices in $S'_e(C_n)$ such that when edge e_i is duplicated then $e'_i = v_i v'_{i+1}$ for odd i and for even i, $e''_i = v''_i v''_{i+1}$. Also e_n is duplicated by $e'_n = v'_n v''_1$ for odd n while $e'_n = v''_n v'_1$ for even n. Thus $|V(S'_e(C_n))| = 3n$ and

$|E(S'_e(C_n))| = 4n$. We consider the following two cases.

Case 1: When n is even.

Here $\chi\,(S'_e(C_n)) = 2$, as the graph $S'_e(C_n)$ is a bipartite graph

Case 2: When n is odd.

In this case, $\chi\,(S'_e(C_n)) \geq 3$ as the graph $S'_e(C_n)$ contains odd cycles. Here the graph $S'_e(C_n) - \{v_1, {v_1}', {v_1}''\}$ is a bipartite graph and hence it is 2-colorable. The three mutually non-adjacent vertices v_1,${v_1}'$ and ${v_1}''$ are adjacent to the vertices of both the color classes. Therefore, the third color must be assigned to v_1,${v_1}'$ and ${v_1}''$ for proper coloring. Hence $\chi(S'_e(C_n)) = 3$. ∎

Theorem 2.13. $\varphi(S'_e(C_n)) = \begin{cases} 3 & n = 3 \\ 4 & n = 4 \\ 5 & n > 4. \end{cases}$

Proof: We continue with the terminology and notations used in Lemma 2.12 and consider the following three cases.

Case 1: For $n = 3$.

We have $|V\,((S'_e(C_3)))| = 9$ and let $V(S'_e(C_3)) = \{v_1, v_2, v_3, v'_1, v'_2, v'_3, {v_1}''\ {v_2}'', {v_3}''\}$. Also the graph $S'_e(C_3)$ has six vertices of degree two and three vertices of degree four. Then by Proposition 1.1, $\varphi(S'_e(C_3)) \leq 5$, as $\Delta\,(S'{}_e\,(C_3)) = 4$.

If $\varphi\,(S'_e(C_3)) = 5$ then according to Proposition 1.2, the graph $S'_e(C_3)$ must have five vertices of degree four, which is not possible as $S'_e(C_3)$ has three vertices of degree four.

If $\varphi(S'_e(C_3)) = 4$ then according to Proposition 1.2, we need minimum four vertices of degree at least three, which is not possible as there are only three vertices of degree four and remaining six vertices are of degree two. Hence it is b-colorable by three colors.

For b-coloring consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = f(v'_1) = f({v_1}'') = 1, f(v_2) = f(v'_2) = f({v_2}'') = 2, f(v_3) = f(v'_3) = f({v_3}'') = 3$.

This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3$.

Therefore $\varphi\,(S'_e(C_3)) = 3$.

Case 2: For $n = 4$.

In $(S'_e(C_4))$ we have $|V(S'_e(C_4))| = 12$ and let $V\,(S'_e(C_4)) = \{v_1, v_2, v_3, v_4, v'_1, v'_2\,, v'_3, v'_4, {v_1}'', {v_2}''\ {v_3}'', {v_4}''\}$. Also the graph $S'_e(C_4)$ has eight vertices of degree two and four vertices of degree four. Then by Proposition 1.1, $\varphi\,(S'_e(C_4)) \leq 5$ as in $S'_e(C_4)$, $\Delta\,(S'_e(C_4)) = 4$.

If $\varphi\,(S'_e(C_4)) = 5$ then according to Proposition 1.2, the graph $S'_e(C_4)$ must have five vertices of degree four, which is not possible as $S'_e(C_4)$ has four vertices of degree four.

If $\varphi\,(S'_e(C_4)) = 4$ then according to Proposition 1.2, the graph $S'_e(C_4)$ must have four vertices of degree three which is possible as the graph $S'_e(C_4)$ has four vertices of degree four. Thus we can color the graph by four colors.

For b-coloring consider the color class $c = \{1,2,3,4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1,2,3,4\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(v_1') = 2, f(v_2') = 1, f(v_3') = 2, f(v_4') = 2, f(v_1{''}) = 4, f(v_2{''}) = 3, f(v_3{''}) = 3, f(v_4{''}) = 2.$
This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4$.
Thus $\varphi\,(S'{}_e(C_4)) = 4$.
Case 3: For $n > 4$.
Subcase 1: For $n = 5$.

We have $|V(S_e'(C_5))| = 15$ and let $V\,(S_e'(C_5)) = \{v_1, v_2, v_3, v_4,\ v_5,\ v_1',\ v_2', v_3', v_4', v_5'\ v_1{''}, v_2{''}, v_3{''}, v_4{''}, v_5{''}\}$. Also $S_e'(C_5)$ has ten vertices of degree two and five vertices of degree four. Then by Proposition 1.1, $\varphi\,(S_e'(C_5)) \leq 5$, as in $S_e'(C_5)$, $\Delta\,(S_e'(C_5)) = 4$.
If $\varphi\,(S_e'(C_5)) = 5$ then according to Proposition 1.2, the graph $S_e'(C_5)$ must have five vertices of degree four which is possible as there are ten vertices of degree two and five vertices of degree four.
For b-coloring consider the color class $c = \{1,2,3,4,5\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1,2,3,4,5\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(v_5) = 5, f(v_1') = 3, f(v_2') = 1, f(v_3') = 2, f(v_4') = 2, f(v_5') = 4, f(v_1{''}) = 5, f(v_2{''}) = 3, f(v_3{''}) = 4, f(v_4{''}) = 5, f(v_5{''}) = 1$
This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v_5$.
Thus $\varphi\,(S'{}_e(C_5)) = 5$.
Subcase 2: For $n = 6$.

In $(S_e'(C_6))$ we have $|V(S_e'(C_6))| = 18$ and let $V\,(S_e'(C_6)) = \{v_1, v_2, v_3, v_4, v_5, v_6,\ v_1', v_2', v_3', v_4', v_5', v_6'\ v_1{''}, v_2{''}, v_3{''}, v_4{''}, v_5{''}, v_6{''}\}$. Also the graph $S_e'(C_6)$ has twelve vertices of degree two and six vertices of degree four. Then by Proposition 1.1, $\varphi\,(S_e'(C_6)) \leq 5$, as in $S_e'(C_6)$, $\Delta\,(S_e'(C_6)) = 4$
If $\varphi\,(S'{}_e(C_6)) = 5$ then according to Proposition 1.2, the graph $S_e'(C_6)$ must have six vertices of degree four which is possible as there are twelve vertices of degree two and six vertices of degree four.
For b-coloring consider the color class $c = \{1,2,3,4,5\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1,2,3,4,5\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(v_5) = 2, f(v_6) = 5, f(v_1') = 5, f(v_2') = 1, f(v_3') = 5, f(v_4') = 1, f(v_5') = 4, f(v_6') = 4, f(v_1{''}) = 3, f(v_2{''}) = 3, f(v_3{''}) = 4, f(v_4{''}) = 5, f(v_5{''}) = 1, f(v_6{''}) = 1.$
This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v_6$.
Thus $\varphi\,(S'{}_e(C_6)) = 5$.
Subcase 3: For $n = 7$.

In $(S_e'(C_7))$ we have $|V(S_e'(C_7))| = 21$ and let $V\,(S_e'(C_7)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_1', v_2', v_3', v_4'\ v_5', v_6', v_7', v_1{''}, v_2{''}, v_3{''}, v_4{''}, v_5{''}, v_6{''}, v_7{''}\}$. Also the graph $S_e'(C_7)$ has fourteen vertices of degree two and seven vertices degree four.

Then by Proposition 1.2, $\varphi\,(S'{}_e(C_7)) \leq 5$ as in $S_e'(C_7)$, $\Delta\,(S_e'(C_7)) = 4$. If $\varphi\,(S_e'(C_7)) = 5$ then

according to Proposition 1.2, the graph $S'_e(C_7)$ must have seven vertices of degree four which is possible as there are fourteen vertices of degree two and seven vertices of degree four.
For b-coloring consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v_1) = 1, f(v_2) = 2, f(v_3) = 3, f(v_4) = 4, f(v_5) = 5, f(v_6) = 2, f(v_7) = 5, f(v'_1) = 5, f(v'_2) = 1, f(v'_3) = 2, f(v'_4) = 1, f(v'_5) = 1, f(v'_6) = 3, f(v'_7) = 4, f(v_1{''}) = 3, f(v_2{''}) = 3, f(v_3{''}) = 4, f(v_4{''}) = 5, f(v_5{''}) = 1, f(v_6{''}) = 3, f(v_7{''}) = 4.$
This proper coloring gives $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_4, cdv(5) = v_5$
Thus $\varphi\,(S'_e(C_7)) = 5$.
Subcase 4: For $n \geq 8$.

In $(S'_e(C_8))$ we have $|V(S'_e(C_8))| = 24$ and let $V\,(S'_e(C_5)) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v'_1, v'_2$ $v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v_1{''}, v_2{''}, v_3{''}, v_4{''}, v_5{''}, v_6{''}, v_7{''}, v_8{''}\}$. Also the graph $S'_e(C_8)$ has sixteen vertices of degree two and eight vertices degree four. Then by Proposition 1.1, $\varphi\,(S'_e(C_8)) \leq 5$ as in $S'_e(C_8)$, $\Delta\,(S'_e(C_8)) = 4$.
If $\varphi\,(S'_e(C_8)) = 5$ then according to Proposition 1.2, the graph $S'_e(C_8)$ must have eight vertices of degree four which is possible as there are sixteen vertices of degree two and eight vertices of degree four.
For b-coloring consider the color class $c = \{1, 2, 3, 4, 5\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3, 4, 5\}$ as $f(v_1) = 5, f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(v_5) = 4, f(v_6) = 5, f(v_7) = 2, f(v_8) = 1, f(v'_1) = 4, f(v'_2) = 5, f(v'_3) = 1, f(v'_4) = 2, f(v'_5) = 1, f(v'_6) = 1, f(v'_7) = 1, f(v'_8) = 1, f(v_1{''}) = 4, f(v_2{''}) = 3, f(v_3{''}) = 3, f(v_4{''}) = 4, f(v_5{''}) = 5, f(v_6{''}) = 1, f(v_7{''}) = 3, f(v_8{''}) = 1$
This proper coloring gives $cdv(1) = v_2, cdv(2) = v_3, cdv(3) = v_4, cdv(4) = v_5, cdv(5) = v_6$. Thus $\varphi\,(S'_e(C_8)) = 5$.

When $n > 8$, we repeat the colors as in the graph $S'_e(C_8)$ for the vertices $\{v_1, v_2, v_3, v_4, v_5\ v_6, v_7$ $v_8, v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8, v_1{''}, v_2{''}, v_3{''}, v_4{''}, v_5{''}, v_6{''}, v_7{''}, v_8{''}\}$ and for the remaining vertices give the colors as

$$\begin{aligned} f(v_{2i+1}) = \quad f(v'_{2i+1}) = \quad f(v''_{2i+1}) = \quad 2; \quad i = 4, 5, 6, ... \\ f(v_{2i}) = \quad f(v'_{2i}) = \quad f(v''_{2i}) = \quad 1; \quad i = 5, 6, 7, ... \end{aligned}$$

The color dominating vertices are same as the color dominating vertices in the case of $S'_e(C_8)$. Therefore $\varphi\,(S'_e(C_n)) = 5$, for all $n > 8$. ∎

Corollary 2.14. $S'_e(C_n)$ is b-continuous.

Proof: To prove the result we consider the following cases.
Case 1: $n = 3$.

In this case the graph $S'_e\,(C_3)$ is b-continuous as $\chi(S'_e(C_3)) = \varphi\,(S'_e(C_3))$=3.
Case 2: $n = 4$.

In this case $\chi(S'_e(C4)) = 2$ by Lemma 2.12 and $\varphi\,(S'_e(C_4)) = 4$ by Theorem 2.13. It is obvious

that b-coloring for the graph $S_e'(C_4)$ is possible using the number of colors $k = 2, 4$.
Now for $k = 3$ the b-coloring for the graph $S_e'(C_4)$ is as follows. Consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = 1, f(v_1') = 1, f(v_1'') = 1, f(v_2) = 2, f(v_2') = 2, f(v_2'') = 3, f(v_3) = 1, f(v_3') = 2, f(v_3'') = 3, f(v_4) = 3, f(v_4') = 2, f(v_4'') = 3$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_4$. Thus the graph $S_e'(C_4)$ is three colorable. Hence b-coloring exists for every integer k satisfying $\chi\left(S'_e\left(C_4\right)\right) \leq k \leq \varphi\left(S'_e\left(C_4\right)\right)$(Here $k = 2, 3, 4$). Hence $S_e'(C_4)$ is b-continuous.

Case 3: When n is odd, $n \geq 5$.

For $n = 5$ by Lemma 2.12, $\chi\left(S_e'(C_5)\right) = 3$ and by Theorem 2.13, $\varphi\left(S_e'(C_5)\right) = 5$. It is obvious that b-coloring for the graph $S_e'(C_5)$ is possible using the number of colors $k = 3, 5$.
Now for $k = 4$ the b-coloring for the graph $S_e'(C_5)$ is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_1') = 1, f(v_1'') = 2, f(v_2) = 2, f(v_2') = 2, f(v_2'') = 3, f(v_3) = 3, f(v_3') = 2, f(v_3'') = 4, f(v_4) = 1, f(v_4') = 3, f(v_4'') = 4, f(v_5) = 4, f(v_5') = 2, f(v_5'') = 2$.
This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3, cdv(4) = v_5$. Thus the graph $S_e'(C_5)$ four colorable. Hence b-coloring exists for every integer k satisfying $\chi\left(S'_e\left(C_5\right)\right) \leq k \leq \varphi\left(S'_e\left(C_5\right)\right)$(Here $k = 3, 4, 5$). Therefore $S_e'(C_5)$ is b-continuous.
When $n > 5$, repeat the colors as in the graph $S_e'(C_5)$ for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_1', v_2', v_3', v_4', v_5', v_1'', v_2'', v_3'', v_4'', v_5'',\}$ and for the remaining vertices give the colors as follows.
<u>When $k = 4$:</u>

$$\begin{array}{llll} f(v_i) = 1 & f(v_i') = 1 & f(v_i'') = & 2; \quad i \quad even \\ f(v_i) = 4 & f(v_i') = 1 & f(v_i'') = & 2; \quad i \quad odd \end{array}$$

Therefore, $S_e'(C_n)$,for all $n > 5$ is b-continuous.

Case 4: When n is even, $n > 5$.

For $n = 6$, by Lemma 2.12, $\chi\left(S_e'(C_6)\right) = 2$ and by Theorem 2.13, $\varphi\left(S_e'(C_6)\right) = 5$. It is obvious that b-coloring for the graph $S_e'(C_6)$ is possible using the number of colors $k = 2, 5$.
Now for $k = 3$ the b-coloring for the graph $S_e'(C_6)$ is as follows. Consider the color class $c = \{1, 2, 3\}$ and to assign the proper coloring to the vertices, we define the color function $f : V \to \{1, 2, 3\}$ as $f(v_1) = 1, f(v_1') = 1, f(v_1'') = 2, f(v_2) = 2, f(v_2') = 1, f(v_2'') = 2, f(v_3) = 3, f(v_3') = 1, f(v_3'') = 1, f(v_4) = 2, f(v_4') = 2, f(v_4'') = 2, f(v_5) = 1, f(v_5') = 1, f(v_5'') = 1, f(v_6) = 3, f(v_6') = 2, f(v_6'') = 2$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_3$.
Now for $k = 4$ the b-coloring for the graph $S_e'(C_6)$ is as follows. Consider the color class $c = \{1, 2, 3, 4\}$ and to assign the proper coloring to the vertices define the color function $f : V \to \{1, 2, 3, 4\}$ as $f(v_1) = 1, f(v_1') = 3, f(v_1'') = 3, f(v_2) = 2, f(v_2') = 1, f(v_2'') = 3, f(v_3) = 4, f(v_3') =$

$2, f(v_3'') = 3, f(v_4) = 3, f(v_4') = 4, f(v_4'') = 3, f(v_5) = 1, f(v_5') = 1, f(v_5'') = 1, f(v_6) = 4, f(v_6') = 4, f(v_6'') = 4$. This proper coloring gives the color dominating vertices as $cdv(1) = v_1, cdv(2) = v_2, cdv(3) = v_4, cdv(4) = v_3$. Thus the graph $S_e'(C_6)$ is three and four colorable. Hence b-coloring exists for every integer k satisfying $\chi(S'_e(C_6)) \leq k \leq \varphi(S'_e(C_6))$(Here $k = 2, 3, 4, 5$). Therefore $S_e'(C_6)$ is b-continuous. When $n > 5$,repeat the colors as in the graph $S_e'(C_6)$ for the vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_1', v_2', v_3', v_4', v_5', v_6', v_1'', v_2'', v_3'', v_4'', v_5'', v_6''\}$ and for the remaining vertices assign the colors as follows.

When $k = 3$:

$$\begin{array}{llllll} f(v_i) = & f(v_i') = & f(v_i'') = & 3; & i & even \\ f(v_i) = & f(v_i') = & f(v_i'') = & 1; & i & odd \end{array}$$

When $k = 4$:

$$\begin{array}{llllll} f(v_i) = & f(v_i') = & f(v_i'') = & 4; & i & even \\ f(v_i) = & f(v_i') = & f(v_i'') = & 1; & i & odd \end{array}$$

Hence $S_e'(C_n)$ is b-continuous for all even $n > 5$. ■

3 Concluding Remarks

The clustering problems occurring in the process of data mining, web-services classifications and decomposition of large distributed systems can be handled with *b*-coloring of graphs. Here we have investigated *b*-chromatic numbers for the larger graphs obtained by means of graph operations on cycle C_n and also discuss their *b*-continuity. To investigate similar results for other graph families is an open area of research.

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