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Some new even harmonious graphs

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Abstract

Let $G(V, E)$ be a graph with p vertices and q edges. A function f is called even harmonious labeling of a graph $G(V, E)$ if $f : V \to \{0, 1, 2, \ldots, 2q\}$ is injective and the induced function $f^* : E \to \{0, 2, 4, \ldots, 2(q-1)\}$ defined as $f^*(uv) = (f(u) + f(v))\ (mod\ 2q)$ is bijective. In this paper we establish an even harmonious labeling for the graphs $C_n \odot mK_1$(n is odd), $P_n \odot mK_1$(n is odd), $C_n@K_1$ (n is even), P_n (n is even) with $n-1$ copies of mK_1, the shadow graph $D_2(K_1, n)$ and the splitting graph $spl(K_1, n)$.

Keywords: Even harmonious labeling, even harmonious graphs, corona graph, shadow graph, splitting graph.
AMS Subject Classification(2010): 05C78.

1 Introduction

Graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serves as useful mathematical models for a broad range of applications including coding theory. Throughout this paper we use modular arithmetic which has been used in cryptography.

A finite undirected simple graph will be considered in this paper. The notations and terminology are taken from Bondy and Murthy [1]. Let $G(V, E)$ be a (p, q) graph with $p = |V|$ vertices and $q = |E|$ edges. Harmonious graphs arose in the study by Graham and Sloane [3] of modular versions of additive base problems stemming from error - correcting codes. Zhi - He Liang, Zhan - Li Bai [4] and S.K.Vaidya, N H Shah [6] discussed odd harmonious graphs with applications. The results about graph labeling are collected and updated regularly in a survey by Gallian [2].

A function f is called even harmonious labeling [5] of a graph $G(V, E)$ if $f : V \to \{0, 1, 2, \ldots, 2q\}$ is injective and the induced function $f^* : E \to \{0, 2, 4, \ldots, 2(q-1)\}$ defined as $f^*(uv) = (f(u) + f(v))\ (mod\ 2q)$ is bijective, the resulting edge labels are distinct. A graph which admits even harmonious labeling is called an even harmonious graph. A corona graph $G \odot H$ is obtained from two graphs G and H taking one copy of G, which is supposed to have p vertices and p copied of the graph H and joining by an edge the k^{th} vertex of G to every vertex in the k^{th} copy of H. In other words, given two graphs G and H, the corona of G with H denoted by $G \odot H$ is the graph with vertex set $V(G) \cup \{\cup_{i \in G} V(H_i)\}$ and the edge set $E(G) \cup_{(i \in V(G))} \{(i, u_i) : i \in V(G)\ and\ u_i \in V(H_i)\}$. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G_1 and G_2 and join each vertex v_i in G_1 to the adjacent vertices of the corresponding vertex u_i in G_2. The splitting graph $Spl(G)$ is obtained from the graph G by adding to each vertex v of G a new vertex u such that u is adjacent to every vertex that is adjacent to v in G. The graph $G_1@G_2$ is nothing but one point union of G_1 and G_2, that is any two graphs G_1 and G_2 are connected by a single edge between any one of the vertex of G_1 and any one of the vertex of G_2.

In this paper we establish an even harmonious labeling for the graphs $C_n \odot mK_1$(n is odd), $P_n \odot mK_1$(n is odd), $C_n@K_1$ (n is even), P_n (n is even) with $n-1$ copies of mK_1, the shadow graph $D_2(K_{1,n})$ and the splitting graph $Spl(K_{1,n})$.

2 Main results

Theorem 2.1. The corona graph $C_n \odot mK_1$ is an even harmonious graph, where n is an odd integer and m is any positive integer.

Proof: Let n be an odd number and $G(V,E)$ be the corona graph $C_n \odot mK_1$ (Figure 1) with $p = q = n(m+1)$.

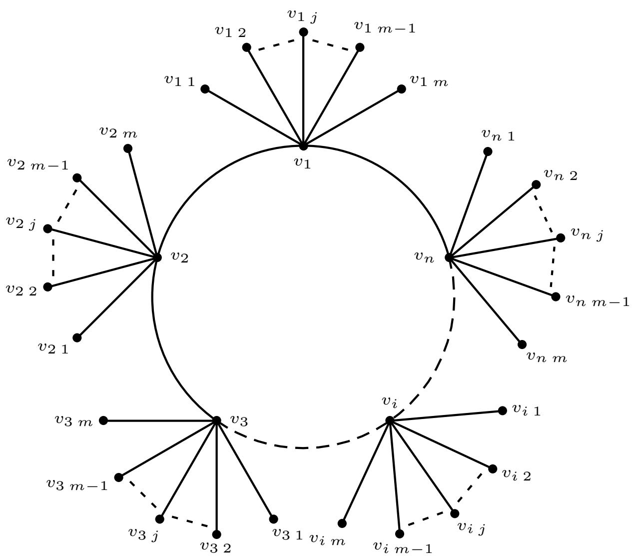

Figure 1: The corona graph $C_n \odot mK_1$.

Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and $v_{i1}, v_{i2}, \ldots, v_{im}$ be the vertices of the i^{th} copy of mK_1 incident with the vertex v_i, where $1 \le i \le n$. Then the edges of the corona graph $C_n \odot mK_1$ are given by $v_1v_2, v_2v_3, \ldots, v_iv_{i+1}, \ldots, v_{n1}v_n, v_nv_1$ and $v_iv_{i\,j}$, where $1 \le i \le n,\ 1 \le j \le m$.

We define the function f from V to $\{0, 1, 2, \ldots, 2q\}$ and assign the numbers $0, 2, 4, \ldots, 2(n-1)$ to the vertices $v_1, v_2, \ldots, v_n$ of the cycle C_n, the numbers $2(2n-1), 2(3n-1), \ldots, 2[n(m+1)-1]$ to the vertices $v_{1\,1}, v_{1\,2}, \ldots, v_{1\,m}$ of the first copy of mK_1 incident with v_1 and the numbers $2(n+i-2), 2(2n+i-2), \ldots, 2(mn+i-2)$ to the vertices $v_{i\,1}, v_{i\,2}, \ldots, v_{i\,m}$ of the i^{th} copy of mK_1, where $2 \le i \le n$.

Then f induces a bijection $f^* : E \to \{0, 2, 4, \ldots, 2(nm + n - 1)\}$. That is f admits an even harmonious labeling for the corona graph $C_n \odot mK_1$. Hence the corona graph $C_n \odot mK_1$ is an even harmonious graph. ■

Example 2.2. An even harmonious labeling for the corona graph $C_3 \odot 6K_1$ is given in Figure 2.

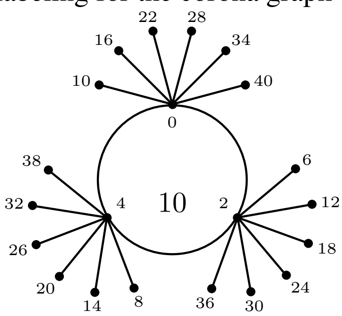

Figure 2: An even harmonious labeling of the corona graph $C_3 \odot 6K_1$.

Theorem 2.3. The corona graph $P_n \odot mK_1$ is an even harmonious graph where $n \geq 3$ is an odd integer and m is any positive integer.

Proof: Let $G(V, E)$ be the corona graph $P_n \odot mK_1$ with $p = n(m+1)$ vertices and $q = n(m+1) - 1$ edges. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n (n is odd)and $v_{i\,1}, v_{i\,2}, \ldots, v_{i\,m}$ be the vertices of the i^{th} copy of mK_1 incident with the vertex v_i, where $1 \leq i \leq n$. Then the edges of the corona graph $P_n \odot mK_1$ are given by v_iv_{i+1} for $1 \leq i \leq n$ and $v_iv_{i\,j}$ for $1 \leq i \leq n, 1 \leq j \leq m$.

We define the function f from V to $\{0, 1, 2, \ldots, 2q\}$ and assign the numbers $0, 2, 4, \ldots, n - 1$ to the vertices $v_1, v_3, \ldots, v_n$ of the path P_n, the numbers $n + 1, n + 3, \ldots, 2(n - 1)$ to the vertices $v_2, v_4, ..., v_{n-1}$ and the numbers $2(2n-i), 2(3n-i), \ldots, 2[n(m+1)-i]$ to the vertices $v_{i\,1}, v_{i\,2}, \ldots, v_{i\,m}$ of the i^{th} copy of mK_1, where $1 \leq i \leq n$.

From the above construction pattern f induces a bijection $f^* : E \to \{0, 2, 4, \ldots, 2(nm + n - 2)\}$. Hence f admits an even harmonious labeling for the corona graph $P_n \odot mK_1$. Hence the corona graph $P_n \odot mK_1$ is an even harmonious graph. ■

Example 2.4. Figure 3 illustrates the even harmonious labeling for the corona graph $P_5 \odot 6K_1$.

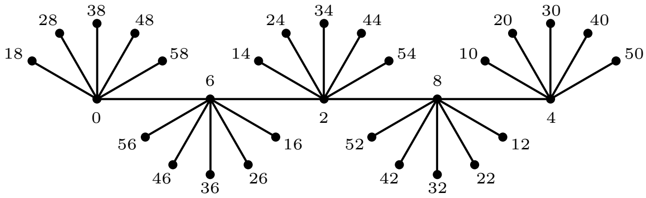

Figure 3: An even harmonious labeling of $P_5 \odot 5K_1$.

Theorem 2.5. The shadow graph $D_2(K_1, n)$ is an even harmonious graph.

Proof: Let $G(V, E)$ be the shadow graph $D_2(K_{1,n})$ (Figure 4) with two copies of the star graph $K_{1,n}$. Let v be the apex vertex and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices of the first copy of the star graph $K_{1,n}$, u be the apex vertex and $u_1, u_2, u_3, \ldots, u_n$ be the pendant vertices of the second copy of the star graph $K_{1,n}$. Hence the edges of the shadow graph $D_2(K_{1,n})$ are $vv_1, vv_2, vv_3, \ldots, vv_n, vu_1, vu_2, vu_3, \ldots, vu_n, uv_1, uv_2, uv_3, \ldots, uv_n$ and $uu_1, uu_2, uu_3, \ldots, uu_n$. Therefore $p = 2n + 2$ and $q = 4n$.

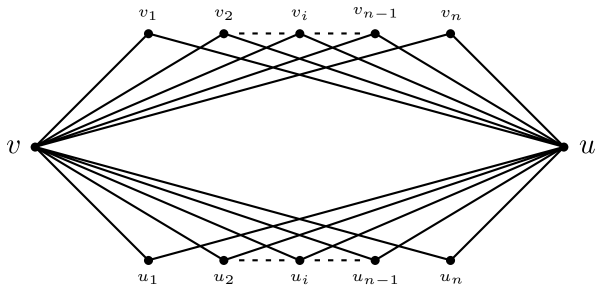

Figure 4: The shadow graph $D_2(K_{1,n})$.

Define the mapping f from the vertex set V of G to the integer set $\{0, 1, 2, \ldots, 2q\}$ as follows. Assign 0 to the apex vertex v and the numbers $4, 8, 12, \ldots, 4n$ to the vertices $v_1, v_2, \ldots, v_n$ of the first copy of the star graph $K_{1,n}$, assign the number 2 to the apex vertex u and the numbers $4(n+1), 4(n+2), \ldots, 8n$ to the vertices $u_1, u_2, \ldots, u_n$ of the second copy of the star graph $K_{1,n}$.

That is $f(v) = 0$, $f(v_i) = 4i$ for $1 \leq i \leq n$, $f(u) = 2$ and $f(u_i) = 4(n + i)$ for $1 \leq i \leq n$. From the above construction pattern f induces a bijection $f^* : E \to \{0, 2, 4, \ldots, 2(4n - 1)\}$. Thus f admits an even harmonious labeling for the shadow graph $D_2(K_{1,n})$. Hence the shadow graph $D_2(K_{1,n})$ is an even harmonious graph. Even harmonious labeling of the shadow graph is $D_2(K_{1,3})$ given in Figure 5. ∎

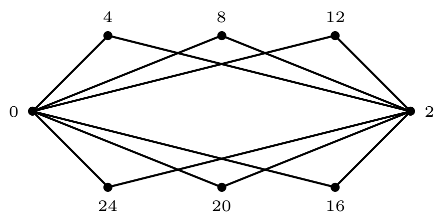

Figure 5: An even harmonious labeling of the shadow graph $D_2(K_{1,3})$.

Theorem 2.6. The splitting graph $Spl(K_{1,n})$ is an even harmonious graph.

Proof: Let $G(V, E)$ be the splitting graph $Spl(K_{1,n})$ (Figure 6) and v be the apex vertex and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices of the star graph $K_{1,n}$ then $u, u_1, u_2, u_3, \ldots, u_n$ are the added vertices corresponding to $v, v_1, v_2, v_3, \ldots, v_n$ respectively. Thus the edges of the splitting graph $Spl(K_{1,n})$ are vv_i, vu_i, uv_i where $1 \leq i \leq n$. Hence $p = 2n + 2$ and $q = 3n$.

Define the mapping f from the vertex set V of G to the integer set $\{0, 1, 2, \ldots, 2q\}$ as follows. Assign 0 to the apex vertex v and the numbers $2(n + 2), 2(n + 4), \ldots, 6n$ to the vertices $v_1, v_2, \ldots, v_n$,

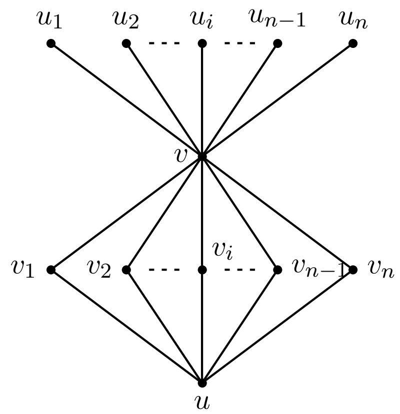

Figure 6: The splitting graph $Spl(K_{1,n})$.

assign 2 to the vertex u and $4, 6, 8, \ldots, 2(n+1)$ to the vertices $u_1, u_2, \ldots, u_n$ respectively of the splitting graph $Spl(K_{1,n})$.

That is $f(v) = 0$, $f(u) = 2$, $f(v_i) = 2(n + 2i)$ for $1 \leq i \leq n$ and $f(u_i) = 2(i + 1)$ for $1 \leq i \leq n$. Then f induces a bijective function $f^* : E \to \{0, 2, 4, \ldots, 2(3n-1)\}$. Hence the splitting graph $Spl(K_{1,n})$ is an even harmonious graph. Figure 7 illustrates the even harmonious labeling for the splitting graph $Spl(K_{1,4})$. ∎

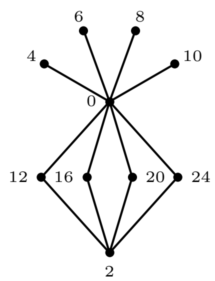

Figure 7: An even harmonious labeling of the splitting graph $Spl(K_{1,4})$.

Theorem 2.7. The graph G obtained from the path graph P_n (n is even) with $n-1$ copies of $\overline{K_m}$ $(m \geq 1)$ incident with first $n-1$ vertices of P_n is an even harmonious graph

Proof: Let G be a graph obtained from the path graph P_n (n is even) with $n-1$ copies of $\overline{K_m}$ incident with first $n-1$ vertices of P_n and $v_1, v_2, \ldots, v_i, \ldots, v_n$ be the vertices of P_n and $v_{i\,1}, v_{i\,2}, \ldots, v_{i\,j}, \ldots, v_{i\,m}$ be the vertices of the i^{th} copy of $\overline{K_m}$ incident with v_i, where $1 \leq i \leq n-1$.

Then the edges are $v_i v_{i+1}$, where $1 \leq i \leq n-1$ and $v_i v_{i\,j}$, where $1 \leq i \leq n-1,\ 1 \leq j \leq m$. Then the graph G has $n(m+1) - m$ vertices and $(n-1)(m+1)$ edges as shown in Figure 8.

We construct a vertex labeling of G as follows. Define the mapping f from the vertex set $V(G)$ to

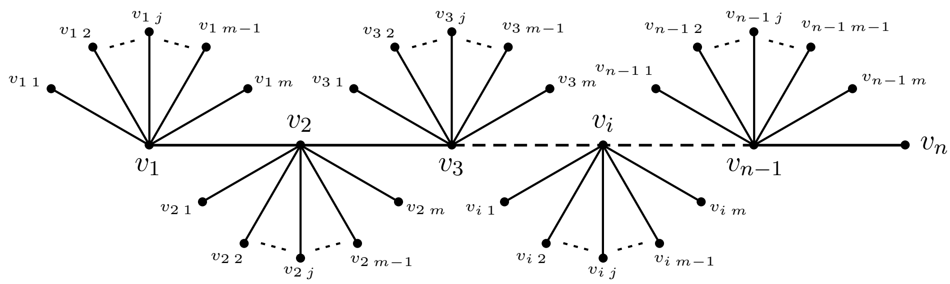

Figure 8

the integer set $\{0, 1, 2, \ldots, 2q\}$ such that

$$f(v_{2i-1}) = 2(i-1) \;\; for \;\; 1 \leq i \leq n/2,$$
$$f(v_{2i}) = n + 2(i-1) \;\; for \;\; 1 \leq i \leq n/2 \;\; and$$
$$f(v_{i\,j}) = 2[n(j+1) - (i+j)] \;\; for \;\; 1 \leq j \leq m, \; 1 \leq i \leq n-1.$$

Therefore f is an injection mapping and f induces a bijective mapping $f^* : E \to \{0, 2, 4, \ldots, 2[(n-1)(m+1) - 1]\}$. From the foregoing discussion, we obtain that the graph G is an even harmonious graph. ■

Example 2.8. Figure 9 shows an even harmonious labeling for the path graph P_4 with 3 copies of $\overline{K_3}$ incident with first 3 vertices of P_4.

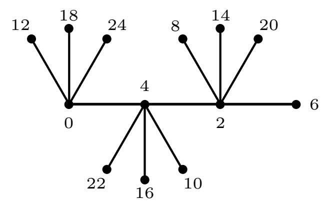

Figure 9

Theorem 2.9. The graph $C_n@K_1(n$ is even) is an even harmonious graph.

Proof: Let $G(V, E)$ be an one point union of the graphs $C_n(n$ is even) and K_1 (Figure 10), that is $C_n@K_1(n$ is even) . Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of C_n and v_n be the vertex of K_1, which is adjacent with the vertex v_0.

Define the mapping f from the vertex set $V(G)$ to the integer set $\{0, 1, 2, \ldots, 2q\}$ as follows. When $n = 4$, $f(v_i) = 2i$ for $0 \leq i \leq 2$, $f(v_3) = 10$ and $f(v_4) = 8$, when $n = 6$, $f(v_0) = 14$, $f(v_1) = 0$, $f(v_5) = 12$, $f(v_6) = 8$ and $f(v_i) = f(v_{i-1}) + 2$ for $\frac{n-2}{2} \leq i \leq n-2$. When $n > 6$, $f(v_0) = n + 8$, $f(v_n) = n + 2$, $f(v_{n-1}) = n + 6$, $f(v_{\frac{n-4}{2}}) = 0$, $f(v_i) = f(v_{i-1}) + 2$ for $1 \leq i \leq \frac{n-6}{2}$ and $\frac{n-2}{2} \leq i \leq n-2$. Therefore f is an injection mapping. This implies that f induces the bijective

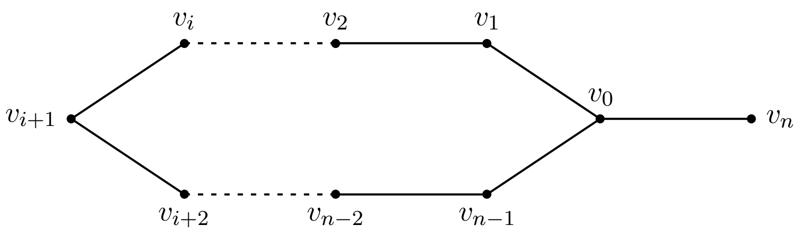

Figure 10: The graph $C_n@K_1$.

function f^* from $E(G)$ to $\{0, 2, 4, \ldots, 2n\}$. From the foregoing discussion, we obtain that the graph G is an even harmonious graph. ∎

Example 2.10. An even harmonious labeling for the graph $C_{12}@K_1$ is given in Figure 11.

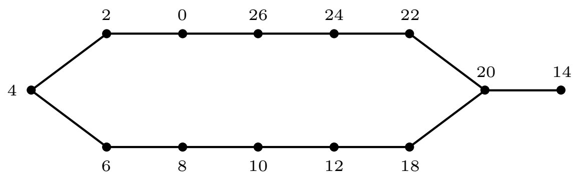

Figure 11: The graph $C_{12}@K_1$.

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References

[1] J.A. Bondy and U.S.R. Murthy, *Graph Theory with Applications*, Macmillan, London (1976).

[2] J.A. Gallian, *A dynamic survey of graph labeling*, The electronics J.of Combinatorics, 16 (2012).

[3] R.L. Graham, and N.J.A. Sloane, *On additive bases and harmonious graphs*, SIAM J.Algebr.Discrete Methods 1, (1980), 382-404.

[4] Liang Z-H. and Bai Z-L, *On the odd harmonious graphs with applications*, J.Appl.Math.Comput.,(2009) 29 105-116.

[5] P.B. Sarasija, R. Binthiya, *Even Harmonious Graphs with Applications*, (IJCSIS) International Journal of Computer Science and Information Security,Vol. 9, No.7(2011), 161-163.

[6] S.K. Vaidya, N. H. Shah, *Some new odd harmonious graphs*, International Journal of Mathematics and Soft Computing Vol.1, No.1 (2011), 9 - 16.