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Topological properties of Sierpiński Gasket Rhombus graphs

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Abstract

In this paper, we propose a new graph namely *Sierpiński Gasket Rhombus* SR_n based on triangular graph like *Sierpiński Gasket graph*. *Sierpiński Gasket Rhombus* SR_n is obtained by identifying two copies of *Sierpiński Gasket graph* along their side edges. Further we study some basic properties of *Sierpiński Gasket Rhombus* SR_n and find hamiltonicity, pancyclicity and chromatic number of SR_n.

Keywords: Sierpiński Gasket Rhombus graph, chromatic number, hamiltonicity, pancyclicity.
AMS Subject Classification(2010): 05C15, 05C45.

1 Introduction

Sierpiński Gasket graphs are strongly related to the well known fractal called the *Sierpiński Gasket*. *Sierpiński Gasket graphs* appear in different areas of graph theory, topology, probability [2, 4], pscychology [5]. *Sierpiński Gasket graphs* S_n are the graphs naturally defined by the finite number of iterations. Grundy, Scorer and Smith introduced this connection in [10]. Teguia and Godbole [11] studied several properties of these graphs, in particular the chromatic number, the domination number and the pebbling number. S_n is uniquely $3-$colorable and the chromatic index of S_n is 4 [9]. S_n contains $1-$perfect code only for $n = 1$ and $n = 3$. The distribution of Euclidean and geodesic distances on the Sierpiński Gasket were studied by Band and Mubarak [1]. Total chromatic number of S_n is studied in [3]. The graph presented in this paper are stimulated from the fractal structure given by Marcelo Epstein and Samer M. Adeeb [6]. The graph is obtained by identifying two copies of *Sierpiński Gasket graphs* along their side edges leading to Rhombus like structure. Hamiltonicity is one of the most important, areas of graph theory. Several papers have been published seeking more sufficient conditions for a graph to contain a Hamilton cycle. In this paper, we establish some basic properties, chromatic number, quotient labeling, hamiltonicity and pancyclicity of *Sierpiński Gasket Rhombus* SR_n.

Definition 1.1. A Path $-$ Sum is a way of combining two graphs by gluing them together along a path. If two graphs G and H, each contains a path of equal length, the Path $-$ Sum of G and H is formed from their disjoint union by identifying pair of vertices in these two paths to form a single shared path.

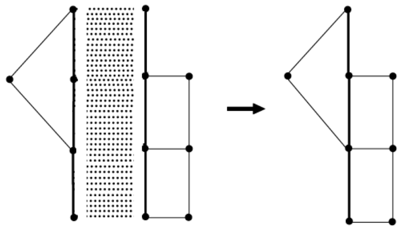


Figure 1: Path $-$ Sum of two graphs.

The graphs are formed by taking the Path $-$ Sum of two triangular graphs, where the path along one of their sides is considered.

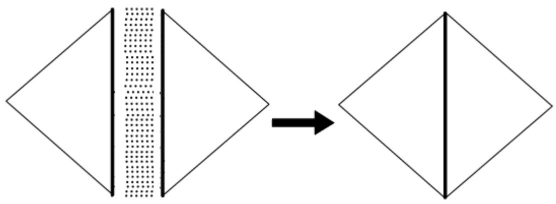


Figure 2: Path $-$ Sum of two triangular graphs.

Definition 1.2. A *Sierpiński Gasket Rhombus* of level n [denoted by SR_n], is obtained by identifying the edges in two *Sierpiński Gasket graphs* S_n along one of their sides.

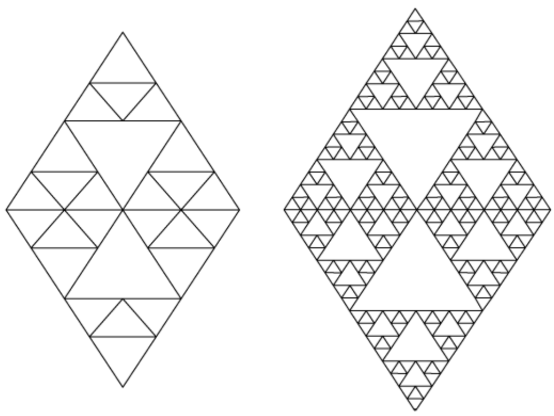


Figure 3: *Sierpiński Gasket Rhombus*, SR_3 and SR_5.

2 Sierpiński Gasket Rhombus SR_n

Sierpiński Gasket Rhombus SR_n consists of two copies of *Sierpiński Gasket* S_n. The four corner vertices of SR_n is denoted by $SR_{n,T,T}$, $SR_{n,R,R}$, $SR_{n,B,B}$ and $SR_{n,L,L}$. SR_n can be decomposed into four parts: The top *Sierpiński Gasket triangle* of level $n-1$ [denoted by $(S_{n,T})$], the left *Sierpiński*

Gasket Rhombus of level $n-1$ [denoted by ($SR_{n,L}$)], the right *Sierpiński Gasket Rhombus* of level $n-1$ [denoted by ($SR_{n,R}$)] and the bottom *Sierpiński Gasket triangle* of level $n-1$ [denoted by ($S_{n,B}$)].

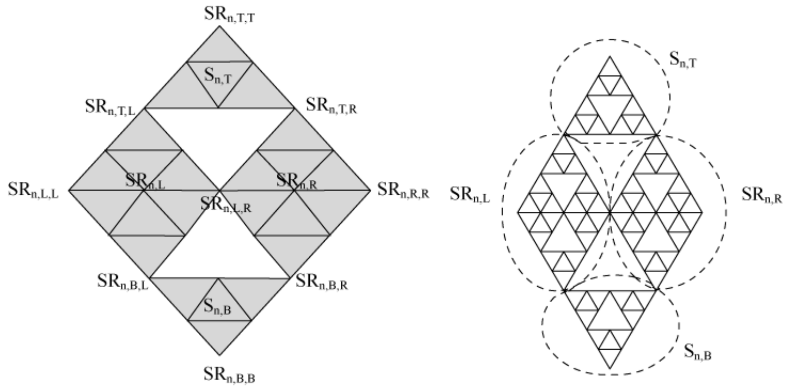


Figure 4: *Sierpiński Gasket Rhombus* SR_n.

Theorem 2.1. *Sierpiński Gasket Rhombus* SR_n has $3^n-2^{n-1}+2$ vertices and $2{\cdot}3^n-2^{n-1}$ edges.

Proof: To construct, SR_{n+1} from SR_n, we add $2{\cdot}3^n$ points and eliminate 2^{n-1} points.

$$\begin{aligned}|V_{n+1}| &= |V_n| + 2.3^n - 2^{n-1}\\ &= |V_1| + 2\sum 3^i - \sum 2^j\\ &= 4 + 2[\tfrac{3}{2}(3^n-1)] - (2^n-1)\\ &= 3^{n+1} - 2^n + 2\end{aligned}$$

as asserted. The number of edges in SR_n may now be easily determined using the fact that the sum of the vertex degrees equals twice the number of edges:

$$\begin{aligned}|E_n| &= \tfrac{1}{2}\sum \deg(v_j)\\ &= \tfrac{1}{2}[2{\cdot}2 + 3{\cdot}2 + 6(2^{n-1}-1) + 4(3^n-2^n-1)]\\ &= 2{\cdot}3^n - 2^{n-1}\end{aligned}$$

completing the proof. ∎

Properties:

From the construction of SR_n, we observe the following,

1. The vertex connectivity is 2 and the edge connectivity is also 2.
2. SR_n is not Eulerian, since it has odd degree vertices.
3. Edge disjoint cycle cover does not exist for SR_n, since it has odd degree.

In order to improve or increase the efficiency of message transmission we need to minimize diameter of the graph. The distance between two vertices x, y in a connected graph G is the length of the shortest path $(x, y)-$ path and is denoted by $d(x, y)$. The diameter $d(G)$ of a connected graph G is defined as *max* {*d(x,y):x,y* ∈ *V(G)*} [7].

Theorem 2.2. The diameter of *Sierpiński Gasket Rhombus* SR_n is 2^n, for $n\geq 1$.

Quotient labeling of SR_n is defined as the graph with four special vertices (1. . .1),(2. . .2), (3. . .3) and (4. . .4) called the extreme vertices of SR_n, together with vertices of the form $(u_0, u_1,\ldots, u_r)\{i, j\}$, $1\leq r\leq n-2$, $i < j$ where all $u_k' s$, i and j are from $\{1, 2, 3, 4\}$ and $(u_0,u_1,\ldots,u_r)$ is called the prefix of $(u_0,\ldots,u_r)\{i, j\}$. (See Figure 5)

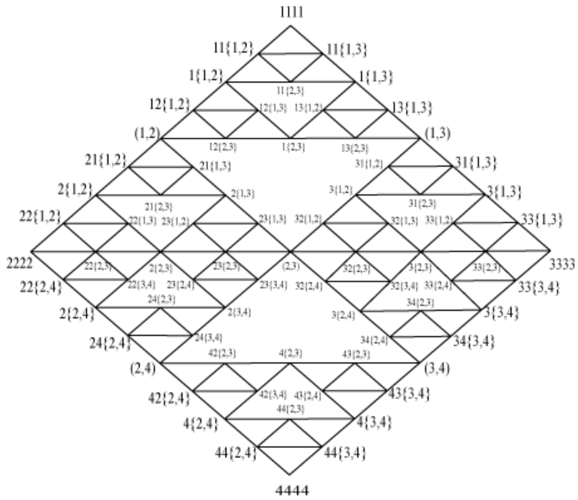


Figure 5: Quotient Labeling of SR_4.

A $k-$vertex coloring of G is an assignment of k colors to the vertices of G. The coloring is proper if no two distinct adjacent vertices have the same color. G is $k-$vertex colorable if G has a proper $k-$vertex coloring abbreviated as $k-$colorable. The chromatic number, $\chi(G)$ of G is the minimum k for which G is $k-$colorable. If $\chi(G) = k$, then G is said to be $k-$chromatic.

Theorem 2.3. The chromatic number of SR_n is 3, for all $n\geq 1$.

Proof: Clearly, $\chi(SR_1)$ = 3. Suppose, $\chi(SR_{n-1})$ = 3. The upper half of SR_n can be colored with exactly three colors. The remaining vertices in lower half of SR_n, are colored as follows: The vertex (4. . .4) will have the same color as (1...1). Other vertices of the lower half of SR_n will be of the form, $(u_0,u_1,\ldots,u_r)$ $\{i, j\}$, $0\leq r\leq n-2$, $i < j$, where

$(u_0,u_1,\ldots,u_r,i,j) \in \{2, 3, 4\}$ $\cdots$ (i)

Replacing the digit ”4” by ”1” we get, $(u_0,u_1,\ldots,u_r)$ $\{i, j\}$, $0\leq r\leq n-2$, $i < j$ where

$(u_0,u_1,\ldots,u_r,i,j) \in \{1, 2, 3\}$ $\cdots$ (ii)

Now the remaining vertices in (i) should take up the same color as in (ii). For instance, replace 4 by 1 in vertex $4\{4, 2\}$ we get the vertex $1\{1, 2\}$.
Therefore, $4\{4, 2\}$ will have the same color as $1\{1, 2\}$. ∎

A path that contains every vertex of G is called a hamiltonian path. A cycle that contains every vertex of G is called a Hamiltonian cycle. G is said to be Hamiltonian if it contains a Hamiltonian cycle. G is Hamiltonian $-$ connected, if a Hamiltonian path exists between every pair of vertices in G [8].

Lemma 2.4. SR_n has two Hamiltonian paths between the top most vertex $SR_{n,T,T}$ to the bottom most vertex $SR_{n,B,B}$. Also, there exists Hamiltonian paths from the top/bottom most vertex $SR_{n,T,T}/SR_{n,B,B}$ to the left/right most vertex $SR_{n,L,L}/SR_{n,R,R}$.

Proof: The lemma is true for SR_1 in both the cases (that is, from $SR_{n,T,T}$ to $SR_{n,B,B}$ and from $SR_{n,T,T}/SR_{n,B,B}$ to $SR_{n,L,L}/SR_{n,R,R}$).

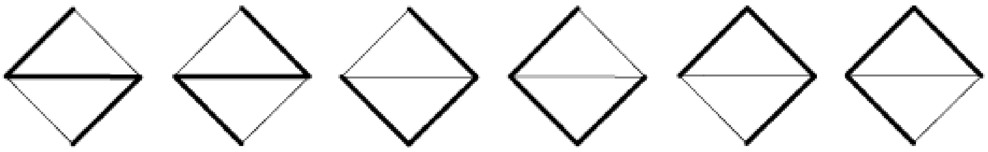


Figure 6: Hamiltonian paths in *Sierpiński Gasket Rhombus*, SR_1.

Suppose it is true for SR_n. Consider SR_{n+1}, which consists of $S_{n,T}$, $SR_{n,L}$, $SR_{n,R}$ and $S_{n,B}$ as mentioned in Section 2.
Case (i): Consider a Hamiltonian path of $S_{n,T}$ moving from the top vertex of $S_{n,T}$ to the top vertex of $SR_{n,L}$. Followed by the Hamiltonian path (guaranteed by the induction hypothesis) we move from the top vertex of $SR_{n,L}$ to the right vertex $SR_{n,L,R}$. Followed by the Hamiltonian path (guaranteed by the induction hypothesis) we move from that vertex to the bottom vertex $SR_{n,B,R}$ of $SR_{n,R}$, but with a critical modification, by avoiding the vertex $SR_{n,T,R}$ of $SR_{n,R}$. Finally, we consider the hamiltonian path of $S_{n,B}$ starting at its right vertex $SR_{n,B,R}$ and ending at its bottom vertex $SR_{n,B,B}$ (with a critical modification, by avoiding the left vertex $SR_{n,B,L}$) of $S_{n,B}$. Thus, we have constructed a Hamiltonian path of SR_{n+1} from its top most vertex $SR_{n,T,T}$ to its bottom most vertex $SR_{n,B,B}$.

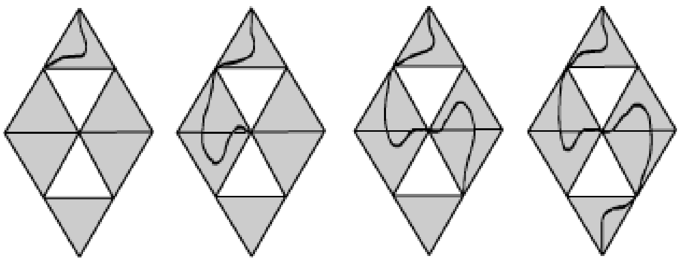


Figure 7: Hamiltonian path from $SR_{n,T,T}$ to $SR_{n,B,B}$.

By considering the vertical reflection of the path obtained in Case (i), we get another Hamiltonian path from the top vertex of SR_{n+1} to the bottom vertex.
Case (ii): Consider a Hamiltonian path of $S_{n,T}$ moving from the top vertex of $S_{n,T}$ to the top vertex of $SR_{n,L}$. Followed by the Hamiltonian path (guaranteed by the induction hypothesis) we move from that vertex to the bottom vertex of $SR_{n,L}$. Followed by the Hamiltonian path (guaranteed by the induction hypothesis) we move from the vertex $SR_{n,B,L}$ to the right vertex $SR_{n,B,R}$ of $S_{n,B}$. Finally, we take the hamiltonian path of $SR_{n,R}$ starting at the vertex $SR_{n,B,R}$ and ending at the vertex $SR_{n,R,R}$ but with

a critical modification, by avoiding the vertices $SR_{n,L,R}$ and $SR_{n,T,R}$ of $SR_{n,R}$. In this method, we have constructed a Hamiltonian path of SR_{n+1} from its top most vertex $SR_{n,T,T}$ to its right most vertex $SR_{n,R,R}$.

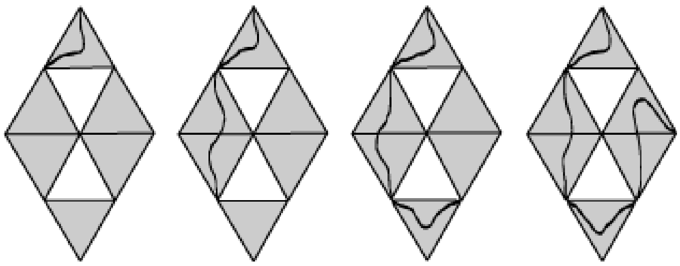
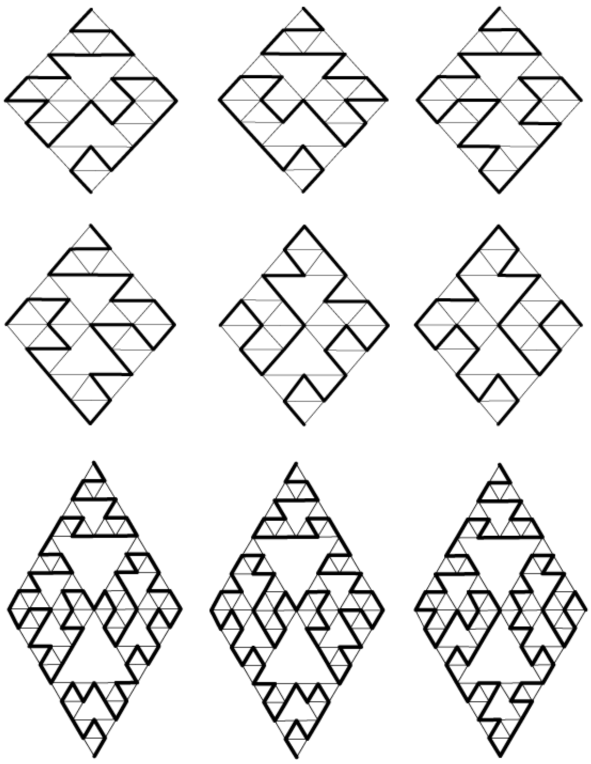


Figure 8: Hamiltonian path from $SR_{n,T,T}$ to $SR_{n,R,R}$.

By similar argument, we can obtain the Hamiltonian paths

- from $SR_{n,T,T}$ to $SR_{n,L,L}$,
- from $SR_{n,L,L}$ to $SR_{n,B,B}$ and
- from $SR_{n,R,R}$ to $SR_{n,B,B}$. ∎



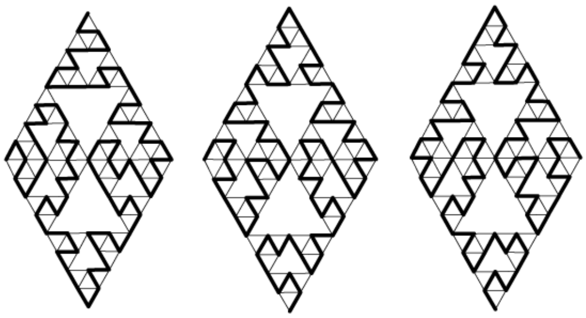


Figure 9: Hamiltonian paths in *Sierpiński Gasket Rhombus*, SR_3 and SR_4.

Theorem 2.5. SR_n is Hamiltonian for each n.

Proof: We take the Hamiltonian path of $S_{n+1,T}$ that moves from $SR_{n+1,T,L}$ to $SR_{n+1,T,R}$. Next, we take the Hamiltonian path of $SR_{n+1,R}$ that moves from $SR_{n+1,T,R}$ to $SR_{n+1,B,R}$. Next, we take the Hamiltonian path of $S_{n+1,B}$ that moves from $SR_{n+1,B,R}$ to $SR_{n+1,B,L}$. And finally, we take the Hamiltonian path of $SR_{n+1,L}$ that moves from $SR_{n+1,B,L}$ to $SR_{n+1,T,L}$. This gives the Hamiltonian cycle for SR_n.

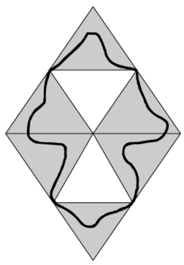


Figure 10: Hamiltonian cycle of SR_n.

∎

Lemma 2.6. [11] Each Hamiltonian path of S_n say, from $S_{n,L,L}$ to $S_{n,R,R}$, can be sequentially reduced in length by one at each step, while maintaining the starting and ending vertices, with the process ending in a path from $S_{n,L,L}$ to $S_{n,R,R}$ along the base of S_n.

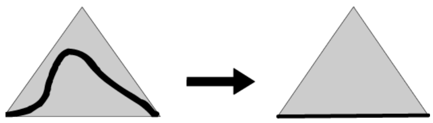


Figure 11: Hamiltonian path of S_1.

Lemma 2.7. The Hamiltonian path of SR_n as defined in Lemma 2.4, from $SR_{n,T,T}$ to $SR_{n,B,B}$ and $SR_{n,T,T}$ to $SR_{n,L,L}$ can be sequentially reduced in length by one at each step, while maintaining the starting and ending vertices, with the process ending in a path from $S_{n,T,T}$ to $S_{n,B,B}$ and $S_{n,T,T}$ to $S_{n,L,L}$ along the left side of SR_n.

Proof: We prove this lemma by induction. We note that the result is clearly true for $n = 1$. Assume that the result is true for SR_{n-1}. Now we prove the result for SR_n.
Case(i): There exists a hamiltonian path from $SR_{n,T,T}$ to $SR_{n,B,B}$ via $SR_{n,T,R}$, $SR_{n,L,R}$ and $SR_{n,B,L}$. The four hamiltonian paths in $S_{n,T}$, $SR_{n,R}$, $SR_{n,L}$ and $S_{n,B}$ are reduced to paths along their sides by using induction and Lemma 2.6, resulting to a path as in Figure 12(b). We modify the reduced path $SR_{n,T,T} \to SR_{n,T,R} \to SR_{n,L,R} \to SR_{n,B,L} \to SR_{n,B,B}$ as follows to achieve the further required reduction: (i) In Figure 12(b), $SR_{n,T,T} \to SR_{n,T,R} \to SR_{n,L,R}$ of length $2s$ (where s is the length of side in S_n) is replaced by the path, $SR_{n,T,T} \to SR_{n,T,L} \to SR_{n,L,R}$ is also of length $2s$ (See Figure 12(c)). (ii) In Figure 12(c), $SR_{n,T,L} \to SR_{n,L,R} \to SR_{n,B,L}$ of length $2s$ is replaced by the path $SR_{n,T,L} \to SR_{n,L,L} \to SR_{n,B,L}$ is also of length $2s$ (See Figure 12(d)). This yields the required path from $SR_{n,T,T}$ to $SR_{n,B,B}$ of length $4s$.

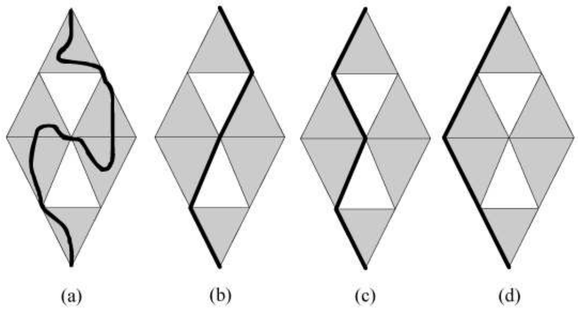


Figure 12

Case (ii): There exists a hamiltonian path from $SR_{n,T,T}$ to $SR_{n,L,L}$ via $SR_{n,T,R}$, $SR_{n,B,R}$ and $SR_{n,B,L}$. The four hamiltonian paths in $S_{n,T}$, $SR_{n,R}$, $SR_{n,L}$ and $S_{n,B}$ are reduced to paths along their sides by using induction and Lemma 2.6 resulting to a path as in Figure 13(b). We modify the reduced path $SR_{n,T,T} \to SR_{n,T,R} \to SR_{n,R,R} \to SR_{n,B,R} \to SR_{n,B,L} \to SR_{n,L,L}$ as follows to achieve the further required reduction: (i) In Figure 13(b), the path $SR_{n,R,R} \to SR_{n,B,R} \to SR_{n,B,L}$ of length $2s$ is replaced by the path $SR_{n,R,R} \to SR_{n,L,R} \to SR_{n,B,L}$ is also of length $2s$ (See Figure 13(c)). (ii) In Figure 13(c), the path $SR_{n,L,L} \to SR_{n,B,L} \to SR_{n,L,R}$ of length $2s$ is reduced using Lemma 2.6 to the path $SR_{n,L,L} \to SR_{n,L,R}$ of length $2s$ (See Figure 13(d)). Also the path $SR_{n,T,R} \to SR_{n,R,R} \to SR_{n,L,R}$ of length $2s$ is reduced using Lemma 2.6 to the path $SR_{n,T,R} \to SR_{n,L,R}$ of length s (See Figure 13(d)). (iii) In Figure 13(d), the path $SR_{n,T,R} \to SR_{n,L,R} \to SR_{n,L,L}$ of length $2s$ is replaced by the path $SR_{n,T,R} \to SR_{n,T,L} \to SR_{n,L,L}$ is also of length $2s$. (iv) In Figure 13(e), the path $SR_{n,T,R} \to SR_{n,T,L} \to SR_{n,L,L}$ of length $2s$ is reduced using Lemma 2.6 by the path $SR_{n,T,L} \to SR_{n,L,L}$ of length s. This yields the required path from $SR_{n,T,T}$ to $SR_{n,L,L}$ of length $2s$. ∎

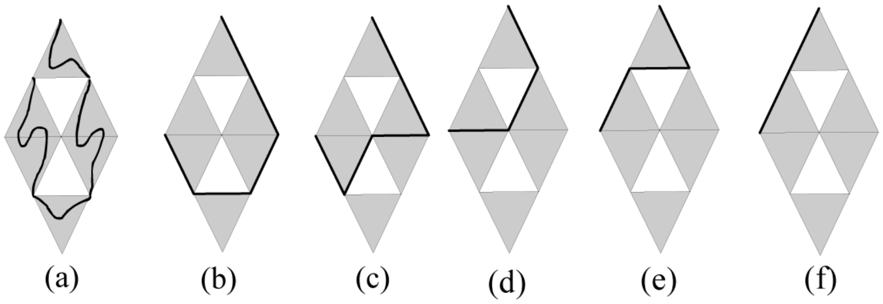


Figure 13

Theorem 2.8. SR_n is pancyclic for each n.

Proof: The proof is by induction. Assume that the result is true for all $m \leq n$. The Hamiltonian cycle $SR_{n,T,L} \rightarrow SR_{n,T,R} \rightarrow SR_{n,B,R} \rightarrow SR_{n,B,L} \rightarrow SR_{n,T,L}$ of SR_n consists of four Hamiltonian paths in $S_{n,T}$, $SR_{n,R}$, $S_{n,B}$ and $SR_{n,L}$ respectively. By Lemma 2.6, applied to $S_{n,T}$ and $S_{n,B}$; also by Lemma 2.7, applied to $SR_{n,R}$ and $SR_{n,L}$, we reduce these as necessary to get cycles of all sizes $\geq 6s$ [See Figure 14]. Cycles of smaller sizes are obtained by invoking the induction hypothesis on SR_{n-1}, noting that $|SR_{n-1}| \geq 6s$. ■

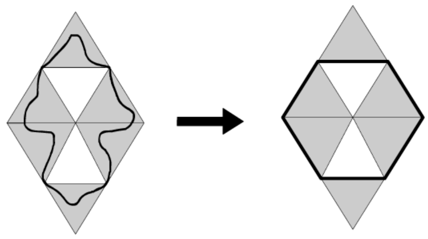


Figure 14

Conclusion

In this paper, we introduced *Sierpiński Gasket Rhombus graph* based on *Sierpiński Gasket graph*. The chromatic number, hamiltonicity and pancyclicity of SR_n are discussed. Further determining other graph theory problems of *Sierpiński Gasket Rhombus* SR_n are under investigation.

Acknowledgement

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