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# On Balance Index Set of Double graphs and Derived graphs

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**Abstract**

In this paper we obtain Balance Index Set of the double graphs and the derived graphs of path graphs, cycle graphs, wheel graphs, complete bipartite graphs, star graphs, double star, crown graphs, helm graphs and flower graphs.

**Keywords**: Friendly labeling, partial edge labeling, balance index set, double graphs and derived graphs.
**AMS Subject Classification(2010):** 05C78.

## 1 Introduction

We begin with simple, finite, connected and undirected graph $G$=$(V, E)$. Here elements of set $V$ and $E$ are known as vertices and edges respectively. For all other terminologies and notations we follow Harary [1].

**Definition 1.1.** Double graph of a connected graph $G$ is constructed by taking two copies of G say $G'$ and $G''$, join each vertex $u'$ in $G'$ to the neighbour of the corresponding $u''$ in $G''$. Double graph of $G$ is denoted by $D_2(G)$.

**Definition 1.2.** The derived graph of $G$, denoted by $G^\dagger$ is a graph with vertex set $V(G)$, in which two vertices are adjacent if and only if their distance in $G$ is two.

**Definition 1.3.** The crown $C_n \odot K_1$ is obtained by joining a pendant edge to each vertex of $C_n$.

**Definition 1.4.** The wheel $W_n$ is defined to be the join $K_1 + C_n$. The vertex corresponding to $K_1$ is known as apex vertex, the vertices corresponding to the cycle are known as rim vertices while the edges corresponding to the cycle are known as rim edges and the edges joining the apex and the vertices of cycle are known as spoke edges.

**Definition 1.5.** The helm $H_n$ is the graph obtained from a wheel $W_n$ by attaching a pendant edge to each rim vertex.

**Definition 1.6.** The flower $Fl_n$ is the graph obtained from a helm $H_n$ by joining each pendant vertex to the apex of the helm.

If $f$ is a binary vertex labeling of a graph $G$, Lee, Liu, and Tan [5] defined a partial edge labeling of the edges of G by

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0 \\ 1, & \text{if } f(u) = f(v) = 1 \end{cases}$$

Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled by 0 and 1 under the mapping $f$ respectively. Such a labeling $f$ is called friendly if $|v_f(0) - v_f(1)| \leq 1$. Likewise, let $e_{f^*}(0)$ and $e_{f^*}(1)$ denote the number of edges of $G$ that are labeled by 0 and 1 under the induced partial function $f^*$ respectively.

In [3] Kim, Lee, and Ng define the balance index set of a graph $G$ as
BI($G$) = $\{|e_{f^*}(0) - e_{f^*}(1)| : f^*$ runs over all friendly labelings $f$ of $G\}$.

For a graph with a vertex labeling $f$, we denote $e_{f^*}(X)$ to be the subset of $E(G)$ containing all the unlabeled edges. In [4] Kwong and Shiu developed an algebraic approach to attack the balance index set problems. It shows that the balance index set depends on the degree sequence of the graph.

**Lemma 1.7.** [6] For any graph $G$,

1. $2e_{f^*}(0) + e_{f^*}(X) = \sum_{v\epsilon v(0)} deg(v)$.
2. $2e_{f^*}(1) + e_{f^*}(X) = \sum_{v\epsilon v(1)} deg(v)$.
3. $2|E(G)| = \sum_{v\epsilon v(G)} deg(v) = \sum_{v\epsilon v(0)} deg(v) + \sum_{v\epsilon v(1)} deg(v)$.

**Corollary 1.8.** [6] For any friendly labeling $f$, the balance index is
$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right)$.

In number theory and combinatorics, a partition of a positive integer $n$, also called an integer partition, is a way of writing $n$ as a sum of positive integers. Two sums that differ only in the order of their summands are considered to be the same partition; if order matters then the sum becomes a composition. For example, 4 can be partitioned in five distinct ways: 4 + 0, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.

Let $G$ be any graph with $p$ vertices. Partition of $p$ in to ($p_0$, $p_1$), where $p_0$ and $p_1$ are the number of vertices labeled with 0 and 1 respectively.

In this paper we obtain Balance Index Set of the double graphs, derived graphs of path graphs, cycle graphs, wheel graphs, complete bipartite graphs, star graphs, double star, crown graphs, helm graphs and flower graphs.

To prove our results in this paper we use Lemma 1.7 and Corollary 1.8.

## 2 Main results

**Theorem 2.1.** Balance index set of the double graph of cycles of order $n$ is, BI($D_2(C_n)$) =$\{0\}$.

**Proof:** In double graph of $C_n$, $|V(D_2(C_n))|$ = 2n, $|E(D_2(C_n))| = 4n.$ It is a regular graph of order $2n$ with regularity 4. For friendly labeling, $n$ vertices are labeled 0 and remaining $n$ vertices are labeled 1. Then, by Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)|$=$\frac{1}{2}\,|4n - 4n| = 0.$ Therefore, BI($D_2(C_n)$) =$\{0\}$. ∎

**Example 2.2.** Balance index set of the double graph of cycle $C_5$ is $\{0\}$.

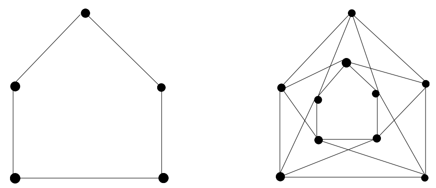

**Figure 1:** The cycle $C_5$ and its double graph $D_2(C_5)$.

**Theorem 2.3.** Balance index set of the double graph of path graphs of order $n$ is,

$$BI(D_2(P_n)) = \begin{cases} \{0\}, & \text{if } n = 2 \\ \{0, 2\}, & \text{if } n = 3 \\ \{0, 2, 4\}, & \text{if } n \geq 4 \end{cases}$$

**Proof:** In double graph of $P_n$, $|V(D_2(P_n))| = 2n$, $|E(D_2(P_n))| = 4n - 4$.

**Case 1:** If $n = 2$, then $D_2(P_2)$ is isomorphic to $C_4$.
Therefore, by Theorem 2.1, we get $BI(D_2(P_n)) = \{0\}$.

**Case 2:** If $n = 3$, then $D_2(P_3)$ is a biregular graph with four vertices of degree 2 and the remaining two vertices of degree 4. To satisfy friendly labeling we partition four vertices of degree 2 and two vertices of degree 4 into ($4 - i,\ i$) and ($i - 1,\ 3 - i$) respectively, where $i = 1, 2$.

By Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\,|2(4 - i) + 4(i - 1) - 2i - 4(3 - i)| = |2i - 4|,\ i = 1, 2.$

**Case 3:** If $n \geq 4$, then $D_2(P_n)$ is a biregular graph with 4 vertices are of degree 2 and the remaining $2n - 4$ vertices are of degree 4. To satisfy friendly labeling these vertices of degree 2 and vertices of degree 4 are partitioned into $(4 - i, i)$ and $(n - 4 + i, n - i)$ respectively, where $i$=0,1,2. Therefore, by Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)|$=$\frac{1}{2}\,|(2(4 - i) + 4(n - 4 + i)) - 2i - 4(n - i)|$=$|2i - 4|,\ i = 0, 1, 2.$
Considering all the possible friendly labeling, we get

$$BI(D_2(P_n)) = \begin{cases} \{0\}, & \text{if } n = 2 \\ \{0, 2\}, & \text{if } n = 3 \\ \{0, 2, 4\}, & \text{if } n \geq 4 \end{cases}$$

∎

**Example 2.4.** Balance index set of the double graph of path $P_4$ is $\{0, 2, 4\}$.

**Figure 2:** The path $P_4$ and its double graph $D_2(P_4)$.

**Theorem 2.5.** Balance index set of the double graph of wheel of order $n$ is, BI$(D_2(W_n))$=$\{0, 2n-6\}$.

**Proof:** In double graph of $W_n$, $|V(D_2(W_n))| = 2n+2$, $|E(D_2(W_n))| = 8n$, also it is a biregular graph with two apex vertices of degree $2n$ and the remaining $2n$ rim vertices with degree 6.

**Case 1:** If the two apex vertices are labeled with 0, then to satisfy friendly labeling, $n-1$ rim vertices are labeled with 0 and the $n+1$ rim vertices are labeled with 1.

Therefore, by Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)|=\frac{1}{2}\,|2(2n) + (n-1)6 - 6(n+1)| = |2n-6|$. Since $n \geq 3$, $|e_{f^*}(0) - e_{f^*}(1)| = 2n-6$.

**Case 2:** If one apex vertex is labeled with 0 and another apex vertex is labeled with 1, then to satisfy friendly labeling $n$ rim vertices are labeled 0 and remaining $n$ rim vertices are labeled 1.
Again by Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)|=\frac{1}{2}|(2n+6n) - (2n+6n)| = 0$.
Therefore considering all possible friendly labelings, we get BI$(D_2(W_n))$ =$\{0,\ 2n-6\}$. ■

**Example 2.6.** Balance index set of the double graph of wheel $W_3$ is $\{0\}$.

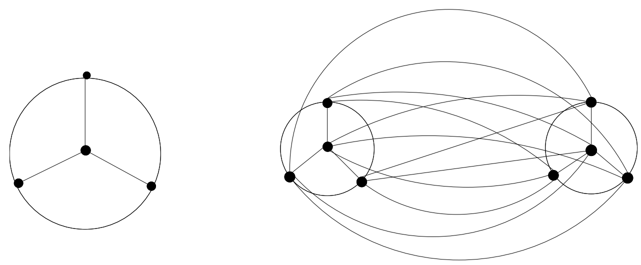

**Figure 3:** The wheel $W_3$ and its double graph $D_2(W_3)$.

**Theorem 2.7.** Balance index set of the double graph of complete bipartite graphs is,
BI$(D_2(K_{m,n})) = \{|2(n-i)(n-m)| : i = 0, 1, 2, \ldots, n\}$.

**Proof:** In double graph of $K_{m,n}$, $|V(D_2(K_{m,n}))| = 2(m+n)$ and $|E(D_2(K_{m,n}))| = 8mn$. Also the double graph of $K_{m,n}$ is a biregular graph with $2m$ vertices of degree $2n$ and the remaining $2n$ vertices of degree $2m$.

Without loss of generality, consider $m \geq n$. To satisfy friendly labeling we partition $2m$ and $2n$ into $(m+n-i,\ m-n+i)$ and $(i,\ 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n$.

Therefore, by Corollary 1.8,
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|2n(m+n-i) + 2mi - 2n(m-n+i) - 2m(2n-i)| = |2(n-i)(n-m)|$, where $i = 0, 1, 2, \ldots, n$.

Therefore, considering all possible friendly labeling, we get
BI$(D_2(K_{m,n})) = \{|2(n-i)(n-m)| : i = 0, 1, 2, \ldots, n\}$. ■

**Example 2.8.** Balance index set of the double graph of complete bipartite graph $K_{2,3}$ is $\{0, 2, 4\}$.

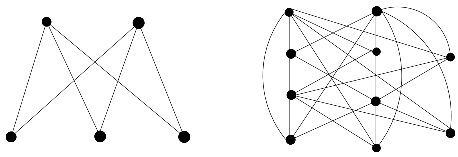

**Figure 4:** The complete bipartite graph $K_{2,3}$ and its double graph $D_2(K_{2,3})$.

**Corollary 2.9.** Balance index set of the double graph of star graph is, BI$(D_2(K_{m,1})) = \{0, 2m-2\}$.

**Proof:** The star graph is a special case of complete bipartite graph when $n$ = 1. Therefore by Theorem 2.7, we get BI$(D_2(K_{m,1})) = \{0, 2m-2\}$. ■

**Example 2.10.** Balance index set of the double graph of star $K_{1,3}$ is $\{0, 4\}$.

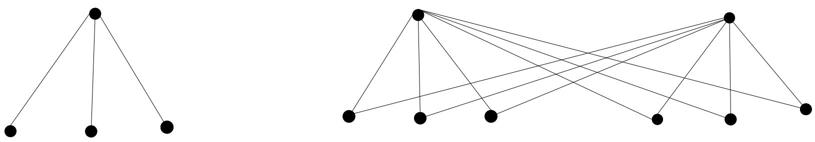

**Figure 5:** The star $K_{1,3}$ and its double graph $D_2(K_{1,3})$.

**Theorem 2.11.** Balance index set of the double graph of crown graph is,
BI$(D_2(C_n \odot K_1))$ = $\{4(n-i) : i = 0, 1, 2, \ldots, n\}$.

**Proof:** Double graph of crown graph is a biregular graph with $2n$ vertices of degree 6 and the remaining $2n$ vertices of degree 2. Also $|V(D_2(C_n \odot K_1))| = 4n$ and $|E(D_2(C_n \odot K_1))| = 8n$.
To satisfy friendly labeling $2n$ vertices of degree 6 and $2n$ vertices of degree 2 are partitioned into $(2n-i, i)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n$.
Then by Corollary 1.8, we get $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|6(2n-i) + 2i - 6i - 2(2n-i)| = 4(n-i)$, where $i = 0, 1, 2, \ldots, n$.
Therefore, BI$(D_2(C_n \odot K_1))$ = $\{4(n-i) : i = 0, 1, 2, \ldots, n\}$. ■

**Example 2.12.** Balance index set of the double graph of crown $C_3 \odot K_1$ is $\{0, 4, 8, 12\}$.

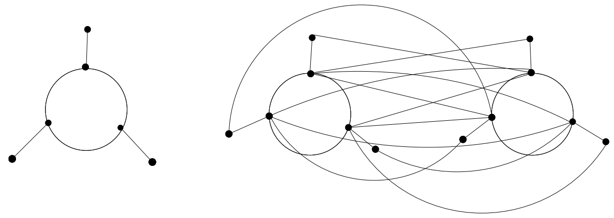

**Figure 6:** The crown $C_3 \odot K_1$ and its double graph $D_2(C_3 \odot K_1)$.

**Theorem 2.13.** Balance index set of the double graph of helm graph is,
$\mathrm{BI}(D_2(H_n)) = \{6(n-i) : i = 0, 1, 2, \ldots, n\} \bigcup$
$\{|8n - 6i - 8|, |4n - 6i + 2|, |4n - 6i - 8|, |8n - 6i + 2| : i = 0, 1, 2, \ldots, n-1\}$.

**Proof:** Double graph of helm graph contains two apex vertex of degree $2n$, $2n$ vertices of degree 8 and $2n$ vertices of degree 2. Also $|D_2(V(H_n))| = 4n + 2$ and $|D_2(E(H_n))| = 12n$.

**Case 1:** One apex vertex is labeled with 0 and the another with 1.
For friendly labeling, the remaining $2n$ vertices of degree 8 and $2n$ vertices of degree 2 are partitioned into $(2n-i, i)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n$. Therefore by Corollary 1.8, we get $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|8(2n-i) + 2i + 2n - 8i - 2(2n-i) - 2n| = 6(n-i)$, where $i = 0, 1, 2, \ldots, n$.

**Case 2:** Both the apex vertices are labeled 0.
**Sub Case 2.1:** If $2n$ vertices of degree 8 and $2n$ vertices of degree 2 are partitioned into $(2n-i-1, i+1)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n-1$. Then by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|8(2n - i - 1) + 2i + 2(2n) - 8(i + 1) - 2(2n - i)| = |8n - 6i - 8|$, where $i = 0, 1, 2, \ldots, n-1$.
**Sub Case 2.2:** If $2n$ vertices of degree 2 and $2n$ vertices of degree 8 are partitioned into $(2n-i-1, i+1)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n-1$.
Then by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\,|2(2n - i - 1) + 8i + 2(2n) - 2(i + 1) - 8(2n - i)|$
$= |-4n + 6i - 2| = |4n - 6i + 2|$, where $i = 0, 1, 2, \ldots, n-1$.

**Case 3:** Both the apex vertices are labeled 1.

**Sub Case 3.1:** If $2n$ vertices of degree 8 and $2n$ vertices of degree 2 are partitioned into $(2n-i-1, i+1)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n-1$. Then by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\,|8(2n - i - 1) + 2i - 8(i + 1) - 2(2n - i) - 2(2n)| = |4n - 6i - 8|$, where $i = 0, 1, 2, \ldots, n-1$.

**Sub case 3.2:** If $2n$ vertices of degree 2 and $2n$ vertices of degree 8 are partitioned into $(2n-i-1, i+1)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n-1$. Then by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\,|2(2n-i-1) + 8i - 2(i+1) - 8(2n-i) - 2(2n)|$
$= |-8n + 6i - 2| = |8n - 6i + 2|$, where $i = 0, 1, 2, \ldots, n-1$.
Therefore, considering all the possible friendly labeling, we get
BI$(D_2(H_n)) = \{6(n-i) : i = 0, 1, 2, \ldots, n\} \bigcup$
$\{|8n - 6i - 8|, |4n - 6i + 2|, |4n - 6i - 8|, |8n - 6i + 2| : i = 0, 1, 2, \ldots, n-1\}$. ■

**Example 2.14.** Balance index set of the double graph of $H_3$ is $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$.

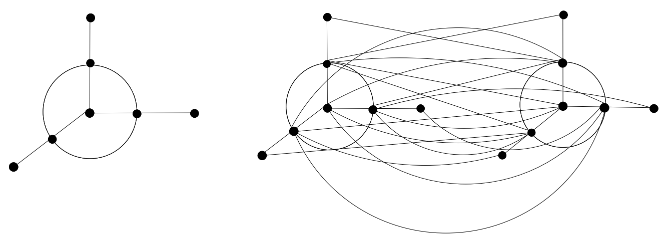

**Figure 7:** The helm $H_3$ and its double graph $D_2(H_3)$.

**Theorem 2.15.** Balance index set double graph of flower graph is,
BI$(D_2(Fl_n)) = \{4(n-i) : i = 0, 1, 2, \ldots, n\} \bigcup$
$\{|8n - 4i - 8|, |4(i-1)|, |4(i+2)|, |8n - 4i + 4| : i = 0, 1, 2, \ldots, n-1\}$.

**Proof:** Double graph of flower graph contains two apex vertices of degree $4n$, $2n$ vertices of degree 8 and $2n$ vertices of degree 4. Also $|D_2(V(Fl_n))| = 4n + 2$ and $|D_2(E(Fl_n))| = 16n$.

**Case 1:** If one apex vertex is labeled 0 and the another apex vertex is labeled 1, then for friendly labeling remaining $2n$ vertices of degree 8 and $2n$ vertices of degree 4 are partitioned into $(2n-i, i)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n$. Therefore by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|8(2n-i) + 4i + 4n - 8i - 4(2n-i) - 4n| = 4(n-i)$, where $i = 0, 1, 2, \ldots, n$.

**Case 2:** Both the apex vertices are labeled 0.

**Sub Case 2.1:** If $2n$ vertices of degree 8 and $2n$ vertices of degree 4 are partitioned into $(2n-i-1, i+1)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, 3, \ldots, n-1$. Then by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|8(2n-i-1) + 4i + 2(4n) - 8(i+1) - 4(2n-i)| = |8n - 4i - 8|$, where $i = 0, 1, 2, \ldots, n-1$.

**Sub Case 2.2:** If $2n$ vertices of degree 4 and $2n$ vertices of degree 8 are partitioned into $(2n-i-1, i+1)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, \ldots, n-1$. Then by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\,|4(2n-i-1) + 8i + 2(4n) - 4(i+1) - 8(2n-i)| = |4(i-1)|$, where $i = 0, 1, 2, \ldots, n-1$.

**Case 3:** Both the apex vertices are labeled 1.

**Sub case 3.1:** If $2n$ vertices of degree 8 and $2n$ vertices of degree 4 are partitioned into $(2n-i-1, i+1)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, 3, \ldots, n-1$. Then by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\,|8(2n-i-1) + 4i - 8(i+1) - 4(2n-i) - 2(4n)| = 4(i+2)$, where $i = 0, 1, 2, \ldots, n-1$.

**Sub case 3.2:** If $2n$ vertices of degree 4 and $2n$ vertices of degree 8 are partitioned into $(2n-i-1, i+1)$ and $(i, 2n-i)$ respectively, where $i = 0, 1, 2, 3, \ldots, n-1$. Then by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\,|4(2n-i-1) + 8i - 4(i+1) - 8(2n-i) - 2(4n)|$
$= |-8n + 4i - 4| = |8n - 4i + 4|$, where $i = 0, 1, 2, \ldots, n-1$.

Considering all possible friendly labeling, we get, BI($D_2(Fl_n)$)= $\{4(n-i) : i = 0, 1, 2, \ldots, n\} \bigcup \{|8n-4i-8|, |4(i-1)|, |4(i+2)|, |8n-4i+4| : i = 0, 1, 2, \ldots, n-1\}$. ■

**Example 2.16.** Balance index set of the double graph of flower $Fl_3$ is $\{0, 4, 8, 12, 16, 20, 24, 28\}$.

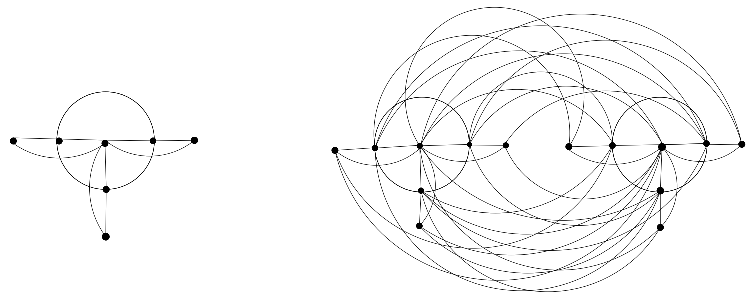

**Figure 8:** The flower $Fl_3$ and its double graph $D_2(Fl_3)$.

**Theorem 2.17.** Balance index set of the derived graph of complete bipartite graph is,

$$\text{BI}(K^{\dagger}_{m,n}) = \begin{cases} \left\{\frac{1}{2}|(m-2i)(m-n)| : i = 0, 1, 2, ..., \left\lfloor \frac{m}{2} \right\rfloor\right\}, & \text{if } m+n \text{ is even} \\ \left\{\frac{1}{2}|(m-2i)(m-n) \pm (n-1)| : i = 0, 1, 2, ..., m\right\}, & \text{if } m+n \text{ is odd} \end{cases}$$

**Proof:** The derived graph of $K^{\dagger}_{m,n}$ is a disjoint union of $K_m$ and $K_n$. Also $|V(K^{\dagger}_{m,n})| = m + n$ and $|E(K^{\dagger}_{m,n})| = \binom{m}{2} + \binom{n}{2}$.
Without loss of generality, let $m \leq n$.

**Case 1:** If $m+n$ is even then to satisfy friendly labeling, consider the partitions of $m$ and $n$ as $(m-i,\ i)$ and $\left(\frac{n-m}{2} + i,\ \frac{n+m}{2} - i\right)$ respectively, where $i = 0, 1, 2, ..., \left\lfloor \frac{m}{2} \right\rfloor$.

By Corollary 1.8,

$$|e_{f^*}(0) - e_{f^*}(1)| = \left|(m-i)(m-1) + \left(\frac{n-m}{2} + i\right)(n-1) - i(m-1) - \left(\frac{n+m}{2} - i\right)(n-1)\right|$$
$$= \frac{1}{2}|(m-2i)(m-n)|, \text{ where } i = 0, 1, 2, ..., \left\lfloor \frac{m}{2} \right\rfloor.$$

Therefore, if $m+n$ is even, then BI($K^{\dagger}_{m,n}$) $= \left\{\frac{1}{2}\,|(m-2i)(m-n)| : i = 0, 1, ..., \left\lfloor \frac{m}{2} \right\rfloor\right\}$.

**Case 2:** If $m+n$ is odd, then for friendly labeling two possibility arises.

**(a)** If $\frac{m+n-1}{2}$ vertices are labeled 0 and $\frac{m+n+1}{2}$ vertices are labeled 1, then the partitions of m and n are $(m-i, i)$ and $\left(\frac{n-m-1}{2} + i,\ \frac{n+m+1}{2} - i\right)$ respectively, where $i = 0, 1, 2, ..., m$.

By Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)|$=
$$\left|(m-i)(m-1) + \left(\frac{n-m-1}{2} + i\right)(n-1) - i(m-1) - \left(\frac{n+m+1}{2} - i\right)(n-1)\right|$$
=$\frac{1}{2}\,|(m-2i)(m-n) - (n-1)|$, where $i = 0, 1, 2, ..., m$.

**(b)** If $\frac{m+n+1}{2}$ vertices are labeled 0 and $\frac{m+n-1}{2}$ vertices are labeled 1, then the partitions of m and n are $(m-i,\ i)$ and $\left(\frac{n-m+1}{2} + i,\ \frac{n+m-1}{2} - i\right)$ respectively, where $i = 0, 1, 2, ..., m$.

By Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)|$=
$$\left|(m-i)(m-1) + \left(\frac{n+m-1}{2} + i\right)(n-1)) - i(m-1) - \left(\frac{n+m-1}{2} - i\right)(n-1)\right|$$
=$\frac{1}{2}|(m-2i)(m-n) + (n-1)|$, where $i = 0, 1, 2, ..., m$.

Therefore, if $m+n$ is odd, then BI($K^{\dagger}_{m,n}$) is $\left\{\frac{1}{2}\,|(m-2i)(m-n) \pm (n-1)| : i = 0, 1, 2, ..., m\right\}$. ∎

**Example 2.18.** Balance index set of the derived graph of $K_{2,3}$ is $\{0, 1, 2\}$.

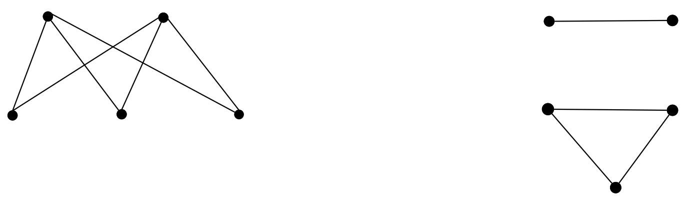

**Figure 9:** The complete bipartite graph $K_{2,3}$ and its derived graph $K^{\dagger}_{2,3}$.

**Corollary 2.19.** Balance index set of the derived graph of $K_{1,n}$ is,

$$\text{BI }(K_{1,n}^{\dagger}) = \begin{cases} \left\{\dfrac{n-1}{2}\right\}, & \text{if } n \text{ is odd} \\ \\ \{0,\ n-1\}\,, & \text{if } n \text{ is even} \end{cases}$$

**Proof:** The star graph is a special case of complete bipartite graph with $m$= 1.

**Case 1:** $n$ is odd.
Then by Theorem 2.17, $BI(K_{1,n}^{\dagger}) = \left\{\frac{1}{2}|(1-2i)(1-n)| : i = 0\right\}$ =$\left\{\frac{1}{2}(n-1)\right\}$

**Case 2:** $n$ is even.
Then by Theorem 2.17, $BI(K_{1,n}^{\dagger}) = \left\{\frac{1}{2}|(1-2i)(1-n) \pm (n-1)| : i = 0, 1\right\}$ = $\{n-1, 0\}$. ■

**Example 2.20.** Balance index set of the derived graph of $K_{1,4}$ is $\{0, 3\}$.

**Figure 10:** The star graph $K_{1,4}$ and its derived graph $K_{1,4}^{\dagger}$.

**Corollary 2.21.** Balance index set of the derived graph of double star $DS(m, n)$ is,

$$\text{BI}(DS(m,n)^{\dagger}) = \begin{cases} \left\{\dfrac{1}{2}|(m-2i+1)(m-n)| : i = 0, 1, 2, ..., \left\lfloor \dfrac{m+1}{2} \right\rfloor \right\}, & \text{if } m+n \text{ is even} \\ \\ \left\{\dfrac{1}{2}|(m-2i+1)(m-n) \pm n| : i = 0, 1, 2, ..., m+1 \right\}, & \text{if } m+n \text{ is odd} \end{cases}$$

**Proof:** The derived graph of $DS(m, n)$ is $K_{m+1} \cup K_{n+1}$. Therefore, the proof follows from Theorem 2.17. ■

**Theorem 2.22.** Balance index set of the derived graph of $C_n$ is,

$$\text{BI}(C_n^{\dagger}) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

**Theorem 2.23.** Balance index set of the derived graph of $W_n$ is,

$$\text{BI}(W_n^{\dagger}) = \begin{cases} \{0\}\,, & \text{if } n \text{ is even} \\ \left\{\dfrac{1}{2}\,|n-3|\right\}, & \text{if } n \text{ is odd} \end{cases}$$

**Proof:** We have $W_n^{\dagger}$=$G' \cup K_1$, where $G'$ is a graph of order $n$ with regularity $n-3$.

**Case 1:** If $n$ is even, say $n = 2m$.
Then $G'$ contains $2m$ vertices of degree $2m-3$. For a friendly labeling, in $G'$ there are $m$ vertices are labeled 0 and remaining $m$ vertices are labeled 1.
Then by Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|m(2m-3) - m(2m-3)|$=0.

**Case 2:** If $n$ is odd, say $n = 2m+1$, then $G'$ contains $2m+1$ vertices of degree $2m-2$. For a friendly labeling, in $G'$ there are $m+1$ vertices labeled 0 and remaining $m$ vertices are labeled 1 or vice versa. Then again by Corollary 1.8, $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|(m+1)(2m-2) - m(2m-2)| = \frac{1}{2}\,|2m-2| = \frac{1}{2}\,|n-3|$. ∎

**Example 2.24.** Balance index set of the derived graph of $W_4$ is $\{0\}$.

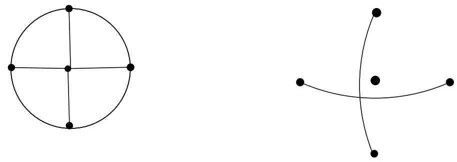

**Figure 11:** The wheel graph $W_4$ and its derived graph $W_4^{\dagger}$.

**Theorem 2.25.** Balance index set of the derived graph of crown graph is,

$$\mathrm{BI}(C_n \odot K_1)^{\dagger} = \begin{cases} \{0\}\,, & \text{if } n = 3 \\ \{0,1,2\}\,, & \text{if } n = 4 \\ \left\{n-2i : i = 0,1,2,\ldots \left\lfloor \frac{n}{2} \right\rfloor\right\}, & \text{if } n \geq 5 \end{cases}$$

**Proof:** Consider the derived graph of crown graph $C_n \odot K_1$.

**Case 1:** If $n$ = 3, then derived graph of $C_3 \odot K_1$ is isomorphic to $C_6$. Therefore proof follows from Theorem 2.22. **Case 2:** If $n = 4$, then

For a friendly labeling, four verices are labeled 0 and remaining four vertices are labeld 1. Therefore by Corollary 1.8, we get
$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|((n-i)(n-1)+2i) - (i(n-1)+2(n-i))|$
$= \frac{1}{2}|(n-2i)(n-3)|$, for $i$ = 0, 1, 2.

**Case 3:** If $n \geq 5$, then for a friendly labeling, $n$ vertices are labeled 0 and remaining $n$ vertices are labeled 1. By Corollary 1.8, we get $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}|4(n-i) + 2i - 4i - 2(n-i)| = n - 2i$, where $i = 0, 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor$. ∎

**Example 2.26.** Balance index set of the derived graph of $C_4 \odot K_1$ is $\{0, 1, 2\}$.

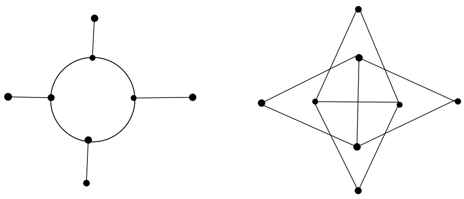

**Figure 12:** The crown graph $C_4 \odot K_1$ and its derived graph $(C_4 \odot K_1)^\dagger$.

**Theorem 2.27.** Balance index set of $H_n^\dagger$ is, BI$(H_n^\dagger)$ = $\left\{\frac{1}{2}\left|(n-2i)(n-4) \pm n\right| : i = 0, 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor\right\}$

**Proof:** Derived graph of helm contains apex vertex of degree $n$, $n$ vertices of degree 3 and the remaining $n$ vertices of degree $n-1$. Also $|V(H_n^\dagger)| = 2n+1$ and $|E(H_n^\dagger)| = \dfrac{n^2+3n}{2}$.

**Case 1:** Apex vertex is labeled with 0.

To satisfy friendly labeling, partition $n$ into $(n-i, i)$ and $(i, n-i)$, where $i = 0, 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

Therefore by Corollary 1.8, we get

$|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\left|n + (n-1)(n-i) + 3i - i(n-1) - 3(n-i)\right|$

$= \frac{1}{2}\left|(n-2i)(n-4) + n\right|$, where $i = 0, 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

**Case 2:** Apex vertex is labeled with 1.

To satisfy friendly labeling, partition $n$ into $(n-i, i)(i, n-i)$.

Therefore $|e_{f^*}(0) - e_{f^*}(1)| = \frac{1}{2}\left|(n-1)(n-i) + 3i - i(n-1) - 3(n-i) - n\right|$

$= \frac{1}{2}\left|(n-2i)(n-4) - n\right|$, where $i = 0, 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$. ∎

**Example 2.28.** Balance index set of the derived graph of $H_5$ is $\{0, 1, 2, 3, 4, 5\}$.

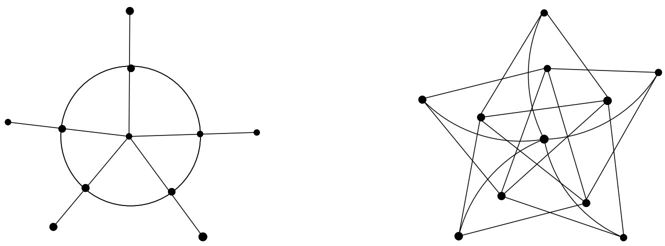

**Figure 13:** The helm graph $H_5$ and its derived graph $H_5^\dagger$.

**Theorem 2.29.** Balance index set of the derived graph of $Fl_n$ is,

BI$(Fl_n^\dagger)$ = $\left\{\frac{1}{2}\left|(n-2i)(n-3)\right| : i = 0, 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$

**Proof:** Derived graph of flower graph contains apex vertex of degree 0, $n$ vertices of degree two and remaining $n$ vertices of degree $n-1$. Also $|V(Fl_n^{\dagger})| = 2n+1$ and $|E(Fl_n^{\dagger})| = \dfrac{n^2+n}{2}$.
To satisfy friendly labeling, partition $n$ vertices of degree two and $n$ vertces of degree $n-1$ in to $(n-i, i)$ and $(i, n-i)$ respectively. Therefore by Corollary 1.8,
$|e_{f^*}(0) - e_{f^*}(1)| = \dfrac{1}{2}\,|(n-i)(n-1) + 2i - i(n-1) - 2(n-i)|$
$= \dfrac{1}{2}\,|(n-2i)(n-3)|$, where $i = 0, 1, 2, \ldots, \left\lfloor \dfrac{n}{2} \right\rfloor$. ∎

**Example 2.30.** Balance index set of the derived graph of $Fl_5$ is $\{1, 3, 5\}$.

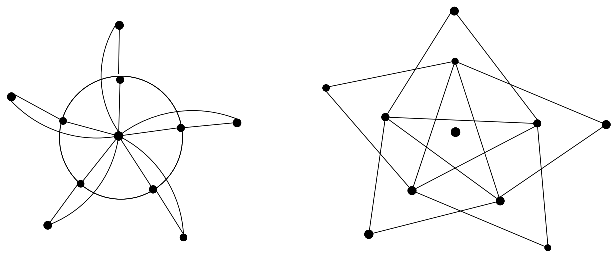

**Figure 14:** The flower graph $Fl_5$ and its derived graph $Fl_5^{\dagger}$.

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