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Stability of a Quartic and Additive functional equation in a 2-Banach space

B. M. Patel
Department of Mathematics
Government Science College
Gandginagar-382016, Gujarat, India.
E-mail: bhavinramani@yahoo.com

Abstract

In this paper, we investigate the generalized Hyers-Ulam stability of the functional equation $f(2x+y)+f(2x-y) = 4(f(x+y)+f(x-y)) - \frac{3}{7}(f(2y)-2f(y)) + 2f(2x) - 8f(x)$ in a 2-Banach space.

Keywords: Hyers-Ulam stability, 2-Banach space, quartic functional equation.
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1 Introduction

Stability of a functional equation for a function from a normed space X to a Banach space Y has been studied by Hyers [3]. He proved that for a function $f : X \longrightarrow Y$, satisfying $\|f(x+y)-f(x)-f(y)\| \leq \delta$ for each $x, y \in X$ and $\delta > 0$, there exists a unique additive function $T : X \longrightarrow Y$ such that $\|f(x) - T(x)\| \leq \delta$ for each $x \in X$. It is a positive answer to a problem raised by Ulam [8] for a functional equation on metric group. In fact several authors have studied the problem for different types of functional equations for functions from a normed space to a Banach space.

Our aim is to study the stability of the functional equation

$$\begin{aligned} f(2x+y)+f(2x-y) =& 4(f(x+y)+f(x-y)) \\ & - \frac{3}{7}(f(2y)-2f(y)) + 2f(2x) - 8f(x). \end{aligned} \tag{1.1}$$

introduced in [2] for a function on a 2-normed space (normed space) to a 2-Banach space. It is easy to see that the function $f(x) = ax^4 + bx$ is a solution of the functional equation (1.1). In the present paper, we investigate the generalized Hyers-Ulam stability of the functional equation (1.1) in a 2-Banach space. We use the following definitions and theorems in the subsequent sections.

Theorem 1.1. [2] Let X and Y be vector spaces, and let $f : X \longrightarrow Y$ be a function satisfying (1.1). Then

1. if f is an even function, then f is quartic.

2. if f is an odd function, then f is additive.

In the 1960s, S. Gähler [1] introduced the concept of a 2-normed space.

Definition 1.2. [1] Let X be a linear space over $\mathbb{R}$ with $\dim X > 1$ and let $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|ax, y\| = |a|\|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for each $x, y, z \in X$ and $a \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and $(X, \|\cdot,\cdot\|)$ is called a 2-normed space.

We introduce a basic property of a 2-normed space as follows. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, $x \in X$ and $\|x, y\|$ = 0 for each $y \in X$. Suppose $x \neq 0$. Since $\dim X > 1$, choose $y \in X$ such that $\{x, y\}$ is linearly independent so we have $\|x, y\| \neq 0$, which is a contradiction. Therefore, we have the following lemma.

Lemma 1.3. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. If $x \in X$ and $\|x, y\| = 0$, for each $y \in X$, then $x = 0$.

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. For $x, z \in X$, let $p_z(x) = \|x, z\|$, $x \in X$. Then for each $z \in X$, p_z is a real-valued function on X such that $p_z(x) = \|x, z\| \geq 0$, $p_z(\alpha x) = |\alpha|\|x, z\| = |\alpha| p_z(x)$ and $p_z(x + y) = \|x + y, z\| = \|z, x + y\| \leq \|z, x\| + \|z, y\| = \|x, z\| + \|y, z\| = p_z(x) + p_z(y)$, for each $\alpha \in \mathbb{R}$ and all $x, y \in X$. Thus p_z is a semi-norm for each $z \in X$.

For $x \in X$, let $\|x, z\| = 0$, for each $z \in X$. By Lemma 1.3, $x = 0$. Thus for $0 \neq x \in X$, there is $z \in X$ such that $p_z(x) = \|x, z\| \neq 0$. Hence the family $\{p_z(\cdot) : z \in X\}$ is a separating family of semi-norms.

Let $x_0 \in X$, for $\varepsilon > 0$, $z \in X$, let $U_{z,\varepsilon}(x_0) := \{x \in X : p_z(x - x_0) < \varepsilon\} = \{x \in X : \|x - x_0, z\| < \varepsilon\}$. Let $S(x_0) := \{U_{z,\varepsilon}(x_0) : \varepsilon > 0, z \in X\}$ and $\beta(x_0) := \{\cap\mathcal{F} : \mathcal{F}$ is a finite subcollection of $S(x_0)\}$. Define a topology τ on X by saying that a set U is open if for every $x \in U$, there is some $N \in \beta(x)$ such that $N \subset U$. That is, τ is the topology on X that has subbase $\{U_{z,\varepsilon}(x_0) : \varepsilon > 0, x_0 \in X, z \in X\}$. The topology τ on X makes X a topological vector space. Since for $x \in X$, the collection $\beta(x)$ is a local base whose members are convex, X is locally convex.

Definition 1.4. [9] A sequence $\{x_n\}$ in a 2-normed space X is called a 2-Cauchy sequence if

$$\lim_{m,n\to\infty} \|x_n - x_m, x\| = 0$$

for each $x \in X$.

Definition 1.5. A sequence $\{x_n\}$ in a 2-normed space X is called a 2-convergent sequence if there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for each $y \in X$. If $\{x_n\}$ converges to x, we write $\lim_{n\to\infty} x_n = x$.

Definition 1.6. We say that a 2-normed space $(X, \|\cdot,\cdot\|)$ is a 2-Banach space if every 2-Cauchy sequence in X is 2-convergent in X.

By using (2) and (4) of Definition 1.2 one can see that $\|\cdot,\cdot\|$ is continuous in each component. More precisely for a convergent sequence $\{x_n\}$ in a 2-normed space X, $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ for each $y \in X$.

2 Stability of a functional equation

Throughout this section, $(X, \|\cdot,\cdot\|)$ a 2-Banach space. For a function
$f : X \longrightarrow X$, define $D_f : X \times X \longrightarrow X$ by

$$\begin{aligned} D_f(x,y) &= 7[f(2x+y) + f(2x-y)] - 28[f(x+y) + f(x-y)] \\ &\quad + 3[f(2y) - 2f(y)] - 14[f(2x) - 4f(x)] \end{aligned}$$

for each $x, y \in X$.

Definition 2.1. The equation $f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y)$ is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

Definition 2.2. The equation $f(x+y) = f(x) + f(y)$ is called an additive functional equation and every solution of the additive functional equation is said to be an additive function.

Theorem 2.3. Let $\varepsilon \geq 0$, $0 < p, q < 4$. If $f : (X, \|\cdot,\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$ is an even function such that

$$\|D_f(x,y), z\| \leq \varepsilon(\|x,z\|^p + \|y,z\|^q) \tag{2.1}$$

for each $x, y, z \in X$. Then there exists a unique quartic function $Q : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - Q(x), z\| \leq \frac{\varepsilon\|x,z\|^q}{3(16 - 2^q)} \tag{2.2}$$

for each $x, z \in X$.

Proof: Let $x = y = 0$ in (2.1). We get, $\|3f(0), z\| = 0$ for each $z \in X$. Hence, $f(0) = 0$. Also substituting $x = 0$ in (2.1), we have

$$\|3f(2y) - 48f(y), z\| \leq \varepsilon\|y,z\|^q \tag{2.3}$$

for each $y, z \in X$. Replacing y by x in (2.3), we have

$$\left\|\frac{f(2x)}{16} - f(x), z\right\| \leq \frac{\varepsilon}{48}\|x, z\|^q \tag{2.4}$$

for each $x, z \in X$. Replacing x by $2x$ in (2.4), we get

$$\left\|\frac{f(4x)}{16} - f(2x), z\right\| \leq \frac{\varepsilon}{48}2^q\|x, z\|^q \tag{2.5}$$

for each $x, z \in X$. By (2.4) and (2.5), we have

$$\begin{aligned}\left\|\frac{f(4x)}{16^2} - f(x), z\right\| &\leq \left\|\frac{f(4x)}{16^2} - \frac{f(2x)}{16}, z\right\| + \left\|\frac{f(2x)}{16} - f(x), z\right\| \\ &\leq \frac{1}{16}\frac{\varepsilon}{48}2^q\|x, z\|^q + \frac{\varepsilon}{48}\|x, z\|^q \\ &= \frac{\varepsilon}{48}\|x, z\|^q\big[1 + \frac{2^q}{2^4}\big] \qquad (2.6)\end{aligned}$$

for each $x, z \in X$. Therefore by using induction on n, we get

$$\begin{aligned}\left\|\frac{f(2^n x)}{16^n} - f(x), z\right\| &\leq \frac{\varepsilon}{48}\|x, z\|^q \sum_{j=0}^{n-1} 2^{(q-4)j} \\ &= \frac{\varepsilon}{48}\|x, z\|^q \frac{1 - 2^{(q-4)n}}{1 - 2^{q-4}} \qquad (2.7)\end{aligned}$$

for each $x, z \in X$. For $m, n \in \mathbb{N}$, and $x \in X$

$$\begin{aligned}\left\|\frac{f(2^m x)}{16^m} - \frac{f(2^n x)}{16^n}, z\right\| &= \left\|\frac{f(2^{m+n-n} x)}{16^{m+n-n}} - \frac{f(2^n x)}{16^n}, z\right\| \\ &= \frac{1}{16^n}\left\|\frac{f(2^{m-n} \cdot 2^n x)}{16^{m-n}} - f(2^n x), z\right\| \\ &\leq \frac{\varepsilon}{48}\|2^n x, z\|^q \cdot \frac{1}{16^n} \sum_{j=0}^{m-n-1} 2^{(q-4)j} \\ &= \frac{\varepsilon}{48}\|x, z\|^q \sum_{j=0}^{m-n-1} 2^{(q-4)(n+j)} \\ &= \frac{\varepsilon}{48}\|x, z\|^q \, 2^{(q-4)n}\Big(\frac{1 - 2^{(q-4)(m-n)}}{1 - 2^{q-4}}\Big) \\ &\longrightarrow 0 \text{ as } n \to \infty\end{aligned}$$

for each $z \in X$. Therefore $\left\{\frac{f(2^n x)}{16^n}\right\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, define $Q : X \longrightarrow X$ as $Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{16^n}$ for each $x \in X$. By (2.7), we have $\|Q(x) - f(x), z\| \leq \frac{\varepsilon\|x,z\|^q}{3(16-2^q)}$ for each $x, z \in X$. Now we show that Q satisfies (1.1). That is $D_Q(x, y) = 0$.

$$\begin{aligned}
\|D_Q(x,y), z\| &= \lim_{n\to\infty} \frac{1}{16^n}\|D_f(2^n x, 2^n y), z\| \\
&\leq \lim_{n\to\infty} \frac{\varepsilon}{16^n}\big[\|2^n x, z\|^p + \|2^n y, z\|^q\big] \\
&= \lim_{n\to\infty} \varepsilon\big[2^{(p-4)n}\|x,z\|^p + 2^{(q-4)n}\|y,z\|^q\big] = 0
\end{aligned}$$

for each $x, y, z \in X$. Therefore, $\|D_Q(x, y), z\| = 0$. for each $z \in X$. So $D_Q(x, y) = 0$. Since f is an even function we have Q is an even function and hence by Theorem 1.1, Q is a quartic function.

Next we show that Q is unique. Let $Q' : X \longrightarrow X$ be another quartic function which satisfies (1.1) and (2.2). Since Q and Q' are quartic $Q(2^n x) = 16^n Q(x)$ and $Q'(2^n x) = 16^n Q'(x)$, for each $x \in X$. Then for $x \in X$

$$\begin{aligned}
\|Q'(x) - Q(x), z\| &= \frac{1}{16^n}\|Q'(2^n x) - Q(2^n x), z\| \\
&\leq \frac{1}{16^n}\big[\|Q'(2^n x) - f(2^n x), z\| + \|f(2^n x) - Q(2^n x), z\|\big] \\
&\leq \frac{2}{16^n}\frac{\varepsilon\|2^n x, z\|^q}{3(16-2^q)} \\
&= \frac{2\varepsilon\|x,z\|^q \cdot 2^{(q-4)n}}{3(16-2^q)} \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty
\end{aligned}$$

for each $z \in X$. Therefore for $x \in X$, $\|Q'(x) - Q(x), z\| = 0$ for each $z \in X$. Hence, $Q'(x) = Q(x)$, for each $x \in X$. ∎

Corollary 2.4. Let $\varepsilon \geq 0, 0 < p, q < 4, r > 0$, $(X, \|\cdot\|)$ be a real normed space. If $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$ is an even function such that

$$\|D_f(x,y), z\| \leq \varepsilon(\|x\|^p + \|y\|^q)\|z\|^r \tag{2.8}$$

for each $x, y, z \in X$. Then there exists a unique quartic function $Q : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - Q(x), z\| \leq \frac{\varepsilon\|x\|^q\|z\|^r}{3(16-2^q)} \tag{2.9}$$

for each $x, z \in X$.

Theorem 2.5. Let $\varepsilon \geq 0, p, q > 4$. If $f : X \longrightarrow X$ is an even function satisfying

$$\|D_f(x,y), z\| \leq \varepsilon\big(\|x,z\|^p + \|y,z\|^q\big) \tag{2.10}$$

for each $x, y, z \in X$. Then there exists a unique quartic function $Q : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - Q(x), z\| \leq \frac{\varepsilon\|x, z\|^q}{3(2^q - 16)} \tag{2.11}$$

for each $x, z \in X$.

Proof: Let $x = y = 0$ in (2.10). Then we get, $\|3f(0), z\| = 0$ for each $z \in X$. Hence, $f(0) = 0$. By (2.3) of Theorem 2.3, we have

$$\|3f(2y) - 48f(y), z\| \leq \varepsilon\|y, z\|^q \tag{2.12}$$

for each $y, z \in X$. Replacing y by $\frac{x}{2}$ in (2.12), we have

$$\left\|f(x) - 16f\left(\frac{x}{2}\right), z\right\| \leq \frac{\varepsilon}{3}2^{-q}\|x, z\|^q \tag{2.13}$$

for each $x, z \in X$. Replacing x by $\frac{x}{2}$ in (2.13), we have

$$\left\|f\left(\frac{x}{2}\right) - 16f\left(\frac{x}{2^2}\right), z\right\| \leq \frac{\varepsilon}{3}2^{-2q}\|x, z\|^q \tag{2.14}$$

for each $x, z \in X$. By (2.13) and (2.14), we obtain

$$\begin{aligned} \left\|16^2 f\left(\frac{x}{4}\right) - f(x), z\right\| &\leq \left\|16^2 f\left(\frac{x}{4}\right) - 16f\left(\frac{x}{2}\right), z\right\| + \left\|16f\left(\frac{x}{2}\right) - f(x), z\right\| \\ &\leq 16 \cdot \frac{\varepsilon}{3}2^{-2q}\|x, z\|^q + \frac{\varepsilon}{3}2^{-q}\|x, z\|^q \\ &= \frac{\varepsilon}{3}\|x, z\|^q\left[2^{-q} + 16 \cdot 2^{-2q}\right] \end{aligned} \tag{2.15}$$

for each $x, z \in X$. By using induction on n, we have

$$\begin{aligned} \left\|16^n f\left(\frac{x}{2^n}\right) - f(x), z\right\| &\leq \frac{\varepsilon}{3}\|x, z\|^q \sum_{j=0}^{n-1} 2^{(-q+j)} \cdot 16^j \\ &= \frac{\varepsilon}{3}\|x, z\|^q \sum_{j=0}^{n-1} 2^{(-q+4)j-q} \\ &= \frac{\varepsilon}{3}\|x, z\|^q \frac{2^{-q}\left(1 - 2^{(-q+4)n}\right)}{1 - 2^{-q+4}} \end{aligned} \tag{2.16}$$

for each $x, z \in X$. For $m, n \in \mathbb{N}$, for $x \in X$

$$\begin{aligned} \left\|16^m f\left(\frac{x}{2^m}\right) - 16^n f\left(\frac{x}{2^n}\right), z\right\| &= \left\|16^{m+n-n} f\left(\frac{x}{2^{m+n-n}}\right) - 16^n f\left(\frac{x}{2^n}\right), z\right\| \\ &= 16^n\left\|16^{m-n} f\left(\frac{x}{2^{m-n} \, . \, 2^n}\right) - f\left(\frac{x}{2^n}\right), z\right\| \end{aligned}$$

$$
\begin{aligned}
&\leq 16^n \frac{\varepsilon}{3}\left\|\frac{x}{2^n}, z\right\|^q \sum_{j=0}^{m-n-1} 2^{(-q+4)j-q} \\
&= \frac{\varepsilon}{3}\|x, z\|^q \sum_{j=0}^{m-n-1} 2^{(-q+4)(n+j)-q} \\
&= \frac{\varepsilon}{3}\|x, z\|^q \, \frac{2^{(-q+4)n}\left(1-2^{(-q+4)(m-n)}\right)}{1-2^{-q+4}} \\
&\longrightarrow 0 \text{ as } n \to \infty
\end{aligned}
$$

for each $z \in X$. Therefore $\left\{16^n f\left(\frac{x}{2^n}\right)\right\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space. Define $Q : X \longrightarrow X$ as $Q(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$ for each $x \in X$. By (2.16), we have $\|Q(x) - f(x), z\| \leq \frac{\varepsilon\|x,z\|^q}{3(2^q-16)}$ for each $x, z \in X$. ∎

Corollary 2.6. Let $\varepsilon \geq 0, r > 0$, $p, q > 4$, $(X, \|\cdot\|)$ be a real normed space. If $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$ is an even function such that

$$\|D_f(x, y), z\| \leq \varepsilon(\|x\|^p + \|y\|^q)\|z\|^r \tag{2.17}$$

for each $x, y, z \in X$. Then there exists a unique quartic function $Q : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - Q(x), z\| \leq \frac{\varepsilon\|x\|^q\|z\|^r}{3(2^q - 16)} \tag{2.18}$$

for each $x, z \in X$.

Theorem 2.7. Let $\varepsilon \geq 0, 0 < p, q < 1$, $f : (X, \|\cdot,\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$ be an odd function satisfying

$$\|D_f(x, y), z\| \leq \varepsilon\left(\|x, z\|^p + \|y, z\|^q\right) \tag{2.19}$$

for each $x, y, z \in X$. Then there exists a unique additive function $A : X \longrightarrow X$ satisfying (1.1) and the inequality

$$\|f(x) - A(x), z\| \leq \frac{\varepsilon\|x, z\|^q}{3(2 - 2^q)} \tag{2.20}$$

for each $x, z \in X$.

Proof: Let $x = 0$ in (2.19). Then we get

$$\|f(2y) - 2f(y), z\| \leq \frac{\varepsilon}{3}\|y, z\|^q \tag{2.21}$$

for each $y, z \in X$. Replacing y by x in (2.21), we get

$$\|f(2x) - 2f(x), z\| \leq \frac{\varepsilon}{3}\|x, z\|^q \tag{2.22}$$

for each $x, z \in X$. Therefore

$$\left\|\frac{f(2x)}{2} - f(x), z\right\| \leq \frac{\varepsilon}{6}\|x, z\|^q \tag{2.23}$$

for each $x, z \in X$. Replacing x by $2x$ in (2.23), we get

$$\left\|\frac{f(4x)}{2} - f(2x), z\right\| \leq \frac{\varepsilon}{6} \cdot 2^q\|x, z\|^q \tag{2.24}$$

for each $x, z \in X$. Now by (2.23) and (2.24), we have

$$\begin{aligned}\left\|\frac{f(4x)}{4} - f(x), z\right\| &\leq \left\|\frac{f(4x)}{4} - \frac{f(2x)}{2}, z\right\| + \left\|\frac{f(2x)}{2} - f(x), z\right\| \\ &\leq \frac{\varepsilon}{6}\|x, z\|^q[1 + \frac{2^q}{2}]\end{aligned}$$

for each $x, z \in X$. Therefore by using induction on n, we get

$$\begin{aligned}\left\|\frac{f(2^n x)}{2^n} - f(x), z\right\| &\leq \frac{\varepsilon}{6}\|x, z\|^q \sum_{j=0}^{n-1} \frac{2^{aj}}{2^j} \\ &= \frac{\varepsilon}{6}\|x, z\|^q \sum_{j=0}^{n-1} 2^{(q-1)j} \\ &= \frac{\varepsilon}{6}\|x, z\|^q \Big(\frac{1 - 2^{(q-1)n}}{1 - 2^{q-1}}\Big)\end{aligned} \tag{2.25}$$

for each $x, z \in X$. Now for $m, n \in \mathbb{N}$, for $x \in X$

$$\begin{aligned}\left\|\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, z\right\| &= \frac{1}{2^n}\left\|\frac{f(2^{m-n} \cdot 2^n x)}{2^{m-n}} - f(2^n x), z\right\| \\ &\leq \frac{1}{2^n}\frac{\varepsilon}{6}\|2^n x, z\|^q \sum_{j=o}^{m-n-1} 2^{(q-1)j} \\ &= \frac{\varepsilon}{6}\|x, z\|^q \sum_{j=0}^{m-n-1} 2^{(q-1)(n+j)} \\ &= \frac{\varepsilon}{6}\|x, z\|^q\, 2^{(q-1)n}\Big(\frac{1 - 2^{(q-1)(m-n)}}{1 - 2^q - 1}\Big) \\ &\longrightarrow 0 \text{ as } n \to \infty\end{aligned}$$

for each $z \in X$. Therefore $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space. $\left\{\frac{f(2^n x)}{2^n}\right\}$ 2-converges in X, for each $x \in X$. Define $A : X \longrightarrow X$ as

$$A(x) := \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n}$$

for each $x \in X$. By (2.25), we have

$$\|f(x) - A(x), z\| \leq \frac{\varepsilon \|x, z\|^q}{3(2 - 2^q)}$$

for each $x \in X$. The rest of the proof is similar to the proof of Theorem 2.3. ∎

Corollary 2.8. Let $\varepsilon \geq 0, r > 0, 0 < p, q < 1$, $(X, \|\cdot\|)$ be a real normed space. If $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, \cdot\|)$ is an odd function satisfying the inequality

$$\|D_f(x, y), z\| \leq \varepsilon(\|x\|^p + \|y\|^q)\|z\|^r \tag{2.26}$$

for each $x, y, z \in X$. Then there exists a unique additive function $A : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - A(x), z\| \leq \frac{\varepsilon \|x\|^q \|z\|^r}{3(2 - 2^q)}$$

for each $x, z \in X$.

Theorem 2.9. Let $\varepsilon \geq 0, p, q > 1$. $f : X \longrightarrow X$ be an odd function satisfying

$$\|D_f(x, y), z\| \leq \varepsilon(\|x, z\|^p + \|y, z\|^q) \tag{2.27}$$

for each $x, y, z \in X$. Then there exists a unique additive function $A : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - A(x), z\| \leq \frac{\varepsilon \|x, z\|^q}{3(2^q - 2)} \tag{2.28}$$

for each $x, z \in X$.

Proof: By (2.22) of Theorem 2.7, we have

$$\|f(2x) - 2f(x), z\| \leq \frac{\varepsilon}{3}\|x, z\|^q \tag{2.29}$$

for each $x, z \in X$. Replacing x by $\frac{x}{2}$ in (2.29), we get

$$\left\|f(x) - 2f\left(\frac{x}{2}\right), z\right\| \leq \frac{\varepsilon}{3} 2^{-q}\|x, z\|^q \tag{2.30}$$

for each $x, z \in X$. Replacing x by $\frac{x}{2}$ in (2.30), we get

$$\left\| f\left(\frac{x}{2}\right) - 2f\left(\frac{x}{4}\right), z \right\| \leq \frac{\varepsilon}{3} 2^{-2q} \|x, z\|^q \tag{2.31}$$

for each $x, z \in X$. Now by (2.30) and (2.31), we have

$$\begin{aligned}\left\| f(x) - 2^2 f\left(\frac{x}{2^2}\right), z \right\| &\leq \left\| f(x) - 2f\left(\frac{x}{2}\right), z \right\| + \left\| 2f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), z \right\| \\ &\leq \frac{\varepsilon}{3} 2^{-q} \|x, z\|^q + 2\frac{\varepsilon}{3} 2^{-2q} \|x, z\|^q \\ &= \frac{\varepsilon}{3} \|x, z\|^q [2^{-q} + 2 \cdot 2^{-2q}]\end{aligned}$$

for each $x, z \in X$. By using induction on n, we get

$$\begin{aligned}\left\| f(x) - 2^n f\left(\frac{x}{2^n}\right), z \right\| &\leq \frac{\varepsilon}{3} \|x, z\|^q \sum_{j=0}^{n-1} 2^{-q(j+1)} \cdot 2^j \\ &= \frac{\varepsilon}{3} \|x, z\|^q \sum_{j=0}^{n-1} 2^{(-q+1)j-q} \\ &= \frac{\varepsilon}{3} \|x, z\|^q \frac{2^{-q}\left(1 - 2^{(-q+1)n}\right)}{1 - 2^{(-q+1)}}\end{aligned} \tag{2.32}$$

for each $x, z \in X$. Now for $m, n \in \mathbb{N}$, for $x \in X$

$$\begin{aligned}\left\| 2^m f\left(\frac{x}{2^m}\right) - 2^n f\left(\frac{x}{2^n}\right), z \right\| &= \left\| 2^{m+n-n} f\left(\frac{x}{2^{m+n-n}}\right) - 2^n f\left(\frac{x}{2^n}\right), z \right\| \\ &= 2^n \left\| 2^{m-n} f\left(\frac{x}{2^{m-n} \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right), z \right\| \\ &\leq 2^n \cdot \frac{\varepsilon}{3} \left\| \frac{x}{2^n}, z \right\|^q \sum_{j=0}^{m-n-1} 2^{(-q+1)j-q} \\ &= \frac{\varepsilon}{3} \|x, z\|^q \sum_{j=0}^{m-n-1} 2^{(-q+1)(n+j)-q} \\ &= \frac{\varepsilon}{3} \|x, z\|^q \frac{2^{(-q+1)n-q}\left(1 - 2^{(-q+1)(m-n)}\right)}{1 - 2^{(} - q + 1)} \\ &\longrightarrow 0 \text{ as } n \to \infty\end{aligned}$$

for each $z \in X$. Therefore $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ 2-converges, for each $x \in X$. Define $A : X \longrightarrow X$ as

$$A(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for each $x, z \in X$. Now by (2.32), we have

$$\|f(x) - A(x), z\| \leq \frac{\varepsilon\|x, z\|^q}{3(2^q - 2)}$$

for each $x, z \in X$. The rest of the proof is similar to that of the proof of Theorem 2.3. ∎

Corollary 2.10. Let $\varepsilon \geq 0, r > 0$, $p, q > 1$, $(X, \|\cdot\|)$ be a real normed space. If $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, \cdot\|)$ is an odd function satisfying

$$\|D_f(x, y), z\| \leq \varepsilon\big(\|x\|^p + \|y\|^q\big)\|z\|^r \tag{2.33}$$

for each $x, y, z \in X$. Then there exists a unique additive function $A : X \longrightarrow X$ satisfying (1.1) and the inequality

$$\|f(x) - A(x), z\| \leq \frac{\varepsilon\|x\|^q\|z\|^r}{3(2^q - 2)} \tag{2.34}$$

for each $x, z \in X$.

Theorem 2.11. Let $\varepsilon \geq 0, 0 < p < 1$, $f : X \longrightarrow X$ be a function satisfying the inequality

$$\|D_f(x, y), z\| \leq \varepsilon(\|x, z\|^p + \|y, z\|^p) \tag{2.35}$$

for each $x, y, z \in X$. Then there exists a unique quartic function $Q : X \longrightarrow X$ and unique additive function $A : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - Q(x) - A(x), z\| \leq \frac{\varepsilon\|x, z\|^p}{3}\Big[\frac{1}{16 - 2^p} + \frac{1}{2 - 2^p}\Big] \tag{2.36}$$

Proof: Let $x = y = 0$ in (2.35), we have $\|3f(0), z\| = 0$ for each $z \in X$, so we have $f(0) = 0$. Define $f_e : X \longrightarrow X$, $f_o : X \longrightarrow X$ as
$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$, $\;f_o(x) = \frac{1}{2}[f(x) - f(-x)]$.
Then f_e is an even function and f_o is an odd function. Since $f(0) = 0$, we have $f_e(0) = 0 = f_o(0)$. Also

$$\|D_{f_e}(x, y), z\| \leq \varepsilon(\|x, z\|^p + \|y, z\|^p)$$
$$\|D_{f_o}(x, y), z\| \leq \varepsilon(\|x, z\|^p + \|y, z\|^p)$$

for each $x, y, z \in X$. Therefore by Theorem 2.3, there exists a unique quartic function $Q : X \longrightarrow X$

satisfying (1.1) and the inequality

$$\|f_e(x) - Q(x), z\| \le \frac{\varepsilon\|x,z\|^p}{3(16-2^p)} \tag{2.37}$$

for each $x, z \in X$. Also by Theorem 2.7, there exists a unique additive function $A : X \longrightarrow X$ satisfying (1.1) and the inequality

$$\|f_o(x) - A(x), z\| \le \frac{\varepsilon\|x,z\|^p}{3(2-2^p)} \tag{2.38}$$

for each $x, z \in X$. Now by (2.37) and (2.38), we have

$$\begin{aligned}\|f(x) - Q(x) - A(x), z\| &\le \|f_e(x) - Q(x), z\| + \|f_o(x) - A(x), z\| \\ &\le \frac{\varepsilon\|x,z\|^p}{3(16-2^p)} + \frac{\varepsilon\|x,z\|^p}{3(2-2^p)} \\ &\le \frac{\varepsilon\|x,z\|^p}{3}\Big[\frac{1}{16-2^p} + \frac{1}{2-2^p}\Big]\end{aligned}$$

for each $x, z \in X$. So we have

$$\|f(x) - Q(x) - A(x), z\| \le \frac{\varepsilon\|x,z\|^p}{3}\Big[\frac{1}{16-2^p} + \frac{1}{2-2^p}\Big]$$

for each $x, z \in X$. ∎

Corollary 2.12. Let $\varepsilon \ge 0, r > 0, 0 < p < 1$, $(X, \|\cdot\|)$ be a real normed space. Suppose $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$ be a function satisfying the inequality

$$\|D_f(x, y), z\| \le \varepsilon(\|x\|^p + \|y\|^p)\|z\|^r \tag{2.39}$$

for each $x, y, z \in X$. Then there exists a unique quartic function $Q : X \longrightarrow X$ and a unique additive function $A : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - Q(x) - A(x), z\| \le \frac{\varepsilon}{3}\Big[\frac{1}{16-2^p} + \frac{1}{2-2^p}\Big]\|x\|^p\|z\|^q$$

for each $x, z \in X$.

Theorem 2.13. Let $\varepsilon \ge 0, p > 4$. Let $f : X \longrightarrow X$ be a function satisfying the inequality

$$\|D_f(x, y), z\| \le \varepsilon(\|x,z\|^p + \|y,z\|^p)$$

for each $x, y, z \in X$. Then there exists a unique quartic function $Q : X \longrightarrow X$ and an additive function

$A : X \longrightarrow X$ satisfying (1.1) and

$$\|f(x) - Q(x) - A(x), z\| \leq \frac{\varepsilon\|x, z\|^p}{3}\Big[\frac{1}{2^p - 16} + \frac{1}{2^p - 2}\Big]$$

for each $x, z \in X$.

Proof: Proof is similar to the proof of Theorem 2.11. ∎

Corollary 2.14. Let $\varepsilon \geq 0, p > 4$, $(X, \|\cdot\|)$ be a real normed space. Suppose $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, \cdot\|)$ be a function satisfying the inequality $\|D_f(x, y), z\| \leq \varepsilon(\|x\|^p + \|y\|^q)\|z\|^r$ for each $x, y, z \in X$. Then there exists a unique quartic function $Q : X \longrightarrow X$ and an additive function $A : X \longrightarrow X$ satisfying (1.1) and the inequality

$$\|f(x) - Q(x) - A(x), z\| \leq \frac{\varepsilon}{3}\|x\|^q\|z\|^q\Big[\frac{1}{2^p - 16} + \frac{1}{2^p - 2}\Big]$$

for each $x, z \in X$.

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