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Neighborhood Connected Domination and Colouring in Graphs

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Abstract

A dominating set S of a connected graph $G = (V, E)$ is a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S)\rangle$ of G is connected. The neighborhood connected domination number $\gamma_{nc}(G)$ is the minimum cardinality of a ncd-set. The minimum number of colours required to colour all the vertices such that no two adjacent vertices have same colour is the chromatic number $\chi(G)$ of G. In this paper we find an upper bound for sum of the ncd-number and chromatic number and characterize the corresponding extremal graphs.

Keywords: Neighborhood connected domination, chromatic number.
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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = N(S) \cup S$.

A subset S of V is called a dominating set of G if $N[S] = V$. The minimum cardinality of a dominating set is called a domination number of G and is denoted by $\gamma(G)$. S. Arumugam and C. Sivagnanm [1] introduced the concept of neighborhood connected domination. A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S)\rangle$ is connected. The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number (ncd-number) of G and is denoted by $\gamma_{nc}(G)$. The chromatic number $\chi(G)$ of a graph G is defined to be the minimum number of colours required to colour all the vertices such that no two adjacent vertices receive the same colour.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterizing the corresponding extremal graphs. In [4], J. Paulraj Joseph and S. Arumugam proved that $\gamma + \chi \leq n + 1$. They also characterized the class of graphs for the upper bound is attained. In this paper, we obtain upper bounds for the sum of ncd-number and chromatic number and characterize the extremal graphs. We need the following theorems.

Theorem 1.1. [2] For any graph G, $\chi \leq 1 + \Delta$.

Theorem 1.2. [2] If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$.

Theorem 1.3. [1] Let G be a graph with $\Delta = n - 1$. Then $\gamma_{nc}(G) = 1$ or 2. Further $\gamma_{nc}(G) = 2$ if and only if G has exactly one vertex v with $deg\ v = n - 1$ and v is a cut vertex of G.

Theorem 1.4. [1] For the path P_n, $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$.

Theorem 1.5. [1] For any graph G, $\gamma_{nc}(G) \leq \lceil \frac{n}{2} \rceil$.

Theorem 1.6. [1] Let G be a graph with $\Delta < n - 1$. Then $\gamma_{nc}(G) \leq n - \Delta$.

2 Definitions and Notations

Definition 2.1. $H(m_1, m_2, \cdots, m_n)$ denotes the graph obtained from the graph H by attaching m_i edges to the vertex $v_i \in V(H),\ 1 \leq i \leq n$.

Definition 2.2. $H(P_{m_1}, P_{m_2}, \cdots, P_{m_n})$ is the graph obtained from the graph H by attaching the end vertex of P_{m_i} to the vertex v_i in H, $1 \leq i \leq n$.

Definition 2.3. Let $H_1$and H_2 be two copies of C_3 with vertex sets $V(H_1) = \{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\}$ and $V(H_2) = \{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\}$. Then the graph $C_3^{(2)}$ is obtained from $H_1 \cup H_2$ by joining the vertices $v_i^{(1)}$ and $v_i^{(2)}$, $1 \leq i \leq 3$, by an edge.

2.1 Graphs and Notations

We define the following graphs.

G_1 is the graph obtained from $C_3(P_3, P_2, P_2)$ by attaching a P_2 to the vertex of degree 2.

G_2 is the graph obtained from $C_3(P_4, P_2, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2, where C is a cycle in $C_3(P_4, P_2, P_1)$.

G_3 is the graph obtained from $C_3(P_3, P_2, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2, where C is a cycle in $C_3(P_3, P_2, P_1)$.

G_4 is the graph obtained from $C_3(P_4, P_1, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2, where C is a cycle in $C_3(P_4, P_1, P_1)$.

G_5 is the graph obtained from $C_3(P_4, P_1, P_1)$ by attaching a P_3 to the vertex $u \notin V(C)$ of degree 2 and $d(u, C) = 1$, where C is a cycle in $C_3(P_4, P_1, P_1)$.

G_6 is the graph obtained from $C_3(P_3, P_1, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2, where C is a cycle in $C_3(P_3, P_1, P_1)$.

G_7 is the graph obtained from $C_3(P_4, P_1, P_1)$ by attaching a P_2 to the vertex of degree 3.

G_8 is the graph obtained from $C_3(P_3, P_1, P_1)$ by attaching a P_2 to the vertex of degree 3.

G_9 is the graph obtained from $C_3(P_3, P_1, P_1)$ by attaching a P_3 to the vertex of degree 3.

G_{10} is the graph obtained from $C_3(2,1,0)$ by attaching a P_2 to a pendant vertex whose support has degree 4.
G_{11} is the graph obtained from $C_3(2,0,0)$ by attaching a P_3 to the vertex $u \in V(C)$ of degree 2, where C is a cycle in $C_3(2,0,0)$.
G_{12} is the graph obtained from $C_3(2,1,0)$ by attaching a P_2 to the vertices whose support has degree 4.
G_{13} is the graph obtained from $C_3(2,1,1)$ by attaching a P_2 to a pendant vertex whose support has degree 4.

2.2 Sets of graphs

Also we define the sets of graphs.
$B_1 = \{C_3(3,0,0)\}$
$B_2 = \{C_6, C_4(P_3,P_1,P_1,P_1),\ C_4(1,0,0,0)\}$
$B_3 = \{C_7, C_9\}$
$B_4 = \{G_1, C_3(P_3,P_3,P_2)\}$
$B_5 = \{G_2, G_3, C_5(P_3,P_2,P_1,P_1,P_1), C_5(P_3,P_1,P_2,P_1,P_1), C_5(P_2,P_2,P_1,P_1,P_1),$
$C_5(P_2,P_1,P_2,P_1,P_1), C_3(P_5,P_2,P_1), C_3(P_4,P_2,P_1), C_3(P_4,P_3,P_1), C_3(P_3,P_3,P_1)\}$
$B_6 = \{G_4, G_5, G_6, C_7(P_2,P_1,P_1,P_1,P_1,P_1,P_1), C_5(P_4,P_1,P_1,P_1,P_1), C_5(P_3,P_1,P_1,P_1,P_1),$
$C_3(P_6,P_1,P_1), C_3(P_5,P_1,P_1)\}$
$B_7 = \{C_5(2,0,0,0,0), C_3(2,1,0), C_3(2,1,1)$ and $G_i\ :\ 7 \le i \le 13\}$
$B_8 = \{C_4,\ C_3(2,0,0),\ C_3(1,1,1),\ C_3(1,1,0),\ C_3(P_1,P_2,P_3), C_3(P_1,P_1,P_3),\ C_3(P_1,P_1,P_4),$
$C_5(1,0,0,0,0)\}$

2.3 Family of graphs

Let $\mathscr{A}_1$ be the family of unicyclic graphs of order n with odd cycle $C = (v_1, v_2, \cdots, v_k, v_1)$ satisfy the following conditions: (i) $6 \le n \le 9$ (ii) $\Delta = 3$ (iii) C contains s number of maximum degree vertices.

Let $\mathscr{A}_2$ be the family of unicyclic graphs of order n with odd cycle $C = (v_1, v_2, \cdots, v_k, v_1)$ satisfy the following conditions (i) $6 \le n \le 9$ (ii) $\Delta = 4$.

3 Main Results

Theorem 3.1. For any graph G, $\gamma_{nc} + \chi \le n + 1$. Further equality holds if and only if G is isomorphic to C_5 or K_n or $K_n - Y$ where $Y \subseteq E(G)$, $|Y| = n - 2$ and the edge-induced subgraph $\langle Y \rangle$ is a star.

Proof: If $\Delta < n - 1$, then $\gamma_{nc} \le n - \Delta$ and $\chi \le \Delta + 1$. Hence $\gamma_{nc} + \chi \le n + 1$. If $\Delta = n - 1$ then $\gamma_{nc} \le 2$.

If $\gamma_{nc} = 1$, the inequality is obvious. If $\gamma_{nc} = 2$, then G is neither an odd cycle nor a complete graph. Hence $\chi \le \Delta$ so that $\gamma_{nc} + \chi \le 2 + \Delta \le 2 + n - 1 = n + 1$.

Now, suppose G is a graph with $\gamma_{nc} + \chi = n + 1$.

Case (i): $\Delta = n-1$.

Then $\gamma_{nc} \leq 2$. If $\gamma_{nc} = 1$, then $\chi = n$ and hence G is isomorphic to K_n. Suppose $\gamma_{nc} = 2$. It follows from Theorem 1.3 that G has exactly one vertex v with $deg\ v = n-1$ and v is a cut vertex of G. Since $\chi(G) = n-1$, it follows that $G-v$ has exactly two components G_1 and G_2 where $G_1 = K_1$ and $\langle V(G_2) \cup \{v\} \rangle = K_{n-1}$. Hence $G = K_n - Y$ where $Y \subseteq |E(G)|$, $|Y| = n-2$ and the edge-induced subgraph $\langle Y \rangle$ is a star.

Case (ii): $\Delta < n-1$.

Then $\gamma_{nc} \leq n - \Delta$. Since $\gamma_{nc} + \chi = n+1$, it follows that $\chi \geq \Delta + 1$ and hence $\chi = \Delta + 1$. Thus G is an odd cycle and $\gamma_{nc} = n - \Delta = n-2$. Hence $G = C_5$. The converse is obvious. ∎

Theorem 3.2. Let T be a tree of order n. Then $\gamma_{nc} + \chi = n$ if and only if $T = P_4, P_5$ or $K_{1,3}$.

Proof: Let T be a tree with $\gamma_{nc} + \chi = n$. Since $\chi = 2$, we have $\gamma_{nc} = n-2$ and hence $n \geq 4$. If $\Delta = n-1$, then $\gamma_{nc} = 2$ so that $n = 4$ and $T = K_{1,3}$. Suppose $\Delta < n-1$. Then $\gamma_{nc} \leq n - \Delta$. Since $\gamma_{nc} + 2 = n$, it follows that $\Delta \leq 2$, so that T is a path. Now by Theorem 1.4, $\gamma_{nc} = \left\lceil \frac{n}{2} \right\rceil$, so that $n - 2 = \left\lceil \frac{n}{2} \right\rceil$, which gives $n = 4$ or 5. Thus T is either P_4 or P_5. The converse is obvious. ∎

Theorem 3.3. Let G be a unicyclic graph of order n. Then $\gamma_{nc} + \chi = n$ if and only if $G \in B_8$.

Proof: It can be easily verified for all the graphs given in the theorem, $\gamma_{nc} + \chi = n$. Now, let G be a unicyclic graph with cycle $C = (v_1, v_2, \ldots, v_k, v_1)$ and let $\gamma_{nc} + \chi = n$.

Case 1: $\Delta = n-1$.

Then $\gamma_{nc} \leq 2$. If $\gamma_{nc} = 1$ then G is K_3 and $\gamma_{nc} + \chi \neq n$. Hence $\gamma_{nc} = 2$ and $\chi = n-2$. Further if k is even, then $\chi = 2$ and $n = 4$ and in this case $k = 4$ and $\Delta = 2$, which is a contradiction. Hence k is odd, so that $\chi = 3$, $n = 5$ and $\Delta = 4$. Hence G is isomorphic to $C_3(2,0,0)$.

Case 2: $\Delta < n-1$.

If k is even, then $\chi = 2$ and $\gamma_{nc} = n-2$. Also it follows from Theorem 1.5. that $\gamma_{nc} \leq \left\lceil \frac{n}{2} \right\rceil$, and hence $n \leq 5$.

If $n = 5$, then $k = 4$ and there exists a vertex not in C which is adjacent to a vertex in C. However, for this graph $\gamma_{nc} = 2$ and $\chi = 2$, so that $\gamma_{nc} + \chi \neq n$. Thus $n = 4$ and G is isomorphic to C_4.

Now, suppose k is odd. Then $\chi = 3$ and $\gamma_{nc} = n-3$. Also by Theorem 1.5, $\gamma_{nc} \leq \left\lceil \frac{n}{2} \right\rceil$, so that $5 \leq n \leq 7$. Further since $\Delta < n-1$, we have $\gamma_{nc} \leq n - \Delta$ and hence $\Delta \leq 3$.

If $\Delta = 2$, then G is isomorphic to C_5 or C_7 and for these graphs $\gamma_{nc} + \chi \neq n$. Thus $\Delta = 3$. Since $\gamma_{nc} = n-3$, G contains at most three pendant vertices and hence C contains at most three vertices of degree 3. Suppose C contains three vertices of degree 3. Since $n \leq 7$, $k = 3$. Let $C = (v_1, v_2, v_3, v_1)$ and $u_i v_i \in E$, $i = 1,2,3$. If $deg\ u_i = 2$ for some i, then $\gamma_{nc} + \chi \neq n$. Hence $deg\ u_i = 1$ for all i and G is isomorphic to $C_3(1,1,1)$.

Suppose C contain two vertices of degree 3. Then $k = 3$ or 5. We claim that $k = 3$. Suppose $k = 5$ and let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. Since exactly two vertices of G have degree 3, G is isomorphic to the graph obtained from C_5 by attaching P_2 to two distinct vertices. For this graph $\gamma_{nc} = \chi = 3$, so that $\gamma_{nc} + \chi \neq n$, which is a contradiction.

Thus $k = 3$ and let $C = (v_1, v_2, v_3, v_1)$. Let $deg\ v_1 = deg\ v_2 = 3$ and $u_i v_i \in E$, $i = 1, 2$. If $deg\ u_1 = deg\ u_2 = 2$, then G is isomorphic to the graph obtained from C_3 by attaching P_3 to two distinct vertices. For this graph $\gamma_{nc} = \chi = 3$, so that $\gamma_{nc} + \chi \neq n$. Hence at least one of the vertices u_1, u_2 is of degree 1.

If deg $u_1 =$ deg $u_2 = 1$, then G is isomorphic to $C_3(1, 1, 0)$. If $deg\ u_2 = 3$ then $\gamma_{nc} = \chi = 3$, so that $\gamma_{nc} + \chi \neq n$. Hence $deg\ u_2 = 2$. Let $u_2 w \in E(G)$. If $deg\ w = 2$, then G is isomorphic to the graph obtained from C_3 by attaching P_2 to one vertex and attaching P_4 to another vertex. For this graph $\gamma_{nc} = \chi = 3$, so that $\gamma_{nc} + \chi \neq n$. Hence deg $w = 1$ and G is isomorphic to $C_3(P_1, P_2, P_3)$.

Suppose C contains exactly one vertex of degree 3. Then $k = 3$ or 5. Suppose $k = 3$. Let $u_1 v_1 \in E$. Since $n \leq 7$, deg $u_1 \leq 4$. If deg $u_1 = 3$ or 4, then G is a graph with $\gamma_{nc} + \chi \neq n$. Hence deg $u_1 = 2$. Let $u_1 w \in E$. If deg $w = 3$ then G is a graph with $\gamma_{nc} + \chi \neq n$. Hence deg $w \leq 2$ and G is isomorphic to $C_3(P_1, P_1, P_3)$ or $C_3(P_1, P_1, P_4)$.

Suppose $k = 5$. Let $u_1 v_1 \in E$. Then deg $u_1 = 1$ or 2. If deg $u_1 = 2$ then G is a graph with $\gamma_{nc} + \chi \neq n$. Hence $deg\ w = 1$ and G is isomorphic to $C_5(1, 0, 0, 0, 0)$. ■

Remark 3.4. There is no cubic graph of order n with $\gamma_{nc} + \chi = n$.

Proof: Let G be a cubic graph with $\gamma_{nc} + \chi = n$. If G is a complete graph then $\gamma_{nc} + \chi = n + 1$, which is a contradiction. Hence $\chi \leq 3$. Then $\gamma_{nc} \geq n - 3$. It follows from Theorem 1.6, $\gamma_{nc} \leq n - 3$. Thus we have $\gamma_{nc} = n - 3$ and then $\chi = 3$. Theorem 1.5 gives $\gamma_{nc} \leq \lceil \frac{n}{2} \rceil$ which implies $n \leq 7$. Since G is not a complete graph, we have $n = 6$. Then $\gamma_{nc} = 3$ and $\chi = 3$. Since each vertices v of G dominates four vertices, and all the vertices have degree 3, two vertices are sufficient to dominates six vertices. Hence there does not exist a cubic graph with $\gamma_{nc} + \chi = n$. ■

Theorem 3.5. Let T be a tree of order n. Then $\gamma_{nc} + \chi = n - 1$ if and only if T is $K_{1,4}$ or P_6 or P_7 or its obtained from $K_{1,3}$ by subdividing at least two edges once.

Proof: Let T be a tree with $\gamma_{nc} + \chi = n - 1$. Since $\chi = 2$, we have $\gamma_{nc} = n - 3$ and hence $n \geq 5$. If $\Delta = n - 1$ then $\gamma_{nc} = 2$ so that $n = 5$ and $T = K_{1,4}$. Suppose $\Delta < n - 1$. Then $\gamma_{nc} \leq n - \Delta$. Since $\gamma_{nc} + 2 = n - 1$, it follows that $\Delta \leq 3$. Now from Theorem 1.5, $\gamma_{nc} \leq \lceil \frac{n}{2} \rceil$, so that $n - 3 \leq \lceil \frac{n}{2} \rceil$ which gives $n \leq 7$. Hence $5 \leq n \leq 7$. If $\Delta = 2$ then T is a path and hence $T = P_6$ or P_7. If $\Delta = 3$ then $n \neq 5$. Suppose $n = 6$. Let v be a vertex of degree $\Delta = 3$ and $N(v) = \{v_1, v_2, v_3\}$. Let $u_1,\ u_2 \in V - N[v]$. If $u_1,\ u_2 \in N(v_i)$, $1 \leq i \leq 3$, then $\gamma_{nc} = 2$ which is a contradiction. Hence u_1 is adjacent to v_i and u_2 is adjacent to $v_j,\ j \neq i$. Thus T is isomorphic to a graph obtained from $K_{1,3}$ by subdividing two edges once.

Suppose $n = 7$. Let v be a vertex of degree $\Delta = 3$ and $N(v) = \{v_1, v_2, v_3\}$. Let $u_1, u_2, u_3 \in V - N[v]$. Suppose $\deg v_1 = 1$. If $\deg v_2 = 2$ and $\deg v_3 = 3$ then $\gamma_{nc} = 3 \neq n - 3$ which is a contradiction. If $\deg v_2 = 1$ and $\deg v_3 = 3$ then $u_1 v_3,\ u_2 v_3 \in E(T)$. Since T is a tree without loss of generality we assume u_3 is adjacent to u_1 then $\gamma_{nc} = 3 \neq n-3$ which is a contradiction. Let $\deg v_2 = 1$ and $\deg v_3 = 2$ and $u_1 v_3 \in E(T)$. If $u_1 u_2,\ u_1 u_3 \in E(T)$ then $\gamma_{nc} = 3 \neq n-3$ which is a contradiction. If $u_1 u_2,\ u_2 u_3 \in E(T)$ then $\gamma_{nc} \neq n-3$ which is a contradiction. Hence $\deg v_i = 2,\ 1 \leq i \leq 3$. Thus T is isomorphic to a graph obtained from $K_{1,3}$ by subdividing all the edges once. The converse is obvious. ∎

Lemma 3.6. Let G be a unicyclic graph of order n and $\Delta = n-1$. Then $\gamma_{nc} + \chi = n-1$ if and only if $G \in B_1$.

Proof: Let G be an unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_k, v_1)$, $\Delta = n-1$ and let $\gamma_{nc} + \chi = n-1$. Then $\gamma_{nc} \leq 2$. If $\gamma_{nc} = 1$ then G is k_3 and $\gamma_{nc} + \chi = n+1$. Hence $\gamma_{nc} = 2$ and $\chi = n-3$. If $k \geq 4$ then $\Delta < n-1$. Hence $k = 3$. Thus $\chi = 3,\ n = 6$ and $\Delta = 5$. Hence G is isomorphic to $C_3(3,0,0) \in B_1$. The converse is obvious. ∎

Theorem 3.7. Let G be a unicyclic graph with even cycle $C = (v_1, v_2, \cdots, v_k, v_1)$ and $\Delta < n-1$. Then $\gamma_{nc} + \chi = n-1$ if and only if $G \in B_2$.

Proof: Since k is even $\chi = 2$ and $\gamma_{nc} = n-3$. It follows from Theorem 1.5 that $\gamma_{nc} \leq \lceil \frac{n}{2} \rceil$ and hence $n \leq 7$.

Case 1: $n = 7$.

Then $k = 4$ or 6. If $k = 4$ then $C = (v_1, v_2, v_3, v_4, v_1)$. Let u_1, u_2, u_3 be the vertices not on C. If $\deg u_i = 1$ for all $i = 1, 2, 3$ then G is isomorphic to $C_4(1,1,1,0)$ or $C_4(2,1,0,0)$ or $C_4(3,0,0,0)$. For this graphs $\gamma_{nc} \neq n-3$. If $\deg u_3 = 2$ then G is isomorphic to $C_4(P_3, P_2, P_1, P_1)$. But $\gamma_{nc} = 3 \neq n-3$. If $\deg u_3 = 3$ then G is a graph obtained from $C_4(P_3, P_1, P_1, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2. For this graph $\gamma_{nc} = 3 \neq n-3$. If $\deg u_1 = 1$ and $\deg u_i \neq 1,\ i = 2,\ 3$ then $\deg u_2 = \deg u_3 = 2$. Hence G is isomorphic to $C_4(P_4, P_1, P_1, P_1)$. But $\gamma_{nc} = 3 \neq n-3$. If $k = 6$ then G is isomorphic to $C_6(1,0,0,0,0,0)$. But $\gamma_{nc} = 3 \neq n-3$.

Case 2: $n = 6$.

Then $k = 4$ or 6. If $k = 6$ then G is C_6. Let $k = 4$ then G is any one of the following graph $(i)\ H_1 = C_4(1,1,0,0)$ $(ii)\ H_2 = C_4(2,0,0,0)$ $(iii)\ H_3 = C_4(P_3, P_1, P_1, P_1)$. But $\gamma_{nc}(H_1) = \gamma_{nc}(H_2) = 2$. Hence G is isomorphic to $C_4(P_3, P_1, P_1, P_1)$.

If $n = 5$ then $k = 4$ and G is isomorphic to $C_4(1,0,0,0)$. If $n = 4$ then G is C_4 and $\gamma_{nc} \neq n-3$. The converse is obvious. ∎

Lemma 3.8. Let $G \in \mathscr{A}_1$ and $s = 3$. Then $\gamma_{nc} + \chi = n-1$ if and only if $G \in B_4$.

Proof: Since k is odd, $\chi = 3$ and $\gamma_{nc} = n-4$, also $s = 3$ gives $k = 3$ or 5. If $k = 5$ then G is isomorphic to $C_5(P_3, P_2, P_2, P_1, P_1)$ or $C_5(P_2, P_2, P_2, P_1, P_1)$ or $C_5(P_2, P_2, P_1, P_2, P_1)$ or $C_5(P_3, P_1, P_2, P_2, P_1)$ or $C_5(P_3, P_2, P_1, P_1, P_2)$. For this graphs $\gamma_{nc} \neq n-4$ which is a contradiction. Thus $k = 3$.

Let $C = (v_1, v_2, v_3, v_1)$ and $u_i\, v_i \in E,\ 1 \leq i \leq 3$. If $\deg u_i = 1,\ 1 \leq i \leq 3$ then G is isomorphic to $C_3(1,1,1)$. But $\gamma_{nc}[C_3(1,1,1)] = 3 \neq n-4$ which is a contradiction. If $\deg u_1 = 3$ then at most one of u_2 and u_3 has degree 2. Let $\deg u_2 = 2$. Then $\deg u_3 = 1$. For this graph $\gamma_{nc} = 4 \neq n-4$ which is a contradiction. If $\deg u_2 = 1$ and $\deg u_3 = 1$ then the graph G is isomorphic to G_1. Suppose $\deg u_1 = 2$. Then ($\deg u_2 = 2$ and $\deg u_3 = 2$) or ($\deg u_2 = 2$ and $\deg u_3 = 1$) or ($\deg u_2 = 1$ and $\deg u_3 = 1$).

If $\deg u_1 = 2,\ \deg u_2 = 2$ and $\deg u_3 = 2$ then G is isomorphic to $C_3(P_3, P_3, P_3)$. But $\gamma_{nc}[C_3(P_3, P_3, P_3)] = 4 \neq n-4$ which is a contradiction. If $\deg u_1 = 2,\ \deg u_2 = 2$ and $\deg u_3 = 1$ then G is isomorphic to $C_3(P_3, P_3, P_2)$. If $\deg u_1 = 2,\ \deg u_2 = 1$ and $\deg u_3 = 1$ then G is isomorphic to $C_3(P_3, P_1, P_1)$. But $\gamma_{nc}[C_3(P_3, P_1, P_1)] \neq n-4$ which is a contradiction.

The converse is obvious. ∎

Lemma 3.9. Let $G \in \mathscr{A}_1$ and $s = 2$. Then $\gamma_{nc} + \chi = n-1$ if and only if $G \in B_5$.

Proof: Since k is odd $\chi = 3$ and $\gamma_{nc} = n-4$. Also $s = 2$ gives $k = 3$ or 5 or 7.

Case 1: $k = 7$.

Then G is isomorphic to the graph obtained from C_7 by attaching P_2 to any two vertices. But $\gamma_{nc} < n-4$ which is a contradiction.

Case 2: $k = 5$.

Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$ and let x be a pendant vertex in G. It is clear that $d(x, c) \leq 3$. If $d(x, C) = 3$ then G is isomorphic to $C_5(P_4, P_2, P_1, P_1, P_1)$ or $C_5(P_4, P_1, P_2, P_1, P_1)$. For this graphs $\gamma_{nc} \neq n-4$ which is a contradiction.

Suppose $d(x, C) = 2$. Then $n = 8$ or 9. Suppose $n = 8$. Let $\deg v_1 = 3$ and (v_1, x_1, x) be a path in G. Since $n = 8$ there is a vertex x_2 in $V(G)$ and x_2 is adjacent to v_2 or v_3 or v_4 or v_5. Hence G is isomorphic to $C_5(P_3, P_2, P_1, P_1, P_1)$ or $C_5(P_3, P_1, P_2, P_1, P_1)$. Suppose $n = 9$. Then there are two vertices $x_2,\ x_3 \in V(G)$. If $\deg x_2 = \deg x_3 = 1$ then $x_2\, v_2$ or $x_2 v_3 \in E$ and $x_1\, x_3 \in E$. For this graph $\gamma_{nc} \neq n-4$ which is a contradiction. If $\deg x_2 = 2$ then $\deg x_3 = 1$ and $x_2\, x_3 \in E$. Hence G is isomorphic to $C_5(P_3, P_3, P_1, P_1, P_1)$ or $C_5(P_3, P_1, P_3, P_1, P_1)$. For this graph $\gamma_{nc} \neq n-4$.

If $d(x, C) = 1$ then G is isomorphic to $C_5(P_2, P_2, P_1, P_1, P_1)$ or $C_5(P_2, P_1, P_2, P_1, P_1)$.

Case 3: $k = 3$.

Let $C = (v_1, v_2, v_3, v_1)$ and let x be a pendant vertex in G. Then $d(x, C) \leq 5$. If $d(x, C) = 5$ then $n = 9$ and G is isomorphic to $C_3(P_6, P_2, P_1)$. But $\gamma_{nc} \neq n-4$ which is a contradiction.

Sub Case 3.1: $d(x, C) = 4$.

Let (v_1, x_1, x_2, x_3, x) be the $v_1 - x$ path. Then $n = 8$ or 9. Suppose $n = 8$ then G is isomorphic to $C_3(P_5, P_2, P_1)$. If $n = 9$ there exist two vertices x_4 and x_5 such that $x_4\, v_2 \in E$ and x_5 is adjacent to any one of x_1, x_2, x_3 and x_4. All these cases $\gamma_{nc} \neq n-4$.

Sub Case 3.2: $d(x, C) = 3$.

Let (v_1, x_1, x_2, x) be the $v_1 - x$ path. Then $7 \leq n \leq 9$. If $n = 7$ then G is isomorphic to $C_3(P_4, P_2, P_1)$. Let $n = 8$ and $x_3\, v_2 \in E$. Then there is a vertex x_4 which is adjacent to any one of

x_1, x_2 and x_3. Hence G is isomorphic to $C_3(P_4, P_3, P_1)$ or G_2. Let $n = 9$ and $x_3 v_2 \in E$. Then there are two vertices x_4 and x_5 with $x_2 x_4$, $x_3 x_5 \in E$ or $x_2 x_4$, $x_1, x_5 \in E$ or $x_1 x_5$, $x_3 x_4 \in E$. All these cases $\gamma_{nc} \neq n - 4$.

Sub Case 3.3: $d(x, C) = 2$.

Let (v_1, x_1, x) be the $v_1 - x$ path and let $x_2 v_2 \in E$. Then $6 \leq n \leq 9$. If $n = 6$ then G is isomorphic to $C_3(P_3, P_2, P_1)$. For this graph $\gamma_{nc} \neq n - 4$. If $n = 7$ then G is isomorphic to $C_3(P_3, P_3, P_1)$ or G_3. If $n = 8$ then G is a graph obtained from $C_3(P_3, P_3, P_1)$ by attaching a P_2 to the vertex $u \notin V(C)$ of degree 2. For this graph $\gamma_{nc} \neq n - 4$. If $n = 9$ then G is a graph obtained from $C_3(P_3, P_3, P_1)$ by attaching a pendant vertex to all the vertices of degree 2 which are not on C. For this graph $\gamma_{nc} \neq n - 4$.

If $d(x, C) = 1$ then G is isomorphic to $C_3(P_2, P_2, P_1)$. But $\gamma_{nc} \neq n - 4$ which is a contradiction. The converse is obvious. ■

Lemma 3.10. Let $G \in \mathscr{A}_1$ and $s = 1$. Then $\gamma_{nc} + \chi = n - 1$ if and only if $G \in B_6$.

Proof: Since k is odd, $\chi = 3$ and $\gamma_{nc} = n - 4$. Also $s = 1$ gives $k = 3$ or 5 or 7.

Case 1: $k = 7$.

Then G is isomorphic to $C_7(P_3, P_1, P_1, P_1, P_1, P_1, P_1,)$ or $C_7(P_2, P_1, P_1, P_1, P_1, P_1, P_1,)$. But $\gamma_{nc}[C_7(P_3, P_1, P_1, P_1, P_1, P_1, P_1,)] \neq n-4$. Hence G is isomorphic to $C_7(P_2, P_1, P_1, P_1, P_1, P_1, P_1,)$.

Case 2: $k = 5$.

Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$ and let x be a pendant vertex in G. Also let us assume $\deg v_1 = 3$. Then $d(x, C) \leq 4$. If $d(x, C) = 4$ then $n = 9$ and G is isomorphic to $C_5(P_5, P_1, P_1, P_1, P_1)$. But $\gamma_{nc} \neq n - 4$. Let $d(x, C) = 3$ and let (v_1, x_1, x_2, x) be the $v_1 - x$ path. If $\deg x_1 = 3$ or $\deg x_2 = 3$ then $\gamma_{nc} \neq n - 4$. Hence G is isomorphic to $C_5(P_4, P_1, P_1, P_1, P_1)$. Let $d(x, C) = 2$ and let (v_1, x_1, x) be the $v_1 - x$ path. Then $n = 7$ or 8. If $n = 7$ then G is isomorphic to $C_5(P_3, P_1, P_1, P_1, P_1)$. If $n = 8$ then there is a vertex x_2 such that $x_2 x_1 \in E$. For this graph $\gamma_{nc} \neq n - 4$. If $d(x, C) = 1$ then G is isomorphic to $C_5(P_2, P_1, P_1, P_1, P_1)$. But $\gamma_{nc} \neq n - 4$.

Case 3: $k = 3$.

Let $C = (v_1, v_2, v_3, v_1)$ and let x be a pendant vertex in G. Also let us assume $\deg v_1 = 3$. Then $d(x, C) \leq 6$. If $d(x, C) = 6$ then $n = 9$ and G is isomorphic to $C_3(P_7, P_1, P_1)$. But $\gamma_{nc} \neq n - 4$.

Let $d(x, C) = 5$ and let $(v_1, x_1, x_2, x_3, x_4, x)$ be the $v_1 - x$ path. If $\deg x_i = 2$, $1 \leq i \leq 4$ then G is isomorphic to $C_3(P_6, P_1, P_1)$. If $\deg x_i = 3$ for some i, $1 \leq i \leq 4$ then $\gamma_{nc} \neq n - 4$.

Let $d(x, C) = 4$ and let (v_1, x_1, x_2, x_3, x) be the $v_1 - x$ path. If $\deg x_i = 2$, $1 \leq i \leq 3$ then G is isomorphic to $C_3(P_5, P_1, P_1)$. If $\deg x_i = 3$ for some i, $1 \leq i \leq 3$ then $\gamma_{nc} \neq n - 4$.

Let $d(x, C) = 3$ and let (v_1, x_1, x_2, x) be the $v_1 - x$ path. If $\deg x_i = 2$ or 3, $1 \leq i \leq 2$ then $\gamma_{nc} \neq n - 4$. Hence at most one of x_1, x_2 has degree 3. Then G is isomorphic to G_4 or G_5.

Let $d(x, C) = 2$ and let (v_1, x_1, x) be the $v_1 - x$ path. If $\deg x_1 = 2$ then $\gamma_{nc} \neq n - 4$. Thus $\deg x_1 = 3$ and hence G is isomorphic to G_6.

If $d(x, C) = 1$ then $n = 5$ which is a contradiction. The converse is obvious. ■

Lemma 3.11. Let $G \in \mathscr{A}_2$. Then $\gamma_{nc} + \chi = n - 1$ if and only if $G \in B_7$.

Proof: Since k is odd $\chi = 3$ and $\gamma_{nc} = n - 4$.

Case 1: C contains a vertex of degree Δ.

Then $k = 3$ or 5 or 7. If $k = 7$ then G is isomorphic to $C_7(2, 0, 0, 0, 0, 0, 0)$ and $\gamma_{nc} \neq n - 4$.

Sub Case 1.1: $k = 5$.

Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. If x is a pendant vertex then $d(x, C) \leq 3$. If $d(x, C) = 3$ then $\gamma_{nc} \neq n - 4$. If $d(x, C) = 2$ then by similar arguments given in previous lemma $\gamma_{nc} \neq n - 4$. If $d(x, C) = 1$ then $n = 7$ or 8 or 9. If $n = 8$ or 9 we have $\gamma_{nc} \neq n - 4$. Then G is isomorphic to $C_5(2, 0, 0, 0, 0)$.

Sub Case 1.2: $k = 3$.

Let $C = (v_1, v_2, v_3, v_1)$. If x is a pendant vertex then $d(x, C) \leq 5$. If $d(x, C) = 5$ or 4 then $\gamma_{nc} \neq n - 4$. If $d(x, C) = 3$ then $n = 7$ or 8 or 9. If $n = 8$ or 9 then by similar arguments given in previous lemma $\gamma_{nc} \neq n - 4$. If $n = 7$ then G is isomorphic to G_7. If $d(x, C) = 2$ then $6 \leq n \leq 9$. If $n = 6$ then G is isomorphic to G_8. If $n = 7$ then G is isomorphic to G_9 or G_{10} or G_{11}. If $n = 8$ then G is isomorphic to G_{12} or G_{13}. If $n = 9$ then no graph exists. If $d(x, C) = 1$ then $6 \leq n \leq 9$. Then by similar arguments given in previous lemma there is no graph of order 8 and 9. If $n = 6$ then G is isomorphic to $C_3(2, 1, 0)$. If $n = 7$ then G is isomorphic to $C_3(2, 1, 1)$.

Case 2: C does not contains maximum degree vertex.

Then $k = 3$ or 5. If $k = 5$ then G is a graph obtained from $C_5(P_3, P_1, P_1, P_1, P_1)$ by attaching two P_2 to the vertex $u \notin V(C)$ of degree 2. For this graph $\gamma_{nc} \neq n - 4$. If $k = 3$ then by similar arguments in previous lemmas there is no graph. The converse is obvious. ■

Theorem 3.12. Let G be a unicyclic graph of order n. Then $\gamma_{nc} + \chi = n - 1$ if and only if $G \in \bigcup_{i=1}^{7} B_i$.

Proof: Let G be a unicyclic graph with cycle $C = (v_1, v_2, \cdots, v_k, v_1)$ and let $\gamma_{nc} + \chi = n - 1$. If $\Delta = n - 1$ or $\Delta < n - 1$ with k is even then $G \in B_1 \cup B_2$.

Suppose $\Delta < n - 1$ and k is odd.Then $\chi = 3$ and $\gamma_{nc} = n - 4$. Also it follows from Theorem 1.5 that $\gamma_{nc} \leq \lceil \frac{n}{2} \rceil$ so that $6 \leq n \leq 9$. Further since $\Delta < n - 1$ we have $\gamma_{nc} \leq n - \Delta$ and hence $\Delta \leq 4$. If $\Delta = 2$ then G is isomorphic to C_7 or C_9. Hence $G \in B_3$. Since $\gamma_{nc} = n - 4$, G contains at most four pendant vertices and hence C contains at most four vertices of degree 3. Then $s \leq 4$. If $s = 4$ then $k = 5$. Let $v \in V(C)$ of degree 2 and let $u_1,\ u_2,\ u_3,\ u_4 \in V(G - C)$. Then $\deg u_i = 1,\ 1 \leq i \leq 4$ and $S = V - \{v_1, u_1, u_2, u_3, u_4\}$ is a ncd set of cardinality $n - 5$ which is a contradiction. Hence $s \leq 3$. Then $G \in \mathscr{A}_1$ or $\mathscr{A}_2$. Hence by lemmas 3.6, 3.7, 3.8, 3.9, 3.10, 3.11 the result follows. The converse is obvious. ■

Theorem 3.13. Let G be a cubic graph of order n. Then $\gamma_{nc} + \chi = n - 1$ if and only if $G = C_3^{(2)}$.

Proof: Let G be a cubic graph with $\gamma_{nc} + \chi = n - 1$. If G is a complete graph then $\gamma_{nc} + \chi = n + 1$ which is a contradiction. Hence $\chi \leq 3$. Then $\gamma_{nc} \geq n - 4$. It follows from Theorem 1.6, $\gamma_{nc} \leq n - 3$. Thus we have $\gamma_{nc} = n - 4$ or $n - 3$.

Case 1: $\gamma_{nc} = n - 3$.

Then $\chi = 2$. It follows from Theorem 1.5 that $\gamma_{nc} \leq \lceil \frac{n}{2} \rceil$ which gives $n \leq 7$. Since G is not a complete graph we have $n = 6$. Then $\gamma_{nc} = 3$ and $\chi = 2$. Since each vertex v of G dominates four vertices and all the vertices having degree three, two vertices are sufficient to dominate six vertices. Hence $\gamma_{nc} \leq 2$ which is a contradiction.

Case 2: $\gamma_{nc} = n - 4$.

Then $\chi = 3$. It follows from Theorem 1.5 that $\gamma_{nc} \leq \lceil \frac{n}{2} \rceil$ which gives $n \leq 9$. Since G is not a complete graph we have $n = 6$ or 8.

Suppose $n = 6$. Then $\gamma_{nc} = 2$ and $\chi = 3$

Let $S = \{v_1, v_2\}$ be the γ_{nc}-set of G and let $V - S = \{u_1, u_2, u_3, u_4\}$

Sub Case 2.1: $\langle S \rangle = \overline{K_2}$.

Let v_1 is adjacent to u_1, u_2 and u_3. If v_2 is also adjacent to $u_1, u_2,$ and u_3 then u_4 is adjacent to $u_1, u_2,$ and u_3. For this graph $\chi = 2$ which is a contradiction. Suppose v_2 is adjacent to u_1, u_2 and u_4. If $u_1\, u_2 \in E$ then G is not a cubic graph. Hence u_1 is adjacent to u_3 or u_4. If $u_1\ u_3 \in E$ then $u_2u_4,\ u_3u_4 \in E$. Then the graph G is isomorphic to the graph G. If $u_1u_4 \in E$ then $u_2u_3,\ u_3u_4 \in E$. Then the graph G is isomorphic to $C_3^{(2)}$.

Sub Case 2.2: $\langle S \rangle = K_2$.

Let v_1 is adjacent to u_1 and u_2. If v_2 is also adjacent to u_1 and u_2 then G is not a cubic graph. Suppose v_2 is adjacent to u_1 and u_3. Then u_4 is adjacent to $u_1,\ u_2$ and u_3 and hence $u_2\, u_3 \in E$. Then G is isomorphic to the graph G_1. Suppose v_2 is adjacent to u_3 and u_4. If $u_1\, u_2 \in E$ then $u_2\, u_4,\ u_3\, u_4 \in E$. Then also G is isomorphic to $C_3^{(2)}$. If $u_1\, u_2 \notin E$ then $u_1\, u_3,\ u_1\, u_4,\ u_2\, u_3,\ u_2u_4\ \in E(G)$. For this graph $\chi = 2$ which is a contradiction.

Suppose $n = 8$ then $\gamma_{nc} = 4$ and $\chi = 3$. Since each vertex v of G dominates four vertices and all the vertices having degree three maximum of three vertices is sufficient to dominate eight vertices. Hence $\gamma_{nc} \leq 3$ which is a contradiction. The converse is obvious. ■

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