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# 2 - Cartesian product of special graphs

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**Abstract**

The cartesian product of two graphs has been studied by many authors and has been generalized by introducing 2 - cartesian product $G_1 \times_2 G_2$ of two graphs $G_1$ and $G_2$. In this paper, we obtain $G_1 \times_2 G_2$, for $P_n$, $C_n$ and $K_{s,t}$.

**Keywords**: Cartesian product of graphs, 2-cartesian product of graphs, grid graph, tied grid graph.
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## 1 Preliminaries

The cartesian product of two graphs had been studied in ([3], [4], [6]). The generalized cartesian product $G_1 \times_r G_2$ has been defined using the idea of distance in [1]. It is not easy to obtain $G_1 \times_r G_2$ for general $G_1$ and $G_2$. So, in this paper we discuss this product for $r = 2$.

Let $G = (V(G), E(G))$ be a finite, simple graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph $G$ is connected, if there is a path between every pair of vertices. If $G$ is a connected graph then $d_G(u, u')$ is the length of the shortest path between $u$ and $u'$ in $G$. For a graph $G$, a maximal connected subgraph is known as components of $G$. If $G$ is a connected graph, then $G$ has only one component, $G$ itself.

Throughout the paper we consider finite, simple and connected graph. We denote the path graph, cycle graph and complete graph with $n$ vertices by $P_n$, $C_n$ and $K_n$ respectively. The complete bipartite graph is denoted by $K_{m,n}$ with $(m + n)$ vertices. The null graph is a graph with empty edge set. If the graph G is a disjoint union of $r$ similar components H, then we denote it by,

$$G = \mathop{\dot{\bigcup}}_{i=1}^{r} H^{(i)}$$

For the basic terminology, concepts and results of graph theory, we refer to ([3], [4], [6]).

We obtain $G_1 \times_2 G_2$ for $P_n$, $C_n$ and $K_{s,t}$. We discuss mainly connectedness of the product graph.

**Definition 1.1.** The $2-$cartesian product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$ with the vertex set $V = V_1 \times V_2$ and the edge set $E$ defined as follows:
Two vertices $(u, v)$ and $(u', v')$ are adjacent in $G$ if one of the following conditions is satisfied:

(i) $d_{G_1}(u,u') = 2$ and $d_{G_2}(v,v') = 0$,

(ii) $d_{G_1}(u,u') = 0$ and $d_{G_2}(v,v') = 2$.

We denote this graph $G$ by $G_1 \times_2 G_2$.

It is clear that if we replace 2 by 1 in the definition, then we get usual cartesian product $G_1 \times G_2$. Note that, if the diameters of both $G_1$ and $G_2$ are less then 2, then $G_1 \times_2 G_2$ is a null graph.

**Definition 1.2.** The grid graph $G = G_{m,n}$ is defined as the graph with vertex set, $V = \{(u_i, v_j) : i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n\}$ and edge set $E = \bigcup\limits_{i=1}^{m} \{(u_i, v_j) \leftrightarrow (u_i, v_{j+1}) : 1 \le j \le n-1\} \cup \bigcup\limits_{j=1}^{n} \{(u_i, v_j) \leftrightarrow (u_{i+1}, v_j) : 1 \le i \le m-1\}$.

The semi tied grid graph $G_{(m),(n^0)}$ is a grid graph with vertex set $V(G)$ and edge set consisting of the following edges:

(i) Each edge of $G_{m,n}$;

(ii) The edges $(u_i, v_1) \leftrightarrow (u_i, v_n)$, for every $i = 1, 2, \dots, m$,
In place of (ii), if we consider (ii)' then we get another semi tied grid graph denoted by $G_{(m^0),(n)}$,

(ii)' The edges $(u_1, v_j) \leftrightarrow (u_m, v_j)$, for every $j = 1, 2, \dots, n$.

The graph containing all the above type of edges is called a *tied grid graph* and is denoted by $G_{(m^0),(n^0)}$.

## 2 2- Cartesian Product

The 2 - cartesian product of path graphs has been obtained in [1]. In this section we discuss $G_1 \times_2 G_2$ with $G_1$ path graph and $G_2$ cycle graph and $G_1 \times_2 G_2$, if both $G_1$ and $G_2$ are cycle graphs.

We fix the following notations. The path graph $P_m$ is the graph with, $V(P_m) = \{u_1, u_2, \dots, u_m\}$ and $E(P_m) = \{(u_1u_2), (u_2u_3), \dots, (u_{m-1}u_m)\}$ and $C_m (m \ge 3)$ is a cycle graph with $V(C_m) = V(P_m)$ and $E(C_m) = E(P_m) \cup \{(u_m u_1)\}$.

We use the following result on path graphs.

**Proposition 2.1.** [1] For $m, n \ge 3$,

(a) If both $m$ and $n$ are even integers then, $P_m \times_2 P_n = \left[ \mathop{\bigcup\limits_{\circ}}_{i=1}^{4} \left( G_{(\frac{m}{2}),(\frac{n}{2})} \right)^{(i)} \right]$.

(b) If $m$ is odd and $n$ is even, then $P_m \times_2 P_n = \left[ \mathop{\bigcup\limits_{\circ}}_{i=1}^{2} \left( G_{(\frac{m+1}{2}),(\frac{n}{2})} \right)^{(i)} \right] \mathop{\bigcup}\limits_{\circ} \left[ \mathop{\bigcup\limits_{\circ}}_{j=1}^{2} \left( G_{(\frac{m-1}{2}),(\frac{n}{2})} \right)^{(j)} \right]$.

(c) If $m$ is even and $n$ is odd, then $P_m \times_2 P_n = \left[ \mathop{\bigcup\limits_{\circ}}_{i=1}^{2} \left( G_{(\frac{m}{2}),(\frac{n+1}{2})} \right)^{(i)} \right] \mathop{\bigcup}\limits_{\circ} \left[ \mathop{\bigcup\limits_{\circ}}_{j=1}^{2} \left( G_{(\frac{m}{2}),(\frac{n-1}{2})} \right)^{(j)} \right]$.

(d) If both $m$ and $n$ are odd integers, then

$$P_m \times_2 P_n = \left[\, G_{(\frac{m+1}{2}),(\frac{n+1}{2})}\right] \bigcup_{\circ} \left[\, G_{(\frac{m+1}{2}),(\frac{n-1}{2})}\right] \bigcup_{\circ} \left[\, G_{(\frac{m-1}{2}),(\frac{n+1}{2})}\right] \bigcup_{\circ} \left[\, G_{(\frac{m-1}{2}),(\frac{n-1}{2})}\right].$$

**Proposition 2.2.** Let $P_m$ and $C_n$ be path graph and cycle graph with m and n vertices respectively.

(a) If $n$ is an even integer, then $P_m \times_2 C_n$ has four components which are semi tied grid graphs.

(b) If $n$ is an odd integer, then $P_m \times_2 C_n$ has two components which are semi tied grid graphs.

**Proof:**

(a) Let $m$ and $n$ be even integers, then from Proposition 2.1(a) we have

$$P_m \times_2 P_n = \left[ \bigcup_{\substack{\circ \\ i=1}}^{4} \left( G_{(\frac{m}{2}),(\frac{n}{2})} \right)^{(i)} \right]$$

Now in graph $P_n$, we add one edge between $v_1$ to $v_n$, then it becomes a cycle graph $C_n$. Also $d(v_1, v_{n-1}) = 2 = d(v_2, v_n)$ in $C_n$ and so the edges between $(u_i, v_1)$ and $(u_i, v_{n-1})$ as well as $(u_i, v_2)$ and $(u_i, v_n)$, for each $i$, will be added in the resultant graph. Thus the grid component $G_{(\frac{m}{2}),(\frac{n}{2})}$ becomes semi tied grid graph $G_{(\frac{m}{2}),((\frac{n}{2})^0)}$ in $P_m \times_2 C_n$.

So we get $P_m \times_2 C_n = \left[ \bigcup_{\substack{\circ \\ i=1}}^{4} \left( G_{(\frac{m}{2}),((\frac{n}{2})^0)} \right)^{(i)} \right]$.

Similarly, $P_m \times_2 C_n = \left[ \bigcup_{\substack{\circ \\ i=1}}^{2} \left( G_{(\frac{m+1}{2}),((\frac{n}{2})^0)} \right)^{(i)} \right] \bigcup_{\circ} \left[ \bigcup_{\substack{\circ \\ j=1}}^{2} \left( G_{(\frac{m-1}{2}),((\frac{n}{2})^0)} \right)^{(j)} \right]$, if $m$ is odd.

(b) Let $m$ be even integers and $n$ be odd integer, then from Proposition 2.1(c),

$$P_m \times_2 P_n = \left[ \bigcup_{\substack{\circ \\ i=1}}^{2} \left( G_{(\frac{m}{2}),(\frac{n+1}{2})} \right)^{(i)} \right] \bigcup_{\circ} \left[ \bigcup_{\substack{\circ \\ j=1}}^{2} \left( G_{(\frac{m}{2}),(\frac{n-1}{2})} \right)^{(j)} \right]$$

Note that here for $i = 1$ and 2 we get two components $G_{(\frac{m}{2}),(\frac{n+1}{2})}$ of same graph. The vertices are joined as, $(u_i, v_1) \longrightarrow (u_i, v_3) \longrightarrow \ldots \longrightarrow (u_i, v_n)$ in one component and $(u_i, v_2) \longrightarrow (u_i, v_4) \longrightarrow \ldots \longrightarrow (u_i, v_{n-1})$ are joined in other component. Again, $d(v_n, v_2) = 2 = d(v_{n-1}, v_1)$ in $C_n$. So, the two edges $(u_i, v_n) \longrightarrow (u_i, v_2)$ and $(u_i, v_{n-1}) \longrightarrow (u_i, v_1)$ will be added in the resultant graph $P_m \times_2 C_n$ and the two components are joined by these edges. So, we get one component $G_{(\frac{m}{2}),(n^0)}$ in place of $\bigcup_{\substack{\circ \\ i=1}}^{2} \left( G_{(\frac{m}{2}),(\frac{n+1}{2})} \right)^{(i)}$. Consequently, $P_m \times_2 C_n$ has only two components as follows:

$$P_m \times_2 C_n = \left[ \bigcup_{\substack{\circ \\ i=1}}^{2} \left( G_{(\frac{m}{2}),(n^0)} \right)^{(i)} \right].$$

Similarly if $m$ is odd, then

$$P_m \times_2 C_n = \left[G_{(\frac{m+1}{2}),(n^0)}\right] \bigcup_{\circ} \left[G_{(\frac{m-1}{2}),(n^0)}\right]$$

■

**Proposition 2.3.** Let $C_m$ and $C_n$ be cycle graphs with $m$ and $n$ vertices respectively.

(a) If both $m$ and $n$ are even integers, then $C_m \times_2 C_n$ has four components which are tied grid graphs.

(b) If $m$ is odd and $n$ is even, then $C_m \times_2 C_n$ has two components which are tied grid graphs.

(c) If both $m$ and $n$ are odd integers, then $C_m \times_2 C_n$ is a connected graph which is a tied grid graph.

**Proof:**

(a) Let $m$ and $n$ be even integers, then from Proposition 2.2 (a),

$$P_m \times_2 C_n = \left[\bigcup_{\substack{\circ \\ i=1}}^{4} \left(G_{(\frac{m}{2}),((\frac{n}{2})^0)}\right)^{(i)}\right].$$

Also $d(u_1, u_{m-1})$ = 2 = $d(u_2, u_m)$ in $C_m$ and so the edges between $(u_1, v_j)$ and $(u_{m-1}, v_j)$ as well as $(u_2, v_j)$ and $(u_m, v_j)$, for each $j$ will be added in the resultant graph. Thus the semi tied grid component $G_{(\frac{m}{2}),((\frac{n}{2})^0)}$ becomes a tied grid graph $G_{((\frac{m}{2})^0),((\frac{n}{2})^0)}$ in $C_m \times_2 C_n$. So we get

$$C_m \times_2 C_n = \left[\bigcup_{\substack{\circ \\ i=1}}^{4} \left(G_{((\frac{m}{2})^0),((\frac{n}{2})^0)}\right)^{(i)}\right].$$

(b) Let m be odd integer and n be even integer. As we have seen in Proposition 2.2 (a), the two components $G_{(\frac{m+1}{2}),((\frac{n}{2})^0)}$; for $i = 1$ and 2 joined by the edges $(u_m, v_j) \leftrightarrow (u_2, v_j)$ and $(u_{m-1}, v_j) \leftrightarrow (u_1, v_j)$ and give one component $G_{(m^0),((\frac{n}{2})^0)}$ in $C_m \times_2 C_n$. Similarly the other two components $G_{(\frac{m-1}{2}),((\frac{n}{2})^0)}$; for $j = 1$ and 2 give one more tied grid graph in the resultant graph. So we get

$$C_m \times_2 C_n = \left[\bigcup_{\substack{\circ \\ i=1}}^{2} \left(G_{(m^0),((\frac{n}{2})^0)}\right)^{(i)}\right].$$

(c) If $m$ and $n$ both are odd integers then by the same argument given in (b), two different components are joined by the edges $(u_i, v_n) \longleftrightarrow (u_i, v_2)$; for every $i$. Also $(u_i, v_1) \longleftrightarrow (u_i, v_{n-1})$ edges are added in the graph $C_m \times_2 C_n$. So it gives only one component, that is, $G_{(m^0),(n^0)}$, which is a tied grid graph in $C_m \times_2 C_n$. So, the resultant graph is connected. ■

**Remark 2.4.** The number of components in $P_m \times_2 P_n$ is fixed (four), whereas in case of $P_m \times_2 C_n$ and $C_m \times_2 C_n$, the number of components depends on the parity of $m$ and $n$.

**Proposition 2.5.** Let $K_{s,t}$ be a complete bipartite graph and $P_m$ be a path graph with $m$ vertices. Then $K_{s,t} \times_2 P_m$ has exactly four components.

**Proof:** Let $K_{s,t}$ be a complete bipartite graph with $U_1$ and $U_2$ partite sets. It is clear that two vertices lie in the same partite set if and only if distance between them is two. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$. $d(v_j, v_{j+2}) = 2$ for $j = 1, 2, \ldots, m-2$ in $P_m$. Two vertices say, $(u, v_j)$ and $(u, v_{j+2})$ in $K_{s,t} \times_2 P_m$ are adjacent for $j = 1, 2, \ldots, m-2$. Also, two vertices say, $(u, v_j)$ and $(u', v_j)$ in $K_{s,t} \times_2 P_m$ are adjacent only if $u$ and $u'$ are in same partite set of $K_{s,t}$ and $j = 1, 2, \ldots, m$. Hence, for each $u \in U_1$, we get $(u, v_j) \longleftrightarrow (u, v_{j+2})$ for $j = 1, 3, \ldots, m-2$ and $(u, v_j) \longleftrightarrow (u, v_{j+2})$ for $j = 2, 4, \ldots, m-3$. Therefore, there are two components each isomorphic to $K_s \times P_{\frac{m}{2}}$ if $m$ is even or two components $K_s \times P_{\frac{m+1}{2}}$ and $K_s \times P_{\frac{m-1}{2}}$ if $m$ is odd.

Similarly, with respect to the other partite set $U_2$ of $K_{s,t}$ there are two components each isomorphic to $K_t \times P_{\frac{m}{2}}$ if $m$ is even or two components $K_t \times P_{\frac{m+1}{2}}$ and $K_t \times P_{\frac{m-1}{2}}$ if $m$ is odd.

Thus in total, $K_{s,t} \times_2 P_m$ has exactly four components. ■

**Proposition 2.6.** Let $K_{s,t}$ be complete bipartite graph and $C_m$ be cycle graph with $m$ vertices.

(a) If $m$ is an even integer then $K_{s,t} \times_2 C_m$ has four components.

(b) If $m$ is odd an integer then $K_{s,t} \times_2 C_m$ has two components.

**Proof:**

(a) Let $m$ be even. By Proposition 2.5, $K_{s,t} \times_2 P_m$ has four components. As $C_m$ is obtained from $P_m$ by adding an edge $v_1 \longleftrightarrow v_m$, in each of the four components of $K_{s,t} \times_2 P_m$ we have to add the edges $(u, v_1) \longleftrightarrow (u, v_{m-1})$ or $(u, v_2) \longleftrightarrow (u, v_m)$ for each vertex $u$ in $K_{s,t}$. Consequently, we get two components isomorphic to $K_s \times C_{\frac{m}{2}}$ and two components isomorphic to $K_t \times C_{\frac{m}{2}}$.

(b) Let $m$ be odd integer, then the two components $K_s \times P_{\frac{m+1}{2}}$ and $K_s \times P_{\frac{m-1}{2}}$ of $K_{s,t} \times_2 P_m$ are joined by adding the edges $(u, v_m) \longleftrightarrow (u, v_2)$ and $(u, v_1) \longleftrightarrow (u, v_{m-1})$ for every vertex $u$ in $K_{s,t}$. Consequently, we have one component $K_s \times C_m$. Hence, $K_{s,t} \times_2 C_m$ has four components.

Similarly, the two components $K_t \times P_{\frac{m+1}{2}}$ and $K_t \times P_{\frac{m-1}{2}}$ of $K_{s,t} \times_2 P_m$ are joined by adding the edges $(u, v_m) \longleftrightarrow (u, v_2)$ and $(u, v_1) \longleftrightarrow (u, v_{m-1})$ for every vertex $u$ in $K_{s,t}$ and consequently, we have one component $K_t \times C_m$. Thus in total $K_{s,t} \times_2 C_m$ has two components. ■

From the above theorem, it is evident that the number of components in $K_{s,t} \times_2 C_m$ depends only on the parity of the integer $m$. All the above results reveal that, even if $G_1$ and $G_2$ are connected the resultant graph $G_1 \times_2 G_2$ need not be connected. So in 2- cartesian product the result similar to the following result is not true.

**Proposition 2.7.** [6] Let $G = G_1 \times G_2$, with $G_1$ and $G_2$ both connected graphs. Then $G$ is connected.

**Proposition 2.8.** [1] If $G$ and $H$ are two connected bipartite graphs, then $G \times_2 H$ has exactly four connected components.

In [2], the derived graph $G^!$ of $G$ is defined in terms of $d(u, v) = 2$. Also, the graph $G^2$ is obtained by considering $2$ - distance. The $2$ - cartesian product can be obtained with the help of $G^!$ or $G^2$ or combination of $G$ and $G^2$ using usual cartesian product. But, in this paper we consider 2-cartesian product separately as we have obtained $G_1 \times_2 G_2$ and their results directly in terms of the factor graphs $G_1$ and $G_2$, without computing the graphs $G^!$ or $G^2$.

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