International Journal of Mathematics and Soft Computing
Vol.4, No.1 (2014), 119 - 128.

IJMSC

ISSN Print : 2249 - 3328
ISSN Online: 2319 - 5215

Ascending Domination Decomposition of Graphs

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Abstract

In this paper, we combine decomposition and domination and introduce the concept Ascending Domination Decomposition (ADD) of a graph G. An ADD of a graph G is a collection $\psi = \{G_1, G_2, \ldots, G_n\}$ of subgraphs of G such that, each G_i is connected, every edge of G is in exactly one G_i and $\gamma(G_i) = i$, $1 \leq i \leq n$. In this paper, we prove K_n, W_n and $K_{1,n}$ admit ADD. We also establish the characterization for the path and cycle that they should admit ADD. We also prove that the corona of path, cycle and star admit ADD.

Keywords: Domination, decomposition, ascending domination decomposition.
AMS Subject Classification(2010): 05C69, 05C70.

1 Introduction

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by p and q respectively. For terms not defined here we refer to Harary [6] .

Definition 1.1. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 (with p_1 vertices) and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2. In particular, the graph $G \odot K_1$ is denoted by G^+. The graph P_n^+ is called comb and the graph C_n^+ is called a crown.

The subgraph induced by the subset $S \subseteq V$, is denoted by $< S >$.

The theory of domination is one of the fastest growing areas in graph theory, which has been investigated by Hedetniemi [5] , and Walikar et al [8] . A set $D \subseteq V$ of vertices in a graph G is a dominating set if every vertex v in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$.

Another important area of graph theory is decomposition of graphs [7]. A decomposition of a graph G is a collection ψ of edge disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G is in exactly one G_i. If each G_i is isomorphic to a subgraph H of G, then ψ is called a H-decomposition. Several authors studied various types of decompositions by imposing conditions on G_i in the decomposition.

We introduce a concept called Ascending Domination Decomposition (ADD) of a graph which is motivated by the concepts of Ascending Subgraph Decomposition (ASD) and Continuous Monotonic Decomposition (CMD) of a graph. The concept of Ascending Subgraph Decomposition was introduced by Alavi et al [1].

Definition 1.2. [1] A decomposition of G into subgraphs G_i (not necessarily connected) such that $|E(G_i)| = i$ and G_i is isomorphic to a proper subgraph of G_{i+1}, is called an Ascending Subgraph Decomposition(ASD).

Definition 1.3. *[4]* A decomposition $\{G_1, G_2, \ldots, G_n\}$ of G is said to be a Continuous Monotonic Decomposition (CMD) if each G_i is connected and $|E(G_i)| = i$ for each $i = 1, 2, \ldots, n$.

The concept of continuous monotonic decomposition was introduced by N. Gnana Dhas [4]. In this paper, we initiate a study on ADD.

2 Main Results

We define Ascending Domination Decomposition (ADD) as follows.

Definition 2.1. An ADD of a graph G is a collection $\psi = \{G_1, G_2, \ldots, G_n\}$ of subgraphs of G such that

(i) Each G_i is connected.

(ii) Every edge of G is in exactly one G_i

(iii) $\gamma(G_i) = i$, $1 \le i \le n$.

If a graph G has an ADD, we say that G admits Ascending Domination Decomposition.

Example 2.2. An ADD $\{G_1, G_2, G_3\}$ of a given graph G is given in Figure 1. Note that $\gamma(G_i) = i$ for $i = 1, 2, 3$.

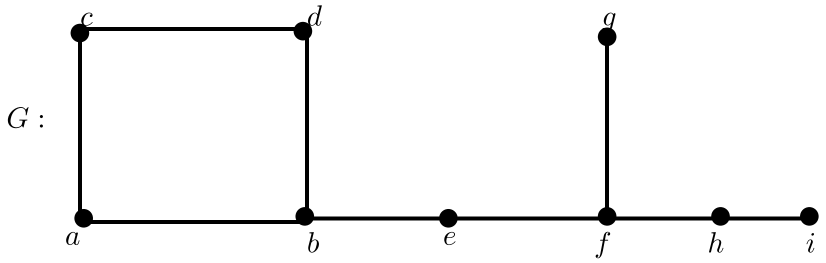

Figure 1(a): *A given graph G.*

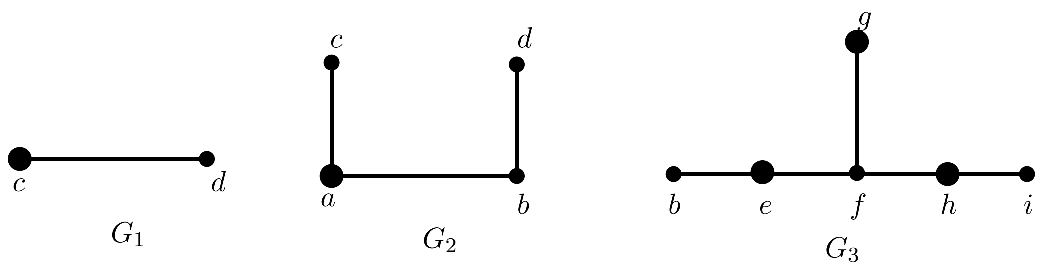

Figure 1(b): *An ADD $\{G_1, G_2, G_3\}$ of G.*

Theorem 2.3. For a graph G, $\gamma(G) = 1$ if and only if $\psi = \{G\}$ is an ADD.

Proof: The proof is obvious. ∎

Corollary 2.4. K_n, W_n and $K_{1,n}$ admit ADD.

Theorem 2.5. Complete bipartite graph $K_{m,n}$ admits ADD.

Proof: Let $V = X \cup Y$ be a bipartition of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Let G_1 be a subgraph obtained from $K_{m,n}$ by taking the vertex x_1 and the edges adjacent to x_1. Then $G_1 \cong K_{1,n}$ and $\gamma(G_1) = 1$. We also see that $G_2 = K_{m,n} - G_1 \cong K_{m-1,n}$ and $\gamma(G_2) = 2$. Hence $\psi = \{G_1, G_2\}$ is a ADD for $K_{m,n}$. ∎

Remark 2.6. [8] A path having $3k-3$, $3k-2$ or $3k-1$ edges has the domination number k.

Using Remark 2.6, we can find bound for q of path if it admits ADD.

Theorem 2.7. A path P_p has an ADD ψ={$G_1, G_2, \ldots, G_n$} if and only if $\frac{3n^2-3n+2}{2} \leq q \leq \frac{3n^2+n}{2}$.

Proof: Let $P_p = v_1 v_2 \ldots v_p$ be a path with $q = p-1$ edges.
Suppose P_p admits ADD $\psi = \{G_1, G_2, \ldots, G_n\}$.
First, we take the minimum possibility

$$G_1 = v_1 v_2$$
$$G_2 = v_2 v_3 v_4 v_5$$
$$G_3 = v_5 v_6 v_7 v_8 v_9 v_{10} v_{11}$$
$$\vdots$$
$$G_k = v_m v_{m+1} \ldots v_r \text{ where } m = \frac{3k^2-9k+10}{2},\ r = \frac{3k^2-3k+4}{2}$$
$$\vdots$$
$$G_n = v_l v_{l+1} \ldots v_p \text{ where } l = \left(\frac{3n^2-9n+10}{2}\right),\ p = \frac{3n^2-3n+4}{2}$$

From Remark 2.6, it is clear that $\gamma(G_i) = i$. So $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD.
Then $\sum_{i=1}^{n} q(G_i) = 1 + 3 + 6 + \ldots + 3n - 3$
$= \frac{3n^2-3n+2}{2}$

For the maximum possibility, we take

$$G_1 = v_1 v_2 v_3$$
$$G_2 = v_3 v_4 v_5 v_6 v_7 v_8$$
$$G_3 = v_8 v_9 v_{10} v_{11} v_{12} v_{13} v_{14} v_{15} v_{16}$$
$$\vdots$$
$$G_k = v_m v_{m+1} \ldots v_r \text{ where } m = \frac{3k^2-5k+4}{2},\ r = \frac{3k^2+k+2}{2}$$
$$\vdots$$
$$G_n = v_l v_{l+1} \ldots v_p \text{ where } l = \left(\frac{3n^2-5n+4}{2}\right),\ p = \frac{3n^2+n+2}{2}.$$

From Remark 2.6, it follows that $\gamma(G_i) = i$ and $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD.
Then $\sum_{i=1}^{n} q(G_i) = 2 + 5 + 8 + \ldots + 3n - 1$

$= \frac{3n^2+n}{2}$.

Thus a path P_p has an ADD which implies that $\frac{3n^2-3n+2}{2} \leq q \leq \frac{3n^2+n}{2}$.

Conversely, suppose P_p does not admit ADD.

Consider the decomposition in (1), which admits ADD.

$$G_1 = v_1v_2v_3$$
$$G_2 = v_3v_4v_5v_6v_7v_8$$
$$G_3 = v_8v_9v_{10}v_{11}v_{12}v_{13}v_{14}v_{15}v_{16}$$
$$\vdots$$
$$G_k = v_mv_{m+1}\dots v_r \text{ where } m = \left(\frac{3k^2-5k+4}{2}\right),\ r = \frac{3k^2+k+2}{2}$$
$$\vdots$$
$$G_n = v_lv_{l+1}\dots v_p \text{ where } l = \left(\frac{3n^2-5n+4}{2}\right),\ p = \frac{3n^2+n+2}{2}.$$

First we prove that if we add $1, 2, \dots, n+1$ edges to each G_i of ψ then the resulting decomposition does not admit ADD.

We add $1,\ 2$ or 3 vertices to each G_i for $i = 1, 2, \dots, n$. then we get $\sum\limits_{i=1}^{n} q(G_i) = \frac{3n^2+n}{2} + n$ or $\frac{3n^2+n}{2} + 2n$ or $\frac{3n^2+n}{2} + 3n$ respectively.

Note that $\gamma(G_i) = i+1$ for $i = 1, 2, \dots, n$, which implies that this does not admit ADD. Next, if we add 4, 5 or 6 vertices to each G_i for $i = 1, 2, \dots, n$ we get $\sum\limits_{i=1}^{n} q(G_i) = \frac{3n^2+n}{2} + 4n$ or $\frac{3n^2+n}{2} + 5n$ or $\frac{3n^2+n}{2} + 6n$. we have that $\gamma(G_i) = i+2$ for $i = 1, 2, \dots, n$ which implies that this does not admit ADD. Continuing in this way, at last if we add $(n+1)$ vertices to each G_i for $i = 1, 2, \dots, n$, we get $\sum\limits_{i=1}^{n} q(G_i) = \frac{3n^2+n}{2} + n(n+1)$.

Note that $\gamma(G_i) = i + \frac{n+1}{3}$ for $i = 1, 2, \dots, n$ which implies this does not admit ADD.

In general, we add $1, 2, \dots, or (n+1)$ edges to each G_i, for $i = 1, 2, \dots, n$, we get $q = \sum\limits_{i=1}^{n} q(G_i) = \frac{3n^2+n}{2} + jn$, for $j = 1, 2, \dots, n+1$. But $\frac{3n^2+n}{2} + jn > \frac{3n^2+n}{2}$, for $j = 1, 2, \dots, n+1$ which contradicts our assumption that $q \leq \frac{3n^2+n}{2}$.

Next, we consider the decomposition in (2) which admits ADD. We remove the last $(n-1)$-edges from G_n, the resulting decomposition does not admit ADD.

Even, if we rearrange the edges in this decomposition in any order it is not an ADD. Hence, $q = \sum\limits_{i=1}^{n} q(G_i) = \frac{3n^2-3n+2}{2} - (n-1) = \frac{3n^2-5n+4}{2} < \frac{3n^2-3n+2}{2}$, which is a contradiction. Hence P_p admits ADD. ∎

Corollary 2.8. If $\frac{3n^2+n}{2} < q < \frac{3n^2+3n+2}{2}$, then P_p does not admit ADD.

Theorem 2.9. A cycle C_p has an ADD $\psi = \{G_1, G_2, \dots, G_n\}$ if and only if $\frac{3n^2-3n+2}{2} \leq q \leq \frac{3n^2+n}{2}$.

Proof: The proof is the same as in Theorem 2.7. ∎

Theorem 2.10. P_p^+ has an ADD $\psi = \{G_1, G_2, \dots, G_n\}$ if and only if P_p has $\frac{n(n+1)}{2}$ vertices.

Proof: Let $P_p = \{v_1, v_2, \dots, v_p\}$ be a path. If we attach the vertices $v_1', v_2', \dots, v_p'$ to $v_1, v_2, \dots, v_p$ respectively, then we get P_p^+.

Suppose $p = \frac{n(n+1)}{2}$.

Let $G_1 =< \{v_1, v_1'\} >$

$G_2 =< \{v_1, v_2, v_3, v_2', v_3'\} >$

$G_3 =< \{v_3, v_4, v_5, v_6, v_4', v_5', v_6'\} >$

$\vdots$

$G_n = \langle \{v_l, v_{l+1}, \ldots v_p, v_{l+1}', \ldots, v_p'\} \rangle$

Clearly $\gamma(G_i) = i$ for $i = 1, 2, \ldots, n$.

We observe that the minimum dominating set of G_n has n vertices and P_p has $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ vertices . Clearly $\gamma(G_i) = i,$ for $i = 1, 2, \ldots, n$ and hence $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD of P_p^+.

Conversely, suppose P_p^+ has an ADD.

Suppose P_p does not have $\frac{n(n+1)}{2}$ vertices.

Then we have the following two possibilities.

(i) Suppose we add j vertices for $j = 1, 2, \ldots,$ or n in P_p. By the above, after constructing $\{G_1, G_2, \ldots, G_n\}$, we have j remaining vertices where $j = 1, 2, \ldots,$ or n. But, we cannot adjust these vertices in the minimum dominating sets of G_i, otherwise $\{G_1, G_2, \ldots, G_n\}$ would not be an ADD for P_p^+.

If these j vertices ($j = 1, 2, \ldots,$ or n) alone contribute a subgraph G_{k_j}, then $\gamma(G_{k_j}) = j$ for $j = 1, 2, \ldots, n$. Then $\{G_1, G_2, \ldots, G_n, G_{k_j}\}$ would not be an ADD.

This is a contradiction to our assumption.

(ii) Suppose we remove $1, 2, \ldots,$ or $n - 1$ vertices in P_p. By the above, after constructing $\{G_1, G_2, \ldots, G_n\}$, we have j remaining vertices where $j = n - 1, n - 2, \ldots,$ or $n - (n - 1)$ in G_n. But we cannot adjust these vertices in the minimum dominating sets of G_i, otherwise $\{G_1, G_2, \ldots, G_n\}$ would not be an ADD for P_p^+ .

If these j vertices ($j = n - 1, n - 2, \ldots, 2,$ or 1) alone contribute a subgraph G_{k_j}, then $\gamma(G_{k_j}) = j$ for $i = n - 1, n - 2, \ldots, 2$ or 1. Then $\{G_1, G_2, \ldots, G_{n-1}, G_{k_j}\}$ would not be an ADD, which is a contradiction to our assumption. ∎

Example 2.11. An ADD $\{G_1, G_2, G_3, G_4\}$ of P_{10}^+ is given in Figure 2.

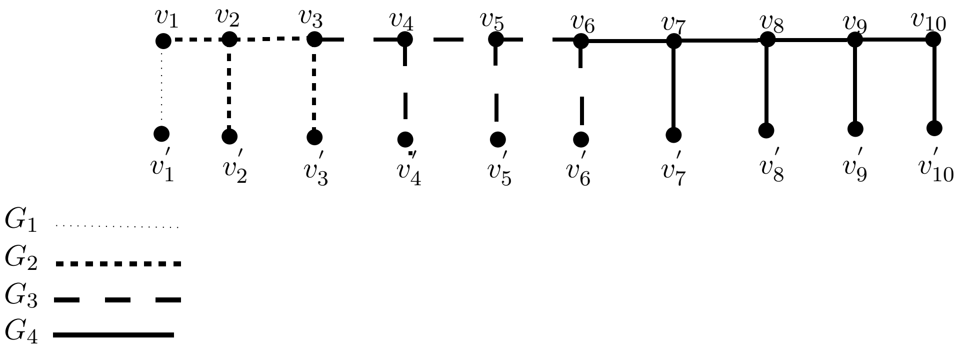

Figure 2: *An ADD* $\{G_1, G_2, G_3, G_4\}$ *of* P_{10}^+.

Theorem 2.12. ${C_p}^+$ has an ADD $\psi = \{G_1, G_2, \ldots, G_n\}$ if and only if C_p has either $\frac{n(n+1)}{2}$ or $\frac{n(n+1)}{2} - 1$ vertices.

Proof: Let $C_p = v_1 v_2 \ldots v_p$ be a cycle. If we attach the vertices $v_1^{'}, v_2^{'}, \ldots, v_p^{'}$ to $v_1, v_2, \ldots, v_p$ respectively, then we get ${C_p}^+$.

Case (i) **:** Suppose $p = \frac{n(n+1)}{2}$.

To prove C_p^+ has an ADD.

Let $G_1 = < \{v_1, v_2, v_3, v_2'\} >$

$G_2 = < \{v_3, v_4, v_5, v_3', v_4'\} >$

$G_3 = < \{v_5, v_6, v_7, v_8, v_5', v_6', v_7'\} >$

$\vdots$

$G_n = < \{v_l, v_{l+1}, \ldots, v_p, v_1, v_l', v_{l+1}', \ldots, v_p', v_1'\} >$

The minimum dominating set of G_n has n vertices and C_p has $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ vertices. Clearly $\gamma(G_i) = i$ for $i = 1, 2, \ldots, n$ and hence $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD of C_p^+.

Case (ii) **:** Suppose $p = \frac{n(n+1)}{2} - 1$.

Let $G_1 = < \{v_1, v_2\} >$

$G_2 = < \{v_2, v_3, v_4, v_5, v_2', v_3'\} >$

$G_3 = < \{v_4, v_5, v_6, v_7, v_4', v_5', v_6', v_7'\} >$

$\vdots$

$G_n = < \{v_l, v_{l+1}, \ldots, v_p, v_1, v_l', v_{l+1}', \ldots, v_p', v_1'\} >$

Note that, the minimum dominating set of G_n has n vertices.

We have already taken $\{v_1\}$ as a minimum dominating set of G_1 and therefore it has been counted in the total number of vertices in C_p. We see that v_1 is one of the elements of the minimum dominating set of G_n. Thus G_1 and G_n have the same vertex v_1 in their respective dominating sets and so we subtract 1 from the total number of vertices in C_p.

Note that C_p has $(1 + 2 + 3 + \ldots + n) - 1 = \frac{n(n+1)}{2} - 1$ vertices. Clearly $\gamma(G_i) = i$ for $i = 1, 2, \ldots, n$. Thus $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD of C_p^+.

Conversely, suppose C_p^+ has an ADD

To prove C_p has $\frac{n(n+1)}{2}$ or $\frac{n(n+1)}{2} - 1$ vertices. Suppose C_p does not have $\frac{n(n+1)}{2}$ and $\frac{n(n+1)}{2} - 1$ vertices.

Case (iii) **:** C_p does not have $\frac{n(n+1)}{2}$ vertices.

We have the following possibilities.

(i) Suppose we add j vertices for $j = 1, 2, \ldots,$ or $n-1$ in C_p, as in the case (i), after constructing $\{G_1, G_2, \ldots, G_n\}$, we have j remaining vertices where $j = 1, 2, \ldots,$ or $n-1$. But, we cannot adjust these vertices in the minimum dominating sets of G_i, otherwise $\{G_1, G_2, \ldots, G_n\}$ would not be an ADD for C_p^+.

If these j vertices ($j = 1, 2, \ldots,$ or $n-1$) alone contribute a subgraph G_{k_j}, then $\gamma(G_{k_j}) = j$ for $j = 1, 2, \ldots,$ or $n-1$. Then $\{G_1, G_2, \ldots, G_n, G_{k_j}\}$ would not be an ADD.

This is a contradiction to our assumption.

(ii) Suppose we remove $2, 3, \ldots,$ or $n-1$ vertices in C_p. As in the case (i), after constructing $\{G_1, G_2, \ldots, G_n\}$, we have j remaining vertices where $j = n-2, n-3, \ldots,$ or $n-(n-1)$ in G_n. But we cannot adjust these vertices in the minimum dominating sets of G_i, otherwise $\{G_1, G_2, \ldots, G_n\}$ would not be an ADD for C_p^+.

If these j vertices ($j = n-2,\ n-3,\ \ldots,$ or $n-(n-1)$) alone contribute a subgraph G_{k_j}, then $\gamma(G_{k_j})=j$ for $j = n-2,\ n-3,\ \ldots,$ or $n-(n-1)$. Then $\{G_1, G_2, \ldots, G_{n-1}, G_{k_j}\}$ would be not an ADD.

This is a contradiction to our assumption.

Case (iv) : C_p does not have $\frac{n(n+1)}{2} - 1$ vertices.

We have the following possibilities.

(i) Suppose we add j vertices for $j = 2,\ 3, \ldots,$ or n in C_p, as in the case (ii) ,after constructing $\{G_1, G_2, \ldots, G_n\}$, we have j remaining vertices where $j = 2,\ 3,\ \ldots,$ or n. But, we cannot adjust these vertices in the minimum dominating sets of G_i, otherwise $\{G_1, G_2, \ldots, G_n\}$ would not be an ADD for C_p^+.

If these j vertices ($j = 2,\ 3,\ \ldots,$ or n) alone contribute a subgraph G_{k_j}, then $\gamma(G_{k_j})=j$ for j $= 2,\ 3,\ \ldots,$ or n. Then $\{G_1, G_2, \ldots, G_n, G_{k_j}\}$ would not be an ADD for C_p^+.

This is a contradiction to our assumption.

(ii) Suppose we remove $1,\ 2,\ 3,\ \ldots,$ or $n-2$ vertices in C_p, as in the case (ii), after constructing $\{G_1, G_2, \ldots, G_n\}$, we have j remaining vertices where $j = n-1,\ n-2,\ \ldots,$ or $n-(n-2)$ in G_n. But we cannot adjust these vertices in the minimum dominating sets of G_i, otherwise $\{G_1, G_2, \ldots, G_n\}$ would not be an ADD for C_p^+.

If these j vertices ($j = n-1,\ n-2, \ldots,$ or $n-(n-2)$) alone contribute a subgraph G_{k_j}, then $\gamma(G_{k_j})=j$ for $j = n-1,\ n-2,\ \ldots,$ or $n-(n-2)$. Then $\{G_1, G_2, \ldots, G_{n-1}, G_{k_j}\}$ would be not an ADD. This is a contradiction to our assumption. ∎

Example 2.13. An ADD $\{G_1, G_2, G_3, G_4\}$ of C_{10}^+ for $\frac{n(n+1)}{2}$ vertices is given in Figure 3.

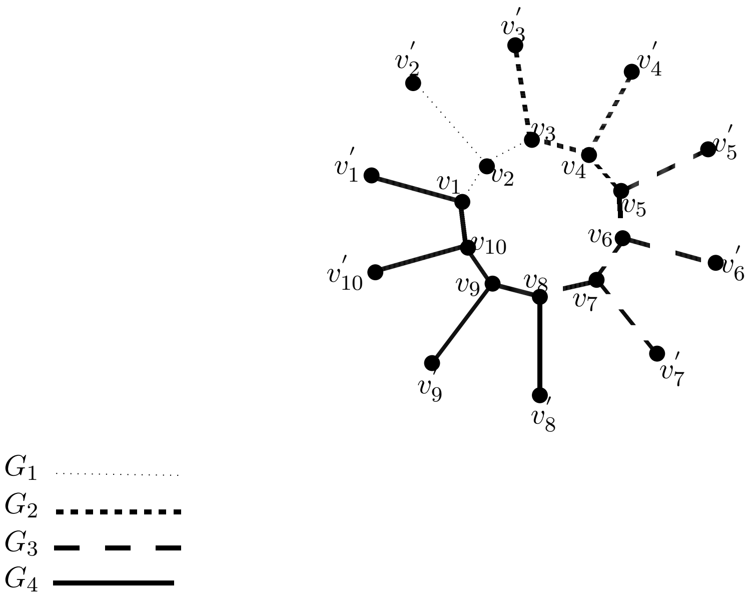

Figure 3: *An* ADD $\{G_1, G_2, G_3, G_4\}$ *of* C_{10}^+.

Example 2.14. An ADD $\{G_1, G_2, G_3, G_4\}$ of C_9^+ for $\frac{n(n+1)}{2} - 1$ vertices is given in Figure 4.

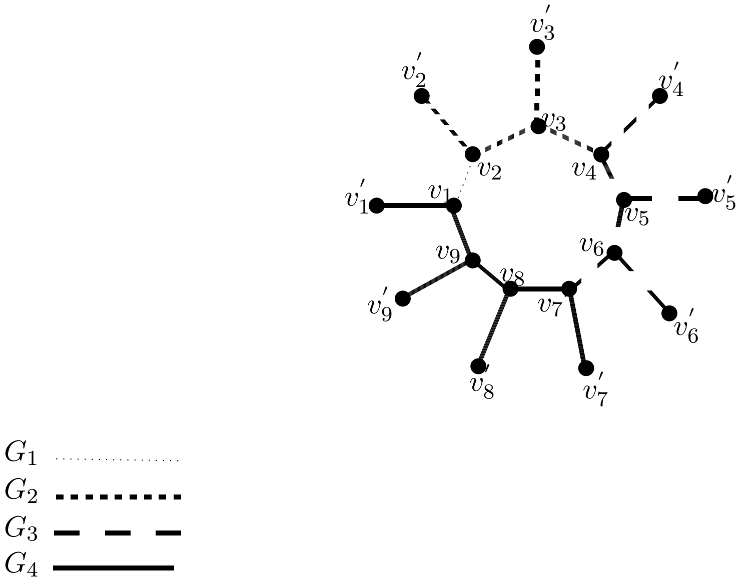

Figure 4: *An* ADD $\{G_1, G_2, G_3, G_4\}$ *of* C_9^+.

Theorem 2.15. $K_{1,p}^+$ has an ADD $\psi = \{G_1, G_2, \ldots, G_n\}$ if and only if $K_{1,p}$ has either $\frac{n(n+1)}{2}$ or $\frac{n(n+1)}{2} - 1$ pendent vertices.

Proof: Let v be the central vertex and $u, v_1, v_2, \ldots, v_p$ be the pendent vertices of $K_{1,p}$. If we attach the vertices $v_1', v_2', \ldots, v_p'$ to $v_1, v_2, \ldots, v_p$ respectively, then we get $K_{1,p}^+$.

Case (i) **:** Suppose $K_{1,p}$ has $\frac{n(n+1)}{2}$ pendent vertices.

Let $G_1 = < \{v, v_1, v_1'\} >$

$G_2 = < \{v, v_2, v_3, v_2', v_3'\} >$

$G_3 = < \{v, v_4, v_5, v_6, v_4', v_5', v_6'\} >$

$\vdots$

$G_n = < \{v, v_l, v_{l+1}, \ldots, v_p, v_l', v_{l+1}', \ldots, v_p'\} >$.

The minimum dominating set of G_n has n vertices and C_p has $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ vertices. Clearly, $\gamma(G_i) = i$ for $i = 1, 2, \ldots, n$. Thus $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD of $K_{1,p}^+$.

Case (ii) **:** Suppose $K_{1,p}$ has $\frac{n(n+1)}{2} - 1$ pendent vertices.

To prove that $K_{1,p}$ has an ADD.

Let $G_1 = < \{v_1, v_1'\} >$

$G_2 = < \{v, v_1, v_2, v_2'\} >$

$G_3 = < \{v, v_3, v_4, v_5, v_3', v_4', v_5'\} >$

$\vdots$

$G_n = < \{v, v_l, v_{l+1}, \ldots, v_p, v_l', v_{l+1}', \ldots, v_p'\} >$

The minimum dominating set of G_n has n vertices.

Clearly, $\gamma(G_i) = i$ for $i = 1, 2, \ldots, n$. Thus $\psi = \{G_1, G_2, \ldots, G_n\}$ is an ADD of $K_{1,p}^+$.

Conversely, suppose $K_{1,p}^{+}$ has an ADD.
To prove $K_{1,p}$ has either $\frac{n(n+1)}{2}$ or $\frac{n(n+1)}{2} - 1$ pendent vertices.
Suppose $K_{1,p}$ does not have $\frac{n(n+1)}{2}$ and $\frac{n(n+1)}{2} - 1$ pendent vertices.
Case (iii)**:** $K_{1,p}$ does not have $\frac{n(n+1)}{2}$ pendent vertices.
Then we have the following possibilities.

(i) Suppose we add j pendent vertices for $j = 1, 2, \ldots,$ or n in $K_{1,p}$, as in the case (iii) of Theorem 2.12, we get a contradiction.

(ii) Suppose we remove $2, 3, \ldots,$ or $n-1$ pendent vertices in $K_{1,p,}$ as in the case (iii) of Theorem 2.12, again we get a contradiction.

Case (iv)**:** $K_{1,p}$ does not have $\frac{n(n+1)}{2} - 1$ vertices.
Then we have the following possibilities.

(i) Suppose we add j vertices for $j = 2, 3, \ldots,$ or n in $K_{1,p}$, as in the proof of case (iv) Theorem 2.12, we get a contradiction to our assumption.

(ii) Suppose we remove $1, 2, 3, \ldots,$ or $n-2$ pendent vertices in $K_{1,p}$, as in the proof of case (iv) of Theorem 2.12, we get a contradiction to our assumption. ∎

Example 2.16. An ADD $\{G_1, G_2, G_3, G_4\}$ of $K_{1,10}^{+}$ for $\frac{n(n+1)}{2}$ vertices is given below.

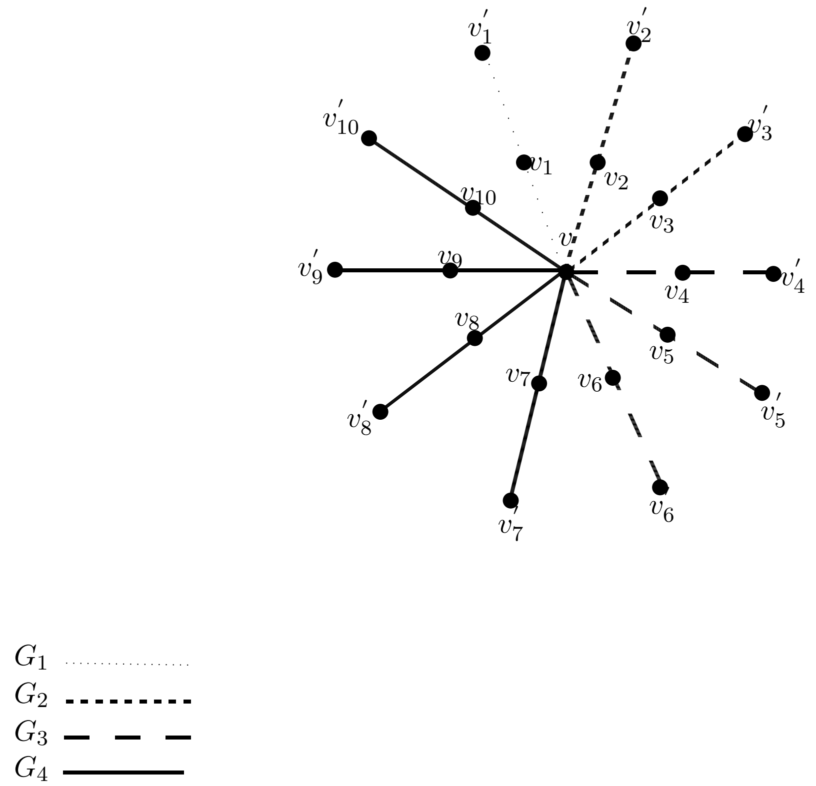

Figure 5: *An* ADD $\{G_1, G_2, G_3, G_4\}$ *of* $K_{1,10}^{+}$.

Example 2.17. An ADD $\{G_1, G_2, G_3, G_4\}$ of $K_{1,9}^{+}$ for $\frac{n(n+1)}{2} - 1$ vertices is given below.

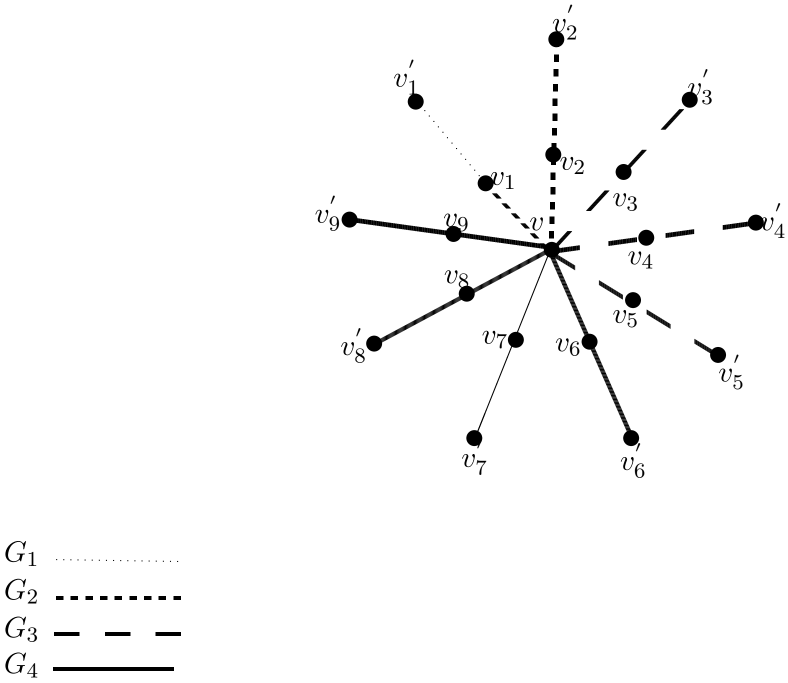

Figure 6: *An* $ADD\ \{G_1, G_2, G_3, G_4\}$ *of* $K_{1,9}^{+}$.

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