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Critical graphs in total K domination

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Abstract

In this paper we study the concept of edge critical graph with respect to total K domination and the characterization of edges whose deletion or addition are responsible for the decrement or increment in the total K domination number of a graph.

Keywords: Total K Dominating Set, minimal total K dominating set , minimum total K dominating set ,Total K domination number.
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1 Introduction

Let G be a graph. The number of elements in a minimum dominating set S of G is called the the domination number of the graph and is denoted by $\gamma(G)$. If we remove an edge from graph or if we add an edge between two non-adjacent vertices then the domination number may increase or decrease or remain unchanged. This topic has been studied by several authors. Reader may refer to references. We shall prove results regarding this phenomenon for total K Dominating Set. Total K domination number was introduced by Ning Li and X. Hou [7].

In this paper, we assume that K is any integer and $K \geq 1$. All graphs considered are simple. If u and v are end vertices of any edge e then e is also be denoted by $e = uv$. We consider only those graphs for which the minimum degree is at least $K + 1$. So when an edge is removed, the resulting graph has minimum degree at least K.

Definition 1.1. Let G be a graph and K be an integer , $K \geq 1$. A subset S of $V(G)$ is said to be a total K dominating set if for every vertex $v \in V(G)$, v is adjacent to at least K vertices of S.

Definition 1.2. A total K dominating set S is said to be a minimal total K dominating set if no proper subset $S' \subseteq S$ is a total K dominating set.

Definition 1.3. A minimal total K dominating set with minimum cardinality is called minimum total K dominating set.

Definition 1.4. The Total K domination number γ_{TK} of a graph G equals the minimum cardinality of a total K dominating set in G.

Every totally K dominating set is a K dominating set , however the converse is not true.

2 Main Results

It is useful to partition the edges of G into two sets according to how their removal affects its total K domination number. Let $E = E^+_{TK} \bigcup E^0_{TK}$ where $E^+_{TK} = \{e \in E(G)/\gamma_{TK}(G-e) > \gamma_{TK}(G)\}$ and $E^0_{TK} = e \in E(G)/\gamma_{TK}(G-e) = \gamma_{TK}(G)$.

We begin with the case of removing an edge from the given graph in which the total K domination number of the graph does not decrease.

Lemma 2.1. Let G be a graph and $e = uv$ be an edge of graph G then $\gamma_{TK}(G-e) \geq \gamma_{TK}(G)$.

Proof: Let G be a graph and $e = uv$ be an edge of graph G. Suppose $\gamma_{TK}(G-e) < \gamma_{TK}(G)$. Let T be γ_{TK} set of graph $G-e$ then $|T| < \gamma_{TK}(G)$. There are four possibilities for γ_{TK} set T of graph $G-e$.

(1) u belongs to T, v belongs to T.

(2) u does not belong to T, v does not belong to T.

(3) u belongs to T,v does not belong to T.

(4) u does not belong to T, v belongs to T.

In all the cases, T is a total K dominating set in G also and $|T| < \gamma_{TK}(G)$, which is a contradiction. Hence, $\gamma_{TK}(G-e) < \gamma_{TK}(G)$ is not possible. Therefore, $\gamma_{TK}(G-e) \geq \gamma_{TK}(G)$. ∎

Theorem 2.2. Let G be a graph and S be a minimum total K domination set of G. Then, an edge $e = uv \in E^+_{TK}$ if and only if one of the following conditions holds:

1. $u \in S$, $v \notin S$ and v is adjacent to exactly K vertices of S including u.
2. $u \in S$, $v \in S$ then v is adjacent to exactly K vertices of S including u or u is adjacent to exactly K vertices of S including v.

Proof: Let S be a minimum total K domination set of a graph G and $e = uv$ be an edge of G. First we prove that the condition is sufficient. Let S be a minimum total K domination set of a graph G and $e = uv$ be an edge of G. Suppose one of these two conditions holds. If we remove an edge $e = uv$ then S is not a total K dominating set in graph $G-e$. So, by lemma 2.1, $\gamma_{TK}(G-e) \geq \gamma_{TK}(G)$.

Suppose $\gamma_{TK}(G-e) = \gamma_{TK}(G)$. Let T be a minimum total K domination set of $G-e$ with $|T| = |S|$. Then we have the following two cases.

Case 1: u belongs to T and v does not belong to T.

By definition of total K dominating set, v is adjacent to at least K vertices of T in $G-e$. T is a total K dominating set of G also. Since $|T| = |S|$, T is a minimum total K domination set of G also. Moreover, u belongs to T, v does not belong to T and v is now adjacent to at least K + 1 vertices of T including u which contradicts condition (1).

Case 2: u belongs to T and v belongs to T.

By definition of total K dominating set, v is adjacent to at least K vertices of T and u is adjacent to at least K vertices of T. Further T is a total K dominating set of graph G and $|T| = |S|$. So, T is a minimum total K domination set of G. Moreover, u belongs to T, v belongs to T and v is now adjacent to at least K+1 vertices of T including u in G and u is now adjacent to at least K+1 vertices of T including v in G, which contradicts condition (2).

So, $\gamma_{TK}(G-e) = \gamma_{TK}(G)$ is not possible. Hence, $\gamma_{TK}(G-e) > \gamma_{TK}(G)$ and $e = uv \in E^{+}_{TK}$.

Next we prove the conditions are necessary. Let $e = uv \in E^{+}_{TK}$. Hence the removal of $e = uv$ increases the total K domination number of G. Let S be minimum total K domination set of G.

Suppose $u \notin S$ and $v \notin S$. Then S is a total K dominating set of $G - e$ also. Hence $\gamma_{TK}(G-e) = \gamma_{TK}(G)$, which is a contradiction. Hence, either $u \in S$ or $v \in S$. Without loss of generality, we assume that $u \in S$. Then the following two cases arise.

Case 1: $u \in S$ and $v \notin S$.

Suppose v is adjacent to at least $K + 1$ vertices of S including u. Then even after removing $e = uv$, v is adjacent to at least K vertices of S and $e = uv \notin E^{+}_{TK}$. This is a contradiction. Suppose v is adjacent to at most $K - 1$ vertices of S including u, then S is not a total K dominating set, which is a contradiction. Hence, $u \in S$, $v \notin S$ and v is adjacent to exactly K vertices of S including u.

Case 2: Let $u \in S$ and $v \in S$.

Suppose v is adjacent to at least $K + 1$ vertices of S including u. Then even after removing $e = uv$, v is adjacent to at least K vertices of S and $e = uv \notin E^{+}_{TK}$, which is a contradiction. Suppose v is adjacent to at most $K - 1$ vertices of S including u, then S is not a total K dominating set, which is again a contradiction. Hence, $u \in S$, $v \in S$ and v is adjacent to exactly K vertices of S including u. Similarly u is adjacent to exactly K vertices of S including v. ■

Theorem 2.3. Let G be a graph and $e = uv$ be an edge of G. Then $e \in E^{0}_{TK}$ if and only if there exist a minimum total K dominating set S of G such that one of the following conditions holds:

1. $u \notin S$ and $v \notin S$.
2. $u \in S$, $v \notin S$ and v is adjacent to at least $K + 1$ vertices of S.
3. $u \in S$, $v \in S$ then u and v are adjacent to at least $K + 1$ vertices of S.

Proof: Suppose one of these conditions holds. Then if we remove the edge $e = uv$ then still S is a total K dominating set of the graph $G - e$ also. Therefore, S is a minimum total K dominating set of $G - e$. Hence, $\gamma_{TK}(G-e) = |S| = \gamma_{TK}(G)$. Therefore, $e = uv$ belongs to E^{0}_{TK}.

To prove the other part, let $e = uv \in E^{0}_{TK}$. That means the removal of the edge $e = uv$ does not affect the total K domination number of G. Let S be a minimum total K dominating set of $G - e$. Then $|S| = \gamma_{TK}(G)$. We have the following three possibilities.

1. Suppose $u \notin S$ and $v \notin S$, then theorem is proved.
2. Suppose $u \in S$ and $v \notin S$. Then v is adjacent to at least K vertices of S in $G-e$. Hence, v is adjacent to at least $K + 1$ vertices including u of S in graph G. Thus, S is a minimum total K domination set of G which satisfies the required conditions.
3. Suppose $u \in S$ and $v \in S$. Now u and v are adjacent to at least K vertices of S in $G - e$. Hence, u and v are adjacent to at least $K + 1$ vertices of S in G. Thus, S is a minimum total K domination set of G which satisfies the required conditions. ■

Example 2.4. Here we give an example of a graph with total K (K=2) dominating set and an edge whose removal does not affect the total K (K=2) domination number.

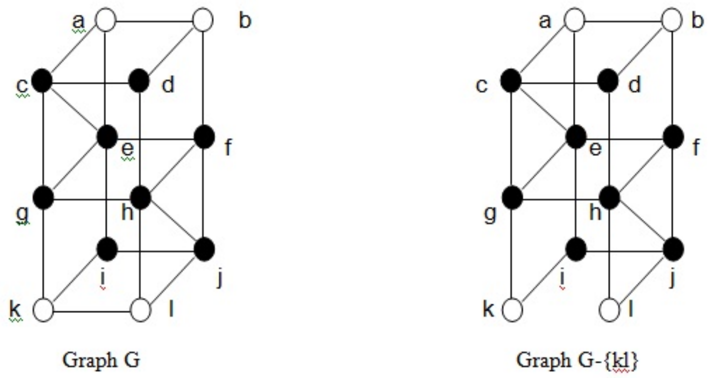

Figure 1.

Example 2.5. Here we give an example of a graph with total K (K=2) dominating set and an edge whose removal increases the total K (K=2) domination number. Here we can see that E^{+}_{T2}= $\{ae, ei\}$.

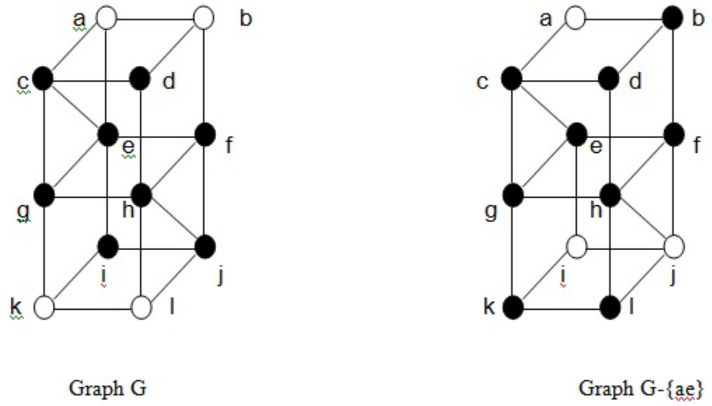

Figure 2.

Now we consider the case of adding an edge between two non-adjacent vertices of a given graph. It is noted in this case that the total K domination number of the graph does not increase. Moreover, we assume that the minimum degree of the given graph is at least K.

Theorem 2.6. Let u and v be two non - adjacent vertices of a graph G. Then $\gamma_{TK}(G+uv) < \gamma_{TK}(G)$ if and only if there exists a total K dominating set S of the subgraph $G-v$ such that $|S| < \gamma_{TK}(G)$ and one of the following conditions holds:

1. $u \in S$, $v \notin S$ and v is adjacent to exactly $K-1$ vertices of S.
2. $u \in S$, $v \in S$ and v is adjacent to at least $K-1$ vertices of S.

Proof: Suppose there exists a total K dominating set S of the subgraph $G-v$ such that $|S| < \gamma_{TK}(G)$ and one of the following conditions holds. Since S is a total K dominating set of $G-v$ we have $S+v$ is a total K dominating set of G. If we add $e = uv$ then S becomes a total K dominating set in $G+uv$ and $|S| < \gamma_{TK}(G)$. Hence, $\gamma_{TK}(G+uv) < \gamma_{TK}(G)$

To prove the other part, let us suppose that S be a minimum total K dominating set in $G+uv$ such that $|S| < \gamma_{TK}(G)$. Since u and v are two non-adjacent vertices we have $\gamma_{TK}(G+uv) < \gamma_{TK}(G)$.

Suppose $u \notin S$ and $v \notin S$. Then S is a total K dominating set in G, which is not possible since $|S| < \gamma_{TK}(G)$. Hence at least one of the two vertices must be in S. Without loss of generality, we assume that $u \in S$. Then the following two cases arise.

Case 1: Suppose $u \in S$ and $v \notin S$. By Theorem 2.2, v is adjacent to exactly K vertices of S including u in $G + uv$. Therefore, v is adjacent to exactly $K - 1$ vertices of S.

Case 2: Suppose $u \in S$ and $v \in S$. By Theorem 2.2 , v is adjacent to exactly K vertices of S including u in $G + uv$. Therefore, v is adjacent to exactly $K - 1$ vertices of S. ■

Example 2.7. For the graph given in Figure 3, $\gamma_{T2}(G + kj) < \gamma_{T2}(G)$.

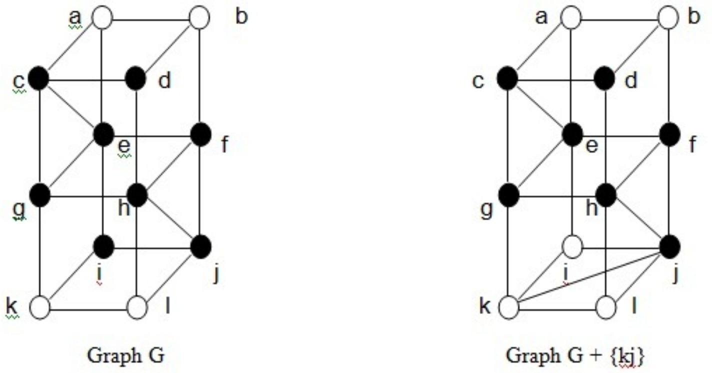

Figure 3.

Definition 2.8. Let G be a graph and $e = uv$ be an edge of G. Then G is said to be γ^{+}_{TK} critical graph if $\gamma_{TK}(G - uv) > \gamma_{TK}(G)$ for all edges of graph G.

Definition 2.9. Let G be a graph and $e = uv$ be an edge of graph G. Then G is said to be γ^{0}_{TK} critical graph if $\gamma_{TK}(G - uv) = \gamma_{TK}(G)$ for all edges of graph G.

Example 2.10. The Peterson graph is a γ^{0}_{TK} critical graph (K=2). Here, $\gamma_{T2}(G - uv) = \gamma_{T2}(G)$ for all edges $uv \in G$.

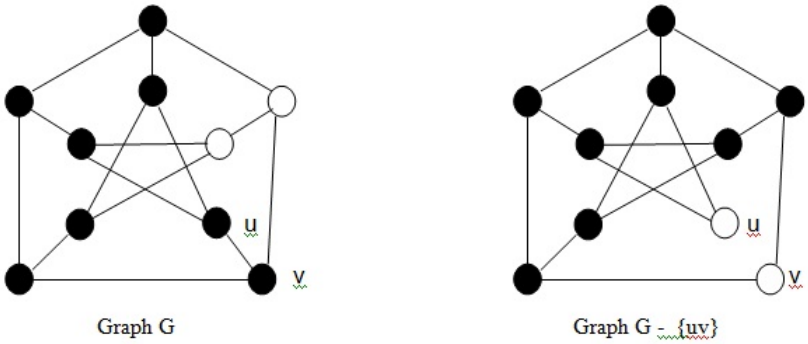

Figure 4.

Lemma 2.11. Let G be a graph and $deg(v) \geq K+1$ for all v and $K \geq 1$. Then there is an edge $e = uv$ such that $\gamma_{TK}(G-e) = \gamma_{TK}(G)$.

Proof: Let G be a graph and $deg(v) \geq K+1$ for all v, $K \geq 1$. Let S be a minimum total K dominating set of graph G. Since $\delta(G) \geq K+1$, S must be a proper subset of $V(G)$. So at least one vertex v does not belong to S . We have $deg(v) \geq K+1$ and let the vertices $u_1, u_2, u_3, , , , , u_K, u_{K+1}$ be adjacent to v . If all vertices $u_1, u_2, u_3, , , , , u_K, u_{K+1}$ are in S then $\gamma_{TK}(G-e) = \gamma_{TK}(G)$ for $e = uv_p$, $p = 1, 2, 3, , , , K+1$. If at least one of the above vertices, say u_r is not in S, then $\gamma_{TK}(G-e) = \gamma_{TK}(G)$ where $e = vu_r$.
Hence, in all cases there exists at least one edge $e = uv$ such that $\gamma_{TK}(G-e) = \gamma_{TK}(G)$. ■

Let G be a graph and S be a minimum total K dominating set of G. If G has x vertices outside of S then $|E^0_{TK}| > x$.

Conclusion: Here we have characterized edges which are responsible to increase or decrease K total domination number of graph . Similarly there are some vertices which are responsible to increase or decrease K total domination number of graph . It will be interesting if one can find the relation between these vertices and edges.

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