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Mathematical analysis of tip speed ratio of a wind turbine and its effects on power coefficient

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Abstract

In this paper we estimate the value of tip speed ratio of a wind turbine and evaluate its effects on power coefficient. The power coefficient, C_p, is achieved at a particular tip speed ratio, β, which is specific to the design of the turbine.

Keywords: Power coefficient, wind energy, tip speed ratio power output.
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1 Preliminaries and Notations

The power coefficient, C_p, is the most important parameter in the case of power regulation [5]. It is a non-linear function whose value is unique to each turbine type and is a function of wind speed that the turbine is operating in. Each turbine manufacturer provides look up tables for C_p for operation purposes. Other than look-up tables from turbine manufactured, models for power coefficient have been developed. For example in [4], the models have C_p as a function of the tip speed ratio and the blade pitch angle, θ, in degrees. The C_p is the fraction of the upstream wind power, which is captured by the rotor blades. The remaining power is discharged or wasted in the downstream wind. The factor C_p is called the power coefficient of the rotor or the rotor efficiency. For a given upstream wind speed, the value of C_p depends on the ratio of the downstream to the upstream wind speeds, that is V_o/V. The plot of power coefficient versus V_o/V shows that C_p is a single, maximum-value [3]. It has the maximum value of 0.59 when the V_o/V is one-third. The maximum power is extracted from the wind at that speed ratio when the downstream wind speed equals one-third of the upstream speed. Although the theoretical maximum value of C_p is 0.59, in practical designs, the maximum achievable C_p is below 0.5 for high-speed, two-blade turbines, and between 0.2 and 0.4 for slow speed turbines with more blades. The power coefficient C_p is also a parameter in the case of power regulation [1]. It is a function of the tip speed ratio β and the blade pitch angle θ in degrees that is,

$$C_p(\beta,\theta) = C_1(C_2\frac{1}{\lambda} - C_3\lambda\theta - C_4\theta^x - C_5)e^{-C_6\frac{1}{\lambda}} \tag{1.1}$$

where the values of the coefficients C_1 to C_6 and x depend on turbine type. θ is defined as the angle between the plane of rotation and the blade cross section chord. For particular turbine types $C_1 = 0.5$, $C_2 = 116$, $C_3 = 0.4$, $C_4 = 0$, $C_5 = 5$ and $C_6 = 21$, λ is defined by:

$$\frac{1}{\lambda} = \frac{1}{\beta + 0.08\theta} - \frac{0.035}{1 + \theta^3} \tag{1.2}$$

Anderson and Bose [2] suggested the following empirical relation for C_p.

$$C_p = \frac{1}{2}(\beta - 0.022\theta^2 - 5.6)e^{-0.17\beta} \tag{1.3}$$

where θ is the pitch angle of the blade in degrees, β is the tip speed ratio of the turbine defined by $\beta = \frac{V_b}{V_w}$ where V_b is the blade tip speed and V_w being the average of the velocities at the entry and exit of rotor blades of turbine.

2 Mathematical Formulation of Turbine Model

From a general power output model,

$$P_i = \frac{1}{2}\rho.A[V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}]^3_w \tag{2.1}$$

where ρ is the density of the moving air(wind), A is the area swept by the rotor blade and $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_w$ is the improved velocity of the wind, α is the wind friction coefficient, h_1 and h_2 are the wind turbine hub heights at level one and two respectively. The actual mechanical power extracted, P_e, by the rotor blades in watts is the difference between upstream and the down stream wind powers [3]. That is;

$$P_e = \frac{1}{2}\rho.A\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_w[\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}^2_u - \{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}^2_d] \tag{2.2}$$

where $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_w$ is the velocity of the wind, $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u$ is the upstream wind velocity at the entrance of the rotor blades in m/s and $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d$ is the down stream wind velocity at the exit of the rotor blades in m/s. In this paper, we assume that the density of air does not vary considerably even with variation in altitude or temperature since power in the wind is given by the rate of change of kinetic energy that is, $P = \frac{dE}{dt} = \frac{1}{2}\frac{dM}{dt}.V_w^2$. But we have mass flow rate $\frac{dM}{dt}$ given by $\frac{dM}{dt} = \rho.A.V_w$. In this work, V_w is improved and is represented by $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_w$, thus the mass flow rate is written as;

$$\rho.A\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_w = \frac{\rho.A[\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u + \{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d]}{2} \tag{2.3}$$

where $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_w$ is the average of the velocities at the entry and exit of the rotor blades of the turbine. With this expression Equation 2.2 becomes:

$$P_e = \frac{1}{2}\rho.A.\frac{[\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u + \{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d][\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}^2_u - \{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}^2_d]}{2} \tag{2.4}$$

which is simplified as:

$$\begin{aligned} P_e &= \frac{1}{2}\rho A\{\frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u[\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^2 - \{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^2]}{2} + \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d[\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^2 - \{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^2]}{2}\} \\ &= \frac{1}{2}\rho A\{\frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^3}{2} - \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^2}{2} + \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^2\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d}{2} - \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^3}{2}\} \\ &= \frac{1}{2}\rho A\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^3 \frac{\{1 - \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^2}{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^2} + \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d}{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u} - \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^3}{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^3}\}}{2} \\ &= \frac{1}{2}\rho A\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^3.C_p. \end{aligned}$$

where

$$C_p = \frac{\{1 - \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^2}{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^2} + \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d}{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u} - \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d^3}{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u^3}\}}{2} \tag{2.5}$$

In other words we can write C_p as $C_p = \frac{(1-\beta^2+\beta-\beta^3)}{2}$ and by factorization,

$$C_p = \frac{(1+\beta)(1-\beta^2)}{2}, \tag{2.6}$$

where,

$$\beta = \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d}{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u}. \tag{2.7}$$

The power coefficient is not static and it varies with the tip speed ratio β of the turbine.

3 Approximation of tip speed ratio

The tip speed ratio can also be expressed as [3]:

$$\beta = \frac{Angular\ speed\ of\ turbine\ (\omega) \times R}{Wind\ speed}, \tag{3.1}$$

but $\omega = \frac{\theta}{t}$, where θ is the angular displacement of rotor blade and t is the time taken. $\theta = \frac{S}{R}$ where S is the linear displacement of the rotor blade and R is the radius of the area swept by the rotor that is, length of the rotor blade.
Therefore; $S = R\theta$ and $\frac{S}{t} = \frac{R\theta}{t} = R\omega$. Thus,

$$\beta = \frac{linear\ velocity\ of\ the\ rotor\ blade}{wind\ velocity}$$

but,

$$linear\ velocity\ = \frac{The\ circumference\ of\ the\ area\ swept\ by\ the\ rotor\ blade}{time\ taken\ for\ one\ complete\ oscillation}$$

that is, linear velocity=$\frac{2\Pi.R}{T}$. Therefore we conclude that the tip speed ratio of the turbine can also be estimated as follows:

$$Tip\ speed\ ratio = \frac{2\Pi.R}{T.V} \tag{3.2}$$

$$Tip\ speed\ ratio = \frac{2\Pi.R}{\mu} \tag{3.3}$$

where R is the length of the rotor blade,μ is the product of T which is the time taken for one complete oscillation of the rotor blade and V the speed of wind.

4 Tip speed ratio analysis and control of the power coefficient

Taking the expression of β as given in Equation 3.2, we approximate the value of tip speed ratio using the length of the rotor blade of a typical wind turbine, $R = 42m$, approximate wind velocity, $V = 11m/s$ and approximate time for one complete oscillation $t = 3s$, $\beta = 2 \times \frac{22}{7} \times 42 \times \frac{1}{3\times 11} = 8$. Taking approximate values of β close to the value 8 above, that is 5,6,7,8,9,10,11 and 12 for verification, then taking the model in Equation 1.3, $C_p = \frac{1}{2}(\beta - 0.022\theta^2 - 5.6)e^{-0.17\beta}$. we use MATLAB to verify and perform an analysis of the variation of C_p against β. By varying θ, that is, for $\theta = 0, 4, 8$ and 12 we obtain graphs below.

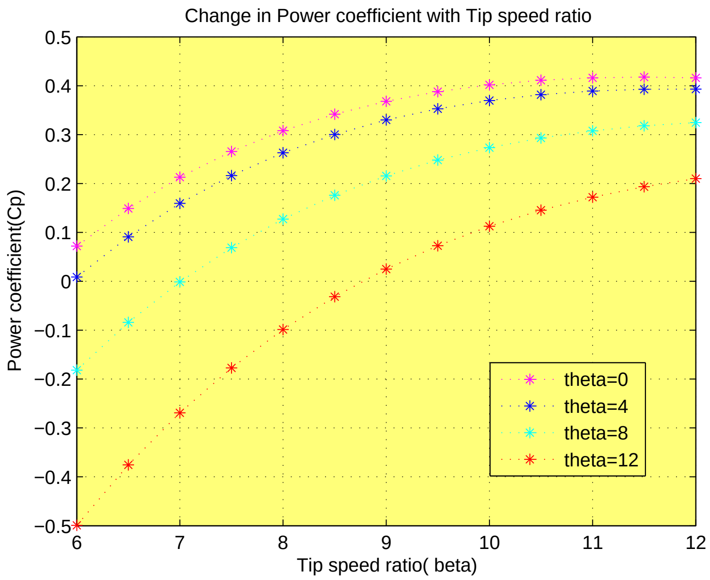

Figure 1: Power coefficient variation with tip speed ratio.

It is observed that the value of power coefficient much depends on the tip speed ratio of the turbine which also vary with the length of rotor blade. From Equation 2.2, the power generated by the turbine increases when the power coefficient increases, that is, power output is directly proportional to the power coefficient. A study of the graphs in Figure 1 shows that the angle of inclination of the turbine blade(pitch angle) also plays an important role in the power output. at $\theta = 0$, the value of C_p is maximum this value reduces as the pitch angle increases. This is the important fact to be considered when pitching the turbine blades.

5 Approximation of downstream velocity

We evaluate the value of β that maximizes C_P. Differentiating Equation 2.6, with respect to β and equating to zero, we find the value of β that makes C_p maximum.

$$\frac{dC_p}{d\beta} = -2\beta(1+\beta) + (1-\beta^2) = 0$$

This yields $\beta = -1$ or $\frac{1}{3}$ We therefore conclude that $\beta = \frac{1}{3}$ makes the value of C_p maximum. Now substituting this into Equation 2.6, we get;

$$C_p = \frac{(1+\frac{1}{3})(1-\frac{1}{9})}{2} = \frac{16}{27} = 59.3.$$

According to the Albert Betz, German physicist no wind turbine can convert more than 59.3 percent of the kinetic energy of the wind into mechanical energy turning a rotor. Thus $C_pmax = 0.59$ which is also referred to as Betz limit. Taking the value of $\beta = \frac{1}{3}$ and substituting in Equation 2.7, assuming an ideal turbine we get;

$$\beta = \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d}{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u} = \frac{1}{3} \tag{5.1}$$

From Equation 5.1, the value of downstream velocity $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d$ is not known while that of upstream $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u$ can be calculated as follows:
Let an approximate value of wind velocity be $10m/s$, hub heights of the turbine be $h_2 = 81m$, $h_1 = 80m$ and friction coefficient $\alpha = 0.1$. Thus upstream velocity $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_u = 10(\frac{81}{80})^{\frac{1}{0.1}} = 11.3227m/s$.

Therefore, $\beta = \frac{\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d}{11.3227} = \frac{1}{3}$ and $\{V(\frac{h_2}{h_1})^{\frac{1}{\alpha}}\}_d = \frac{11.3227}{3} = 3.7742m/s$.

Hence 3.7742m/s is the approximated value of downstream velocity. We have been able to approximate this value which is much less due to the fact that part of the speed of wind has been used to drive the turbine for power production.

An open problem

Turbine blade undergoes some forces as it goes round, one of which is centripetal force. How does centripetal force affect the power output in the turbine?

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