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Distant Divisor Graph of Subdivision of a Graph

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Abstract

Let $G = (V, E)$ be a (p, q) - graph. A shortest path P is called a distant divisor path if $l(P)$ divides q. Distance divisor graph $D(G)$ of a graph $G = (V, E)$ has the vertex set $V = V(G)$ and two vertices in $D(G)$ are adjacent if they have the distant divisor path in G. In this paper, we deal with distant divisor graph of subdivision of some graphs such as path, cycle, star and complete bipartite graph etc. Further we generalize the concept of antipodal graphs and prove that some distant divisor graphs are the generalized antipodal graphs. We also deal with the distant divisor graph of some graphs of diameter 4 and below.

Keywords: Distant divisor path, distant divisor graph, antipodal graph.
AMS Subject Classification(2010): 05C12, 05C76.

1 Introduction

By a graph, we mean a finite undirected graph without loops and multiple edges. For terms not defined here, we refer to Harary[9].

The distance between two vertices in a graph is the length of the shortest path between them. Number theory [1, 8, 10] has a strong impact on graph theory [5, 7, 9]. Using the divisibility concept in the number theory, we introduced the new concept called distant divisor graph[12]. In that paper[12], the distant divisor graphs of some standard graphs such as path, cycle, wheel, star, complete graph, complete bipartite graph and the like are studied.

We begin with the definitions of some number theoretic functions.

Notations 1.1. Let G be a (p, q)-graph. $k_1, k_2, \ldots, k_\tau$ denote the positive divisors of q with $k_1 = 1$ and $k_\tau = q$ and $k_1 < k_2 \ldots < k_\tau$. For $q = 12$, $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 6$ and $k_6 = 12$.

Definition 1.2. A number theoretic function f is said to be multiplicative if $f(mn) = f(m)f(n)$ whenever $gcd(m, n) = 1$.

Note that the functions τ and σ are both multiplicative functions.

Example 1.3. Let $m = 3$ and $n = 8$. Then $mn = 24$ and $gcd(3, 8) = 1$. $\tau(24) = 8$, $\tau(3) = 2$, $\tau(8) = 4$ and clearly $\tau(24) = \tau(3) \cdot \tau(8)$.
Also, we have $\sigma(24) = 60$, $\sigma(3) = 4$ and $\sigma(8) = 15$ and clearly $\sigma(24) = \sigma(3) \cdot \sigma(8)$. Hence, τ and σ are both multiplicative functions.

Definition 1.4. If $e = uv$ is an edge of G and w is not a vertex of G, the edge e is said to be $subdivided$ if it is replaced by edges uw and wv.

Definition 1.5. A subdivision graph $S(G)$ of a graph G is obtained by subdividing each edge of G only once.

One can easily observe that if G has p vertices and q edges then the subdivision graph $S(G)$ has $p + q$ vertices and $2q$ edges. Subdivision graph of a cycle is a cycle and that of a path is a path.

Definition 1.6. The k^{th} power G^k of a graph G is a graph that has the same set of vertices $V(G)$ and two vertices are adjacent when their distance in G is at most k. G^2 is the square of G and G^3 is the cube of G.

Definition 1.7. [3] Two vertices of a graph are said to be antipodal to each other if the distance between them is equal to the diameter of the graph.

Definition 1.8. [3] The antipodal graph of a graph G, denoted by A(G), has the vertex set as in G and two vertices are adjacent in A(G) if and only if they are antipodal in G.

Definition 1.9. [11] Two vertices of a graph are said to be radial to each other if the distance between them is equal to the radius of the graph.

Definition 1.10. [11] The radial graph of a graph G, denoted by R(G), has the vertex set as in G and two vertices are adjacent in R(G) if and only if they are radial in G.

Definition 1.11. [7] A graph is called chordal if every cycle of length strictly greater than 3 possesses a chord , that is, an edge joining two non-consecutive vertices of the cycle.

Definition 1.12. Let $G = (V, E)$ be a (p, q) - graph. A shortest path P is called a distant divisor path if its length $l(P)$ divides q.

Definition 1.13. The distance divisor graph $D(G)$ of a graph $G = (V, E)$ has the vertex set $V = V(G)$ and two vertices in $D(G)$ are adjacent if they have the distant divisor path in G.

Theorem 1.14. [12] For a cycle C_p, $D(C_p) \cong C_p$, if p is a prime. If p is not a prime, then $D(C_p)$ is either $(2\tau - 3)$ - regular or $(2\tau - 2)$ - regular when p is even or odd respectively.

Theorem 1.15. [12] For a path P_p, $q(D(P_p)) = \begin{cases} (\tau - 1)[2k_{\tau-1} + 1] - \sigma(q) + 1 + q, & \text{if } q \text{ is even} \\ \tau(q + 1) - \sigma(q), & \text{if } q \text{ is odd.} \end{cases}$

In this paper, we find the structure of the distant divisor graph of subdivision of graphs such as path, cycle and complete graphs.

2 Main Results

First we introduce the concept of k-chordal graphs as follows.

Definition 2.1. Two vertices of a (p, q)-graph G are said to be $k-chordal$ to each other if the distance between them is equal to k.

Definition 2.2. The $k - chordal$ graph of a graph G has the vertex set $V = V(G)$ and two vertices are said to be adjacent if they are $k - chordal$ in G and is denoted by $C_k(G)$.

Some $k - chordal$ graphs of a given graph G are given in Figure 1.

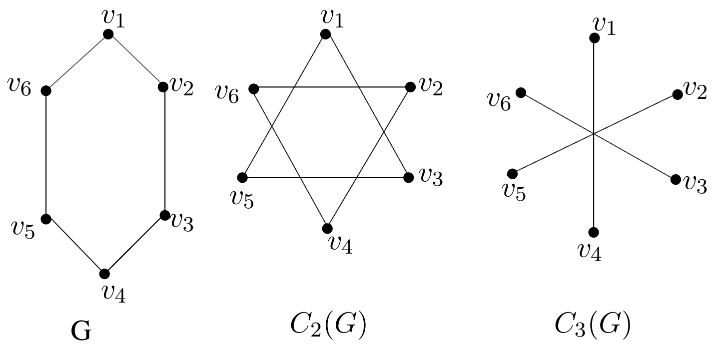

Figure 1.

Observation 2.3. $C_k(G) \cong A(G)$ if and only if $k = diam\, G$ and $C_k(G) \cong R(G)$ if and only if $k = rad\, G$.

Definition 2.4. The $(k_1, k_2, k_3, \ldots, k_r)$- chordal graph of a $(p, q) - graph\ G$ has the vertex set $V = V(G)$ and two vertices are said to be adjacent if the distance between them is $k_i (i = 1, 2, 3, \ldots, r)$ and $k_1 < k_2 < \cdots < k_r$ and it is denoted by $C_{(k_1,k_2,\ldots,k_r)}(G)$. We observe that $k_r \leq diam(G)$.

Observation 2.5. Let G be a (p, q)-graph. Then

1. If $k_1, k_2, \ldots, k_r$ are all divisors of q, then $C_{(k_1,k_2,\ldots,k_r)}(G) \cong D(G)$.

2. $C_{(1,2,\ldots,diam(G))}(G) \cong K_p$.

3. $C_{(1,2,\ldots,k)}(G) \cong G^k$.

4. If $k_1, k_2, \ldots, k_r$ are not consecutive numbers, then $C_{(1,2,3,\ldots,k_1,k_2,\ldots,k_r)}(G)$ contains the subgraph which is isomorphic to G^{k_1}.

5. $C_1(G) \cong G$.

6. $C_{(2,3,\ldots,diam(G))}(G) \cong \bar{G}$.

Theorem 2.6. For a path P_p,

$$q(D(S(P_p))) = \begin{cases} 5q+1, & \text{if } q \text{ is a prime number} \\ 2\tau(2q+1) - 3\sigma(q), & \text{if } q \text{ is an odd composite number} \\ \tau(2q+1) - \sigma(q) + 1, & \text{if } q = 2^{\alpha} \text{ where } \alpha \geq 1 \\ (\alpha+2)\tau(q_1)(2^{\alpha+1}q_1+1) - \sigma(q_1)(2^{\alpha+2}-1), & \text{if } q = 2^{\alpha}q_1 \text{ where} \\ & q_1 \text{ is an odd composite number.} \end{cases}$$

Proof: We observe that $S(P_p)$ is also a path of size $2q$.
From Theorem 1.15, $q(D(P_p)) = (\tau-1)(2k_{\tau-1}+1) - \sigma(q) + 1 + q$, if q is even.
Case (i)**:** Suppose q is prime.
Then $2q$ is even, the only divisors of $2q$ are $1, 2, q$ and $2q$. Since q is a prime,$(2, q) = 1$.
Hence $\tau(2q) = \tau(2)\tau(q) = 4$ and $\sigma(2q) = \sigma(2)\sigma(q) = 3(q+1)$. Here we note that $k_{\tau-1} = q$. Thus, $q(D(S(P_p))) = (4-1)(2q+1) - 3(q+1) + 1 + 2q = 5q+1$.
Case (ii)**:** Suppose q is odd composite.
Then $(2, q) = 1$ and $\tau(2q) = \tau(2)\tau(q) = 2\tau(q)$ and $\sigma(2q) = \sigma(2)\sigma(q) = 3\sigma(q)$. Thus, $q(D(S(P_p))) = (2\tau-1)(2q+1) - \sigma(2q) + 1 + 2q = (2\tau-1)(2q+1) - 3\sigma(q) + 1 + 2q = 2\tau(2q+1) - 3\sigma(q)$.
Case (iii)**:** Suppose q is even and is of the form $q = 2^{\alpha}$ where $\alpha \geq 1$. Then $2q = 2^{\alpha+1}$.
Now $\tau(q) = \tau(2^{\alpha}) = \alpha+1$ and $\tau(2q) = \tau(2^{\alpha+1}) = \alpha+1+1 = \tau(q)+1$. Note that $\sigma(q) = \sigma(2^{\alpha}) = 2^{\alpha+1}-1$ and $\sigma(2q) = \sigma(2^{\alpha+1}) = 2^{\alpha+2}-1 = 2\cdot 2^{\alpha+1}-1 = 2\sigma(q)-1$.
Thus, $q(D(S(P_p))) = (\tau+1-1)(2q+1) - (2\sigma(q)-1) + 1 + 2q = \tau(2q+1) - 2\sigma(q) - 1 + 1 + 2q = (\tau+1)(2q+1) - 2\sigma(q) - 1$.
Case (iv)**:** Suppose $q = 2^{\alpha}q_1$ where $\alpha \geq 1$ and q_1 is an odd composite.
Then $2q = 2^{\alpha+1}q_1$ and $\tau(q) = \tau(2^{\alpha}q_1) = \tau(2^{\alpha})\tau(q_1)$. Since $(2^{\alpha}, q_1) = 1$, $\tau(2^{\alpha})\tau(q_1) = (\alpha+1)\tau(q_1)$ and $\tau(2q) = (\alpha+2)\tau(q_1)$. Now, $\sigma(q) = \sigma(2^{\alpha}q_1) = \sigma(2^{\alpha})\sigma(q_1)$.

Since $(2^{\alpha}, q_1) = 1$, $\sigma(2^{\alpha})\sigma(q_1) = (2^{\alpha+1}-1)\sigma(q_1)$ and $\sigma(2q) = (2^{\alpha+2}-1)\sigma(q_1)$. Also note that $k_{\tau-1} = 2^{\alpha}q_1$. Thus,

$$\begin{aligned} q(D(S(P_p))) &= [(\alpha+2)\tau(q_1)-1](2\cdot 2^{\alpha}q_1+1) - (2^{\alpha+2}-1)\sigma(q_1) + 1 + 2^{\alpha+1}q_1 \\ &= [(\alpha+2)\tau(q_1)-1](2^{\alpha+1}q_1+1) - (2^{\alpha+2}-1)\sigma(q_1) + 1 + 2^{\alpha+1}q_1 \\ &= (\alpha+2)\tau(q_1)2^{\alpha+1}q_1 - (\alpha+2)\tau(q_1) - 2^{\alpha+2}\sigma(q_1) + \sigma(q_1) \\ &= (\alpha+2)\tau(q_1)(2^{\alpha+1}q_1+1) - \sigma(q_1)(2^{\alpha+2}-1). \end{aligned}$$

∎

Theorem 2.7. For a cycle C_p,

$$q(D(S(C_p))) = \begin{cases} 5p, \text{ if } p \text{ is prime} \\ p(4\tau-3), \text{ if } p \text{ is odd composite} \\ p(2\tau-1), \text{ if } p = 2^{\alpha} \text{ where } \alpha \geq 1 \\ p\left[(2\alpha+4)\tau(p_1)-3\right], \text{ if } p = 2^{\alpha}p_1 \text{ where } p_1 \text{ is odd composite} \end{cases}$$

Proof: Observe that $S(C_p)$ is also a cycle of size $2q = 2p$.
From Theorem 1.14, it is clear that,$q(D(C_p)) = \frac{p(2\tau-3)}{2}$ if p is even.
Case(i)**:** Suppose p is prime.
Then $\tau(2p) = 2\tau(p) = 2\tau = 4$. Thus, $q(D(S(C_p))) = \frac{2p(2\cdot 4-3)}{2} = 5p$.
Case(ii)**:** Suppose p is odd composite.
Then $\tau(2p) = \tau(2)\tau(p) = 2\tau$. Thus, $q(D(S(C_p))) = \frac{2p(2\cdot 2\tau-3)}{2} = p(4\tau - 3)$.
Case(iii)**:** Suppose p is even and $p = 2^\alpha$ where $\alpha \geq 1$.
Then $2p = 2^{\alpha+1}$ and $\tau(2p) = \alpha + 1 + 1 = \tau + 1$.
Thus, $q(D(S(C_p))) = \frac{2p(2(\tau+1)-3)}{2} = p(2\tau - 1)$.

Case(iv)**:** Suppose $p = 2^\alpha p_1$ where $\alpha \geq 1$ and p_1 is an odd composite.
Then, $2p = 2^{\alpha+1}$ and $\tau(2p) = (\alpha + 2)\tau(p_1)$.
Thus, $q(D(S(C_p)) = \frac{2p[2(\alpha+2)\tau(p_1)-3]}{2} = p\,[(2\alpha + 4)\tau(p_1) - 3]$. ∎

Theorem 2.8. For a complete graph K_p,

$$D(S(K_p)) \cong \begin{cases} \mathrm{S(K}_p)^2, \text{ if } 2q \ncong 0 \text{ (mod 3) and } 2q \ncong 0 \text{ (mod 4)} \\ \mathrm{S(K}_p)^3, \text{ if } 2q \cong 0 \text{ (mod 3) and } 2q \ncong 0 \text{ (mod 4)} \\ \mathrm{C}_{(1,2,4)}(S(K_p)), \text{ if } 2q \ncong 0 \text{ (mod 3) and } 2q \cong 0 \text{ (mod 4)} \\ \mathrm{K}_{p+q}, \text{ if } 2q \cong 0 \text{ (mod 3) and } 2q \cong 0 \text{ (mod 4).} \end{cases}$$

Proof: Let $v_1, v_2, \ldots, v_p$ be the vertices of K_p and $u_1, u_2, \ldots, u_q$ be the subdivisional vertices of K_p. Note that, $S(K_p)$ has $p + q$ vertices and $2q$ edges and $diam\ S(K_p) = 4$, Then, we check only for the numbers$1, 2, 3$ and 4 if they are divisors of $2q$.
In $S(K_p)$, we observe that
(i) $d(v_i, v_j) = 2$, for all $i \neq j$
(ii) $d(v_i, u_j) =$ either 1 or 3 where $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$.
(iii) $d(u_i, u_j) =$ either 2 or 4, for all $i \neq j$.
From the above observations, we have the following four cases.
Case(i)**:** Suppose $2q \ncong 0$ (mod 3) and $2q \ncong 0$ (mod 4).
Then, the divisors of $2q$ are only 1 and 2. So, we join the vertices u and v such that $d(u, v) \leq 2$. It follows that, $D(S(K_p)) \cong S(K_p)^2$.
Case (ii)**:** Suppose $2q \cong 0$(mod 3) and $2q \ncong 0$(mod 4).
Then the divisors of $2q$ are $1, 2$ and 3 only. So, we join the vertices u and v such that $d(u, v) \leq 3$. It follows that, $D(S(K_p)) \cong S(K_p)^3$.
Case(iii)**:** Suppose $2q \ncong$0(mod 3) and $2q \cong$ 0(mod 4).
Then the divisors of $2q$ are $1, 2$ and 4 only. So, we join the vertices u and v such that $d(u, v) \leq 2$ and $d(u, v) = 4$. For $d(u, v) \leq 2$, we see that $D(S(K_p))$ contains a subgraph $S(K_p)^2$.
For $d(u, v) = 4$ we observe that $D(S(K_p))$ contains the subgraph induced by the subdivisional vertices $u_1, u_2, \ldots, u_q$, which is complete.

Thus, $D(S(K_p)) \cong C_{(1,2,4)}(S(K_p))$.
Case(iv)**:** Suppose $2q \cong 0$(mod 3) and $2q \cong 0$(mod 4).
Then the divisors of $2q$ are 1,2,3 and 4. Hence all the paths of $S(K_p)$ are distant divisor paths and so we join all vertices of $S(K_p)$.
Hence, $D(S(K_p)) \cong K_{p+q}$. ∎

Corollary 2.9. For a complete graph K_p,

$$q(D(S(K_p)))=\begin{cases} \frac{p(p^2-1)}{2}, \text{ if 2q} \ncong 0(mod3) \text{ and 2q} \ncong 0(mod4) \\ \frac{p(p-1)(2p-1)}{2}, \text{ if 2q} \cong 0(mod3) \text{ and 2q} \ncong 0(mod4) \\ \frac{p^4-2p^3+11p^2-10p}{8}, \text{ if 2q} \ncong 0(mod3) \text{ and 2q} \cong 0(mod4) \\ \frac{(p-1)p(p+1)(p+2)}{8}, \text{ if 2q} \cong 0(mod3) \text{ and 2q} \cong 0(mod4) \end{cases}$$

Proof: Let $v_1, v_2, \ldots, v_p$ be the vertices of K_p and let $u_1, u_2, \ldots, u_q$ be the subdivisional vertices of K_p.

The following table shows the number of vertices having the distance 1, 2, 3 or 4 from the vertices of $D(S(K_p))$ which is very useful to find the degree of vertices of $D(S(K_p))$ and hence the number of edges.

Vertex v	Distance d	Number of vertices having distance d from v
v_i	1	$p-1$
	2	$p-1$
	3	$q-(p-1)$
	4	0
u_i	1	2
	2	2(p−2)
	3	p−2
	4	q−2(p−2)-1

From the table it is found that, in $S(K_p)$,the number of vertices having distance 1 from v_i is $p-1$ and that of distance 2 from v_i is also $p-1$ and that of distance 3 from v_i is $q-(p-1)$ and that of distance 4 from v_i is 0. The number of vertices having distance 1 from u_i is 2 and that of distance 2 from u_i is $2(p-2)$ and that of distance 3 from u_i is $p-2$ and that of distance 4 from u_i is $q-2(p-2)-1$.
Case(i)**:** Suppose $2q \ncong 0(mod3)$ and $2q \ncong 0(mod4)$.
Then the divisors of $2q$ are 1 and 2 only.
Hence in $D(S(K_p))$, $d(v_i) = 2(p-1)$, for $i = 1, 2, 3, \ldots, p$ and $d(u_i) = 2(p-2)+2$, for $i = 1, 2, 3, \ldots, q$. Thus, $q(D(S(K_p))) = \frac{p \cdot 2(p-1)+q \cdot [2(p-2)+2]}{2} = (p-1)(p+q) = \frac{p(p^2-1)}{2}$.
Case (ii)**:** Suppose $2q \cong 0(mod3)$ and $2q \ncong 0(mod4)$.
Then the divisors of $2q$ are 1, 2 and 3 only.
Thus, in $D(S(K_p))$, $d(v_i) = 2(p-1)+q-(p-1) = p+q-1$, for $i = 1, 2, 3, \ldots, p$ and $d(u_i) =$

$2(p-2)+2+p-2=3p-4$, for $i=1,2,3,\ldots,q$.
Thus, $q(D(S(K_p)))=\frac{p(p+q-1)+q(3p-4)}{2}=\frac{p(p-1)+4q(p-1)}{2}=\frac{p(p-1)(2p-1)}{2}$.
Case(iii)**:** Suppose $2q \ncong 0(mod 3)$ and $2q \cong 0(mod 4)$.
Then the divisors of $2q$ are 1, 2 and 4 only.
Thus, in $D(S(K_p)),\ d(v_i)=2(p-1)$, for $i=1,2,3,\ldots,p$ and $d(u_i)=q+1$, for $i=1,2,3,\ldots,q$.
Hence, $q(D(S(K_p)))$=$\frac{p\cdot 2(p-1)+q(q+1)}{2}=\frac{p^4-2p^3+11p^2-10p}{8}$.
Case(iv)**:** Suppose $2q \cong 0(mod 3)$ and $2q \cong 0(mod 4)$.
Then the divisors of $2q$ are 1, 2, 3 and 4. In this case, $D(S(K_p)) \cong K_{p+q}$.
Thus, $q(D(S(K_p)))=\frac{(p+q)(p+q-1)}{2}=\frac{(p-1)p(p+1)(p+2)}{8}$. ∎

Example 2.10.
Case(i)**:** Consider the graph K_{11}.
Here $p=11$ and $q=55$ and $S(K_{11})$has 66 vertices and 110 edges.
We note that 110 $\ncong$0(mod 3) and 110 $\ncong$0(mod 4). Thus, $q(D(S(K_{11})))=\frac{11(11^2-1)}{2}=660$.
Case(ii)**:** Consider the graph K_6.
Here $p=6$ and $q=15$ and so $S(K_6)$ has 21 vertices and 30 edges.
Since 30 $\cong 0(mod 3)$ and 30 $\ncong 0(mod 4)$, we have $q(D(S(K_6)))=\frac{6(6-1)(2(6)-1)}{2}=165$.
Case (iii)**:** Consider the graph K_5.
Here $p=5$ and $q=10$ and so $S(K_5)$ has 15 vertices and 20 edges.
Since 20 $\ncong$0(mod 3) and 20 $\cong$0(mod 4), we have $q(D(S(K_5)))=\frac{5^4-2(5^3)+11(5^2)-10(5)}{8}=75$.
Case(iv)**:** Consider the graph K_4.
Here $p=4$ and $q=6$ and so $S(K_4)$ has 10 vertices and 12 edges.
Then 12 $\cong$ 0(mod 3) and 12 $\cong$0(mod 4). Then clearly $D(S(K_4)) \cong K_{10}$.
Thus, $q(D(S(K_4)))=\frac{(4-1)4(4+1)(4+2)}{8}$ = 45.

Now we establish Theorem 2.8 for the complete bipartite graph.

Theorem 2.11. For a complete bipartite graph $K_{m,n}$,

$$D(S(K_{m,n})) \cong \begin{cases} \mathrm{S}(\mathrm{K}_{m,n})^2, \text{ if } 2mn \ncong 0 \text{(mod 3) and } 2mn \ncong 0 \text{(mod 4)} \\ \mathrm{S}(\mathrm{K}_{m,n})^3, \text{ if } 2mn \cong 0 \text{(mod 3) and } 2mn \ncong 0 \text{(mod 4)} \\ \mathrm{C}_{(1,2,4)}(S(K_{m,n})), \text{ if } 2mn \ncong 0 \text{(mod 3) and } 2mn \cong 0 \text{(mod 4)} \\ \mathrm{K}_{m+n+mn}, \text{ if } 2mn \cong 0 \text{ (mod 3) and } 2mn \cong 0 \text{(mod 4).} \end{cases}$$

Proof: Let $v_1,v_2,\ldots,v_m,v_{m+1},\ldots,v_{m+n}$ be the vertices of $K_{m,n}$ and $u_1,u_2,\ldots,u_{mn}$ be the subdivisional vertices of $K_{m,n}$. Note that, $S(K_{m,n})$ has $m+n+mn$ vertices and $2mn$ edges and $diamS(K_{m,n})=4$. Then, we check only the numbers 1, 2, 3 and 4 if they are the divisors of $2mn$.
In $S(K_{m,n})$, we observe that
(i) $d(v_i,v_j)$= either 2 or 4, for $i \neq j$
(ii) $d(v_i,u_j)$= either 1 or 3, for $i=1,2,\ldots,m+n, j=1,2,\ldots,mn$
$(iii) d(u_i,u_j)$= either 2 or 4 for $i \neq j$.

We note that the situations are the same as in Theorem 2.8.
Hence, the result follows from Theorem 2.8. ∎

Now we calculate the edges of the distant divisor graph of subdivision of complete bipartite graph $K_{m,n}$.

Corollary 2.12. For a complete bipartite graph $K_{m,n}$,

$$q(D(S(K_{m,n}))) = \begin{cases} \frac{mn(m+n+4)}{2}, \text{ if } 2mn \ncong 0(\mod 3) \text{ and } 2mn \ncong 0(\mod 4) \\ \frac{3mn(m+n)}{2}, \text{ if } 2mn \cong 0(\mod 3) \text{ and } 2mn \ncong 0(\mod 4) \\ \frac{mn(mn+5)+m(m-1)+n(n-1)}{2}, \text{ if } 2mn \ncong 0(\mod 3) \text{ and } 2mn \cong 0(\mod 4) \\ \frac{(m+n+mn)(m+n+mn-1)}{2}, \text{ if } 2mn \cong 0(\mod 3) \text{ and } 2mn \cong 0(\mod 4). \end{cases}$$

Proof: Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $V(K_{m,n})$ and $v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}$ be the subdivisional vertices of $S(K_{m,n})$. We observe that v_{ij} be the subdivision vertex of the edge x_iy_j.

The following table shows the number of vertices having the distance 1, 2, 3 or 4 from the vertices of $D(S(K_{m,n}))$ which is very useful to find the degree of vertices of $D(S(K_{m,n}))$ and hence the number of edges.

Vertex v	Distance d	Number of vertices having distance d from v
x_i	1	n
	2	n
	3	$mn - n$
	4	$m - 1$
y_j	1	m
	2	m
	3	$mn - m$
	4	$n - 1$
v_{ij}	1	2
	2	$m + n - 2$
	3	$m + n - 2$
	4	$mn - m - n + 1$

From the above table it is found that, in $S(K_{m,n})$, the number of vertices having distance 1 from x_i is n and that of distance 2 from x_i is also n and that of distance 3 from x_i is $(mn - n)$ and that of distance 4 from x_i is $m - 1$.

The number of vertices having distance 1 from y_j is m and that of distance 2 from y_j is also m and that of distance 3 from y_j is $(mn - m)$ and that of distance 4 from y_j is $n - 1$.

The number of vertices having distance 1 from v_{ij} is 2 and that of distance 2 from v_{ij} is $m + n - 2$ and that of distance 3 from v_{ij} is also $m + n - 2$ and that of distance 4 from v_{ij} is $mn - m - n + 1$.

Case(i): Suppose $2mn \ncong 0$(mod 3) and $2mn \ncong 0$(mod 4).

Then the divisors of $2q$ are 1 and 2 only.
Hence in $D(S(K_{m,n}))$, $d(x_i) = 2n$, for $i = 1,2,3,\ldots,m$, $d(y_j) = 2m$, for $j = 1,2,3,\ldots,n$ and $d(v_{ij}) = m+n$, for $i = 1,2,3,\ldots,m, j = 1,2,3,\ldots,n$.
Thus, $q(D(S(K_{m,n}))) = \frac{m(2n)+n(2m)+mn(m+n)}{2} = \frac{mn(m+n+4)}{2}$.
Case(ii)**:** Suppose $2mn \cong 0(\text{mod } 3)$ and $2mn \ncong 0(\text{mod } 4)$.
Then the divisors of $2q$ are 1, 2 and 3 only.
Hence in $D(S(K_{m,n}))$, $d(x_i) = mn+n$, for $i = 1,2,3,\ldots,m$, $d(y_j) = mn+m$, for $j = 1,2,3,\ldots,n$ and $d(v_{ij}) = 2(m+n-1)$, for $i = 1,2,3,\ldots,m, j = 1,2,3,\ldots,n$.
Thus, $q(D(S(K_{m,n}))) = \frac{m(mn+n)+n(mn+m)+mn(2(m+n-1))}{2} = \frac{m^2n+mn+mn^2+mn+2m^2n+2mn^2-2mn}{2} = \frac{3mn(m+n)}{2}$.
Case(iii)**:** Suppose $2mn \cong 0(\text{mod } 4)$ and $2mn \ncong 0(\text{mod } 3)$.
The divisors of $2q$ are 1, 2 and 4 only.
Hence in $D(S(K_{m,n}))$, $d(x_i) = 2n+m-1$, for $i = 1,2,3,\ldots,m$, $d(y_j) = 2m+n-1$, for $j = 1,2,3,\ldots,n$ and $d(v_{ij}) = mn+1$, for $i = 1,2,3,\ldots,m, j = 1,2,3,\ldots,n$.
Thus, $q(D(S(K_{m,n}))) = \frac{m(2n+m-1)+n(2m+n-1)+mn(mn+1)}{2} = \frac{2mn+m^2-m+2mn+n^2-n+m^2n^2+mn}{2} = \frac{mn(mn+5)+m(m-1)+n(n-1)}{2}$.
Case(iv)**:** Suppose $2mn \cong 0(\text{mod } 3)$ and $2mn \cong 0(\text{mod } 4)$.
Then the divisors of $2q$ are 1, 2 and 4 only. Hence, $D(S(K_{m,n})) \cong K_{m+n+mn}$.
Thus, $q(D(S(K_{m,n}))) = \frac{(m+n+mn)+(m+n+mn-1)}{2}$. ■

Corollary 2.13. For $K_{1,p}$,

$$D(S(K_{1,p})) \cong \begin{cases} \mathrm{S}(\mathrm{K}_{1,p})^2, \text{ if } 2q \ncong 0(\text{mod } 3) \text{ and } 2q \ncong 0(\text{mod } 4) \\ \mathrm{S}(\mathrm{K}_{1,p})^3, \text{ if } 2q \cong 0 \text{ (mod 3) and } 2q \ncong 0(\text{mod } 4) \\ \mathrm{C}_{(1,2,4)}(S(K_{1,p})), \text{ if } 2q \ncong 0(\text{mod } 3) \text{ and } 2q \cong 0(\text{mod } 4) \\ \mathrm{K}_{2p+1}, \text{ if } 2q \cong 0(\text{mod } 3) \text{ and } 2q \cong 0(\text{mod } 4). \end{cases}$$

Corollary 2.14. For $K_{1,p}$,

$$q(D(S(K_{1,p}))) = \begin{cases} \frac{p(p+5)}{2}, \text{ if } 2q \ncong 0(\text{mod } 3) \text{ and } 2\text{q} \ncong 0(mod 4) \\ \frac{3}{2}p(p+1), \text{ if } 2q \cong 0(\text{mod } 3) \text{ and } 2\text{q} \ncong 0(mod 4) \\ p(p+2), \text{ if } 2q \ncong 0(\text{mod } 3) \text{ and } 2\text{q} \cong 0(mod 4) \\ p(2p+1), \text{ if } 2\text{q} \cong 0(\text{mod } 3) \text{ and } 2\text{q} \cong 0(mod 4). \end{cases}$$

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