International Journal of Mathematics and Soft Computing
Vol.4, No.1 (2014), 27 - 33.

ISSN Print : 2249 - 3328
ISSN Online: 2319 - 5215

Packing Chromatic Number of Cycle Related Graphs

Albert William, S. Roy
Department of Mathematics,
Loyola College, Chennai 600 034, INDIA.
E-mail: sroysantiago@gmail.com

Indra Rajasingh
School of Advanced Sciences,
VIT University, Chennai 600 127, INDIA.

Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k for which there exists a mapping $\pi : V(G) \longrightarrow \{1, 2, ..., k\}$ such that any two vertices of color i are at distance at least $i+1$. It is a frequency assignment problem used in wireless networks, which is also called Broadcast coloring. It is proved that packing coloring is NP-complete for general graphs and even for trees. In this paper, we give the packing chromatic number for some classes of cycles.

Keywords: Packing chromatic number, kC_n - linear, barbell graph, split graph, total graph, crown graph, lollipop, tadpole.
AMS Subject Classification(2010): 05C15.

1 Introduction

Let G be a connected graph and k be an integer, $k \geq 1$. A packing k-coloring of a graph G is a mapping $\pi : V(G) \longrightarrow \{1, 2, ..., k\}$ such that any two vertices of color i are at distance at least $i+1$. Thus, the vertices of G are partitioned into different color classes $X_1, X_2, .., X_k$, where every X_i is an i-packing of G. The i-packing number of G, denoted by $\rho_i(G)$, is the maximum cardinality of an i-packing that occurs in G. The packing chromatic number $\chi_\rho(G)$ of G is the smallest integer k for which G has packing k-coloring. The concept of packing coloring emerges from the area of frequency assignment in wireless networks and was introduced by Goddard et al. [8] by the name Broadcast coloring. It has several applications such as resource replacement and biological diversity. The term packing chromatic number was introduced by Bresar [3].

Goddard et al. [8] proved that the packing coloring problem is NP-complete for general graphs and Fiala and Golovach [6] proved that it is NP-complete even for trees. It is proved that packing coloring problem is solvable in polynomial time for graphs whose treewidth and diameter are both bounded [6] and for cographs and split graphs [8]. Sloper [14] studied a special type of packing coloring, called eccentric chromatic coloring and proved that the infinite 3-regular tree has packing chromatic number 7. For the infinite planar square lattice $\mathbb{Z}^2$, 10 $\leq \chi_\rho(\mathbb{Z}^2) \leq 17$ [5, 9]. The packing coloring of distance graphs were studied in [4, 15]. For the infinite hexagonal lattice $\mathbb{H}$, $\chi_\rho(\mathbb{H}) = 7$ [3].

Argiroffo et al. [1, 2] proved that the packing coloring is solvable in polynomial time for the class of $(q, q-4)$ graphs, partner limited graphs and for an infinite subclass of lobsters, including caterpillars. In [7, 13] it is proved that the infinite, planar triangular lattice and the three dimensional square lattice have unbounded packing chromatic number. In this paper, we study the packing chromatic number of kC_n - linear, barbell graph, split graph, total graph, crown graph, lollipop and tadpole.

2 Main Results

Lemma 2.1. [8] Let H be a subgraph of G. Then $\chi_\rho(H) \leq \chi_\rho(G)$.

Lemma 2.2. [8] $\chi_\rho(C_n) = \begin{cases} 3 & \text{when } n \text{ is a multiple of } 4 \\ 4 & \text{when } n \text{ is not a multiple of } 4. \end{cases}$

Lemma 2.3. [8] Let G be a complete graph on n vertices. Then $\chi_\rho(G) = n$.

Definition 2.4. A graph is called a kC_{2n}- linear if it is a connected graph with k blocks, each of which is isomorphic to C_{2n}. Moreover, if u and v are cut-vertices in the same block B_i, then $d(u, v) = n$.

Theorem 2.5. For $kC_{2n}, n \geq 4$, $\chi_\rho(kC_{2n}) = 3$ when $n \equiv 0 \ mod \ 4$.

Proof: Case 1: $n = 8r$.
We begin with the cut-vertex common to B_i, B_{i+1} and label it as 2. Label every 4-sequence of consecutive vertices on B_i in the clockwise sense as 2,1,3,1 beginning from the cut-vertex. See Figure 1(a).

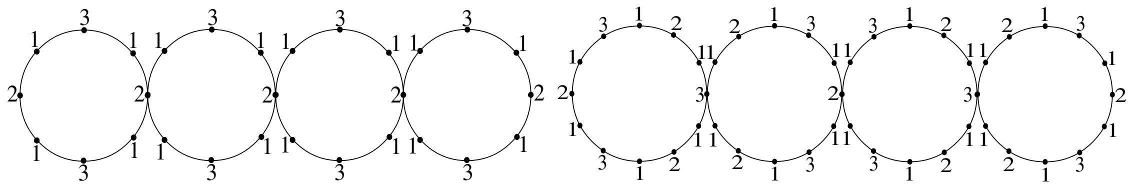

Figure 1: (a) $\chi_\rho(4C_8) = 3$ (b) $\chi_\rho(4C_{12}) = 3$.

Case 2: $n = 8r + 4$.
Label the cut-vertex common to B_i, B_{i+1} as 3 or 2 according as i is odd or even respectively. Label every 4-sequence of consecutive vertices on B_i in the clockwise sense as 3,1,2,1 or 2,1,3,1 beginning from the cut-vertex with label 3 or 2. See Figure 1(b). By Lemma 2.1 and 2.2, $\chi_\rho(kC_{2n}) = 3$. ∎

Definition 2.6. [11] A gear graph is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the n-cycle and it is denoted by G_n, where n is the number of vertices of wheel graph. G_n has $2n + 1$ vertices and $3n$ edges.

Theorem 2.7. For the gear graph G_n, $\chi_\rho(G_n) = 4$, when n is even.

Proof: Let u and v be the vertices of G_n such that $deg(u) = n$ and $deg(v) = 3$. Color the vertices in the outercycle with the sequence 1213,$\cdots$ starting from the vertex v in any direction and fix color 4 to the vertex u. Thus, $\chi_\rho(G_n) \leq 4$.

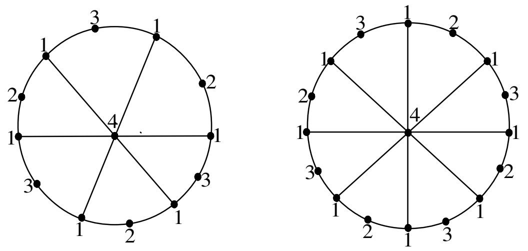

Figure 2 : $\chi_\rho(G_6) = 4$ and $\chi_\rho(G_8) = 4$.

By Lemma 2.1 and 2.2, $\chi_\rho(G_n) = 4$. ■

Definition 2.8. [16] For a graph G, the splitting graph S' of G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Theorem 2.9. Let $K_{1,n}$, $n \geq 2$ be a star on n vertices. Then $\chi_\rho(S'(K_{1,n})) = 3$.

Proof: Let $v_1, v_2, ..., v_n$ be the pendent vertices and v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, ..., u_n$ be the added vertices corresponding to $v, v_1, v_2, ..., v_n$ to obtain $S'(K_{1,n})$.

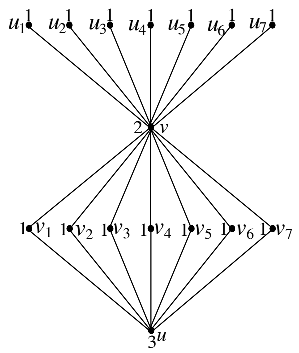

Figure 3: $\chi_\rho(S'(K_{1,7})) = 3$.

Color the vertices $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ with color 1. Assign color 2 to the vertex u and 3 to the vertex v. Thus $\chi_\rho(S'(K_{1,n})) \leq 3$. By Lemma 2.1 and 2.2, $\chi_\rho(S'(K_{1,n})) = 3$. ■

Definition 2.10. [12] The n-barbell graph is a simple graph obtained by connecting two copies of a complete graph K_n by a bridge and it is denoted by $B(K_n, K_n)$.

Theorem 2.11. For $B(K_n, K_n)$, $\chi_\rho(B(K_n, K_n)) = 2n - 2$.

Proof: An n-barbell graph contains two copies of K_n namely $1K_n$ and $2K_n$. Let uv be the connecting bridge where $u \in 1K_n$ and $v \in 2K_n$.
We give an algorithm to color $B(K_n, K_n)$ using exactly $2n - 2$ colors.

Procedure PACKING COLORING $B(K_n, K_n)$
Input: Barbell graph $B(K_n, K_n)$
Algorithm:

Step 1: Label the vertex u with color $2n-2$.
Step 2: Label the vertex v with color $2n-3$.
Step 3: Label any vertex of $1K_n$ and $2K_n$ with color 1.
Step 4: Label any vertex of $1K_n$ and $2K_n$ with color 2.
Step 5: Label the remaining vertices of $1K_n$ and $2K_n$ with color 3,4,$\cdots$, $2n-4$.
Output: $\chi_\rho(B(K_n, K_n))) = 2n-2$.

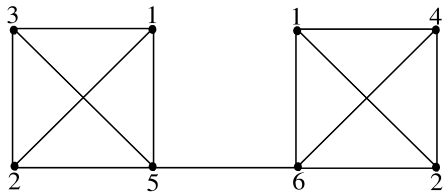

Figure 4: $\chi_\rho(B(K_4, K_4))) = 6$.

Proof of correctness: Let $G \simeq B(K_n, K_n)$. The diameter of G is 3, which implies $\rho_2(G) = 2$ and $\rho_i(G) \leq 1$ for all integer $i \geq 3$.
Case 1: If vertex u or v is labeled by color 1, then $n-1$ vertices at distance greater than one from the vertex u or v form a subgraph, which is isomorphic to K_{n-1}. In this case, at most one vertex receives color 1, because $\rho_1(K_{n-1}) = 1$.
Case 2: If a vertex of G except u and v is labeled by color 1, then n vertices at distance more than one from that vertex received color 1 form a subgraph, which is isomorphic to K_n. In this case, at most one vertex receives color 1, because $\rho_1(K_n) = 1$.
So, $\rho_2(G) = 2$. There is a k-coloring of the graph G with $|X_1| = 2$ and $|X_2| = 2$. As for each of the other $(2n-4)$ vertices, one new color is necessary, which implies that $\chi_\rho(B(K_n, K_n)) = 2n-2$. ∎

Definition 2.12. The graph obtained by joining a pendent edge at each vertex of a cycle C_n is called a crown graph denoted by $C_n \odot K_1$.

Theorem 2.13. For $C_n \odot K_1, n \geq 6$. $\chi_\rho(C_n \odot K_1) \leq 5$, when $n \equiv 0{,}2 \ (mod\ 4)$.

Proof: Let the vertices of C_n be $u_1, u_2, ..., u_n$ and the vertices of pendent edges be $v_1, v_2, ..., v_n$ where u_i and v_j are adjacent for all $i = j$.
Case 1: $n \equiv 0\ mod\ 4$. Color the vertices $u_1, u_2, ..., u_n$ with the sequence 1213,$\cdots$ starting from vertex u_1 in the clockwise sense and $v_1, v_2, ..., v_n$ with the sequence 4151,$\cdots$ starting from vertex v_1 in the clockwise sense. Thus $\chi_\rho(C_n \odot K_1) \leq 5$.
Case 2: $n \equiv 2\ mod\ 4$. Color the vertices $u_1, u_2, ..., u_n$ with the sequence 141315,$\cdots$ starting from vertex u_1 in the clockwise sense and $v_1, v_2, ..., v_n$ with the sequence 21,$\cdots$ starting from vertex v_1 in the clockwise sense. Thus $\chi_\rho(C_n \odot K_1) \leq 5$.

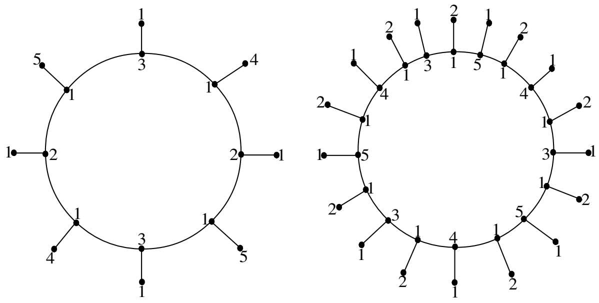

Figure 5: $\chi_\rho(C_8 \odot K_1) = 5$ and $\chi_\rho(C_{18} \odot K_1) = 5$.

■

Definition 2.14. The tadpole graph $T(m,n)$ is obtained from a cycle C_m and a path P_n, by joining one of the end vertices of P_n to a vertex of C_m.

Theorem 2.15. For $T(m,n)$, $n \geq 4$, $\chi_\rho(T(m,n)) = \begin{cases} 3, & \text{when } m \equiv 0 \; mod \; 4 \\ 4, & \text{when } m \not\equiv 0 \; mod \; 4 \end{cases}$.

Proof: Case 1: $m \equiv 0 \; mod \; 4$.
Let u be a vertex in $T(m,n)$ with $deg(u) = 3$. Color the vertices of cycle C_m with the sequence 2131,$\cdots$ starting from the vertex u in the clockwise sense. Let v be a vertex of path P_n which is adjacent to vertex u. Color the path P_n with the sequence 1213,$\cdots$ starting from the vertex v. Thus $\chi_\rho(T(m,n)) \leq 3$. By Lemma 2.1 and 2.2, $\chi_\rho(T(m,n)) = 3$.

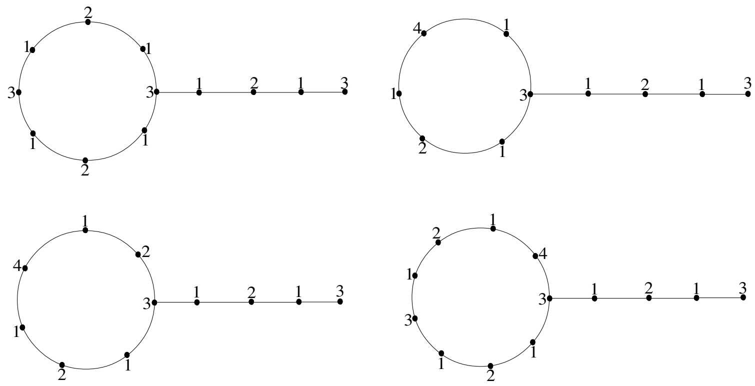

Figure 6: χ_ρ(T(8,5)) = 3,χ_ρ(T(6,5)) = 4,χ_ρ(T(7,5)) = 4 and χ_ρ(T(9,5)) = 4.

Case 2: $m \not\equiv 0 \; mod \; 4$
Let u be a vertex in $T(m,n)$ with $deg(u) = 3$. Color the vertices of cycle C_m with color sequence in the clockwise sense starting from vertex u with repeated blocks of 3,1,2,1 with an adjustment at the very end as shown below:

3121,3121,...,31214 when $n = 4r + 1$

3121,3121,...,312141 when $n = 4r + 2$

3121,3121,...,3121412 when $n = 4r + 3$ starting from the vertex u in the clockwise sense. Let v be the vertex of path P_n which is adjacent to vertex u. Color the path

P_n with the sequence 1213,$\cdots$ starting from the vertex v. Thus $\chi_\rho(T(m,n)) \leq 4$.
By Lemmas 2.1 and 2.2, $\chi_\rho(T(m,n)) = 4$. ∎

Definition 2.16. [10] The lollipop graph $LP(n,k)$ is obtained from a complete graph K_k and a path P_{n-k+1}, by joining one of the end vertices of P_{n-k+1} to a vertex of K_k.

Theorem 2.17. For lollipop graph $LP(n,k)$, $\chi_\rho(LP(n,k)) = n$.

Proof: Let u be the vertex in $LP(n,k)$ with $deg(u) = n$. Give color n to the vertex u and colors 1 to $n-1$ to the remaining vertices in K_k.

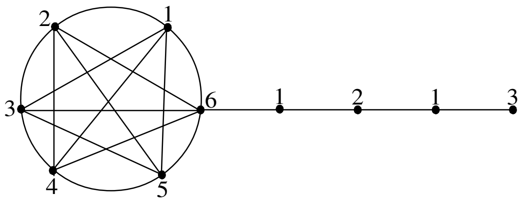

Figure 7: $\chi_\rho(LP(6,2)) = 6$.

Let v be the vertex of path P_{n-k+1} which is adjacent to the vertex u. Color the path P_{n-k+1} with the sequence 1213,$\cdots$ starting from the vertex v. Thus $\chi_\rho(LP(n,k)) \leq n$.
By Lemma 2.1 and 2.3, $\chi_\rho(LP(n,k)) = n$. ∎

Acknowledgment

This work is supported by Maulana Azad Fellowship F1-17.1/2011/MANF-CHR-TAM-2135 of the University Grants Commission, New Delhi, India.

References

[1] G. Argiroffo, G. Nasini and P. Torres, *The Packing Coloring of Lobsters and Partner Limited Graphs*, Discrete Applied Mathematics. Preprint (2012).

[2] G. Argiroffo, G. Nasini and P. Torres, *The Packing Coloring Problem for $(q, q-4)$ Graphs, Combinatorial Optimization*, Lecture Notes in Computer Science 7422(2012), 309-319.

[3] B. Brešar, S. Klavžar and D. F. Rall, *On the Packing Chromatic Number of Cartesian Products, Hexagonal Lattice and Trees*, Discrete Appl. Math, 155 (2007), 2303-2311.

[4] J. Ekstein, P. Holub and B. Lidický, *Packing Chromatic Number of Distance Graphs*, Discrete Applied Mathematics, 160 (2012), 518-524.

[5] J. Fiala, S. Klavžar, B. Lidický, *The Packing Chromatic Number of Infinite Product G*, European J. Combin., 30 (5) (2009), 1101-1113.

[6] J. Fiala and P. A. Golovach, *Complexity of the packing chromatic problem for trees*, Discrete Appl. Math., 158 (2010), 771-778.

[7] A. Finbow and D. F. Rall, *On the Packing Chromatic Number of Some Lat- tices*, Discrete Appl. Math. 158 (2010), 1224-1228.

[8] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris and D.F. Rall,*Broadcast Chromatic Numbers of Graphs*, Ars Combin. 86 (2008) 33-49.

[9] P. Holub, R. Soukal, *A Note on Packing Chromatic Number of the Square Lattice*, Electronic Journal of Combinatorics, 17 (2010), Note 17.

[10] A.Ilič, *On the Extremal Properties of the Average Eccentricity*, arXiv:1106.2987v1 [math.CO] 15 Jun 2011.

[11] R. M. J. Jothi and A. Amutha, *Analysation of Super Strongly Perfect Graphs in Certain Graphs*, Proceedings of National Conference on Mathematical Sciences and Applications, Karunya University, Coimbatore, India, March 14-15, 2013.

[12] G. Jothilakshmi, A. P. Pushpalatha, S. Vallirani and K. SathyaBama, γ_{wc}*-excellent Graphs*, Proceedings of National Conference on Mathematical Sciences and Applications, Karunya University, Coimbatore, India, March 14-15, 2013.

[13] D. Rall, B. Brešar, A. Finbow and S. Klavžar, *On the Packing Chromatic Number of Trees, Cartesian Products and Some Infinite Graphs*, Electronic Notes in Discrete Mathematics, 30 (2008), 571.

[14] C. Sloper,*An Eccentric Coloring of Trees*, Australas. J. Combin. 29 (2004), 309-321.

[15] O. Togni, *On Packing Colorings of Distance Graphs*, arXiv:1011.0971v1 [cs.DM] 3 Nov 2010.

[16] S K Vaidya and P L Vihol, *Discussion on Some Interesting Topics in Graph Theory*, Ph.D Thesis, Saurashtra University, Rajkot, April 2011.