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On super edge-antimagic total labeling of generalized extended w-trees

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Abstract

This paper deals with the construction of generalized extended w-trees denoted by $GEwt(n, m, r, k)$ and the existence of a super (a, d)-edge-antimagic total labeling on them for $d \in \{0, 1, 2\}$.

Keywords: super (a, d)-EAT labeling, w-trees, extended w-trees, generalized extended w-trees.
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1 Introduction

Let G be a finite, simple and undirected graph with vertex-set $V(G)$ and edge-set $E(G)$. Moreover, suppose that $|V(G)| = v$ and $|E(G)| = e$. A general reference for graph-theoretic ideas can be seen in [25]. A *labeling* (or *valuation*) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper, the domain will be the set of all vertices and edges and such a labeling is called a *total labeling*.

Labeling of graphs has its origin in the works of Kotzig and Rosa [18, 19]. There are several types of graph labelings such as harmonius, cordial, graceful, magic, antimagic and the like. The complete survey of graph labelings can be found in [8]. In this paper, we focus on antimagic total labeling. More details on antimagic total labeling can be found in [1].

Definition 1.1. An (s, d)-*edge-antimagic vertex* ((s, d)-EAV) labeling of a graph G is a bijective function $\lambda : V(G) \to \{1, 2, \dots, v\}$ such that the set of edge-sums of all edges in G, $\{w(xy) = \lambda(x)+\lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{s, s + d, s + 2d, \dots, s + (e - 1)d\}$, where $s > 0$ and $d \geq 0$ are two fixed integers.

Definition 1.2. An (a, d)-*edge-antimagic total* ((a, d)-EAT) labeling of a graph G is a bijective function $\lambda : V(G) \cup E(G) \to \{1, 2, \dots, v + e\}$ such that the set of edge-weights of all edges in G, $\{w(xy) = \lambda(x)+\lambda(xy)+\lambda(y) : xy \in E(G)\}$, forms an arithmetic progression $\{a, a+d, a+2d, \dots, a+(e-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers. If such a labeling exists then G is said to be an (a, d)-*EAT graph*.

Definition 1.3. An (a, d)-*EAT* labeling λ is called a *super* (a, d)-*edge-antimagic total* (super (a, d)-EAT) labeling of G if $\lambda(V(G)) = \{1, 2, \dots, v\}$. Thus, a *super* (a, d)-*EAT graph* is a graph that admits a super (a, d)-EAT labeling.

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In the above definition, if $d = 0$ then a *super $(a, 0)$-EAT* labeling is called a *super edge-magic total* (SEMT) labeling and a is called a magic constant. For $d \neq 0$, a is called minimum edge-weight. The definition of an (a, d)-EAT labeling was introduced by Simanjuntak, Bertault and Miller in [24] as a natural extension of an *edge-magic total* labeling defined by Kotzig and Rosa. A super (a, d)-EAT labeling is a natural extension of the notion of a super $(a, 0)$-EAT labeling defined by Enomoto, Lladó, Nakamigawa and Ringel [6]. They also proposed the conjecture that every tree is a super $(a, 0)$-EAT graph.

In favour of this conjecture, several authors derived different results on a super (a, d)-EAT labeling for many classes of trees. In particular, stars [21], path-like trees [3] banana trees [9, 23] w-trees [10], extended w-trees [11, 14, 15], subdivided stars [12, 13, 16], caterpillars [22], subdivided caterpillars [17], disjoint union of caterpillars [4], subdivided w-trees [13, 16], fire crackers and unicyclic graphs [23] are proved super (a, d)-EAT graphs. Lee and Shah [20] verified this conjecture by a computer search for trees with at most 17 vertices. However, this conjecture is still open.

The following proposition presents the relation between a (s, d)-EAV labeling and a super (a, d)-EAT labeling. It is more important as we use frequently in the new results.

Proposition 1.4. [2] If a (v, e)-graph G has an (s, d)-EAV labeling then
(i) G has a super $(s + v + 1, d + 1)$-EAT labeling,
(ii) G has a super $(s + v + e, d - 1)$-EAT labeling.

2 Construction of generalized extended w-trees

Chen et al. (1997) defined banana trees. Swaminathan, Jeyanthi (2006) and Hussain et al. (2009) proved different results on super $(a, 0)$-edge-antimagic total labeling of banana trees. Javaid et al. (2011) defined w-trees and extended w-trees as new classes of trees and proved that these classes are super (a, d)-edge-antimagic total. In this section, we present the concept of w-trees, extended w-trees and the construction of generalized extended w-trees.

Definition 2.1. [10] Let n be a positive integer. Consider a path P on 5 vertices as $V(P) = \{b, c_1, w_1, c_2, d\}$. A w-graph $W(n)$, is a graph derived from the path P by hanging n leaves $x_i^1, x_i^2, ..., x_i^n$ from each vertex c_i. Consider k copies of w-graph $W(n)$ with end vertices $d_1, d_2, ..., d_k$ respectively. A w-tree $Wt(n, k)$ is obtained by joining all the vertices $d_1, d_2, ..., d_k$ to a further vertex a.

Definition 2.2. [11, 14, 15] Let n and r be positive integers. Consider a path P on $2r + 1$ vertices as $V(P) = \{b, c_1, w_1, c_2, w_2, ..., w_{r-2}, c_{r-1}, w_{r-1}, c_r, d\}$. An extended w-graph $Ew(n, r)$, is a graph derived from the path P by hanging n leaves $x_i^1, x_i^2, ..., x_i^n$ from each vertex c_i. Consider k copies of extended w-graph $Ew(n, r)$ with end vertices $d_1, d_2, ..., d_k$ respectively. An extended w-trees $Ewt(n, r, k)$ is obtained by joining all the vertices $d_1, d_2, ..., d_k$ to a further vertex a.

Definition 2.3. Let n, m, r and k be positive integers. Consider a path P on r vertices as $V(P) = \{c_1, c_2, ..., c_r\}$ with n hanging leaves $x_i^1, x_i^2, ..., x_i^n$ (respectively, m leaves $y^1, y^2, ..., y^m$) from each vertex c_i if $1 \le i \le r - 1$ (respectively, if $i = r$). Consider k copies of such path $P_1, P_2, ..., P_k$ with $y_1^m, y_2^m, ..., y_k^m$ as a last hanging leaf from c_r respectively. A generalized extended w-tree is obtained by joining all the vertices $y_1^m, y_2^m, ..., y_k^m$ to a further vertex a.

Let $V(G) = \{a\}$
$\cup \{c_s^i : 1 \leq i \leq k,\ \ 1 \leq s \leq r\}$
$\cup \{x_{is}^l : 1 \leq i \leq k,\ \ 1 \leq s \leq r-1,\ \ 1 \leq l \leq n\}$
$\cup \{y_i^p : 1 \leq i \leq k,\ \ 1 \leq p \leq m\}$ and
$E(G) = \{ay_i^m : 1 \leq i \leq k\}$
$\cup \{c_{s-1}^i C_s^i : 1 \leq i \leq k,\ \ 1 \leq s \leq\ \ r-1\}$
$\cup \{x_{is}^l c_s^i : 1 \leq i \leq k,\ \ 1 \leq s \leq r-1,\ \ 1 \leq l \leq\ \ n\}$
$\cup \{y_i^p c_r^i : 1 \leq p \leq m,\ \ 1 \leq i \leq k\}$
be the vertex-set and edge-set of $G \cong GEwt(n, m, r, k)$ respectively. Thus, $v = |V(G)| = k(rn - n + m + r) + 1$ and $e = |E(G)| = v - 1$.

An example of a generalized extended w-tree is shown in Figure1 for $n = 2$, $m = 3$, $r = 4$ and $k = 3$.

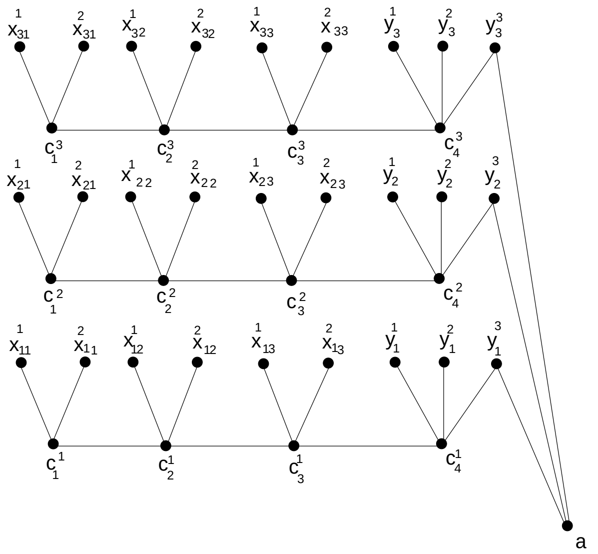

Figure 1: GEwt(2,3,4,3)

3 Super (a, d)-EAT labeling of generalized extended w-trees

In this section, for different values of d, we prove some results related to a super (a, d)-EAT labeling of generalized extended w-trees denoted by $GEwt(n, m, r, k)$ under some certain conditions on n, m, r and k.

Theorem 3.1. If $n \geq 1$, $k \geq 3$, r even and $m \geq \frac{r}{2}(n+1)k + 1$ then $G \cong GEwt(n, m, r, k)$ admits super $(a, 0)$-EAT and super $(a', 2)$-EAT labelings.

Proof: Define the vertex-labeling $\lambda : V(G) \rightarrow \{1, 2, ..., v\}$ as follows:

$$\lambda(a) = \frac{r}{2}(n+1) + 1.$$

For $s = 1, 3, ..., r-1$ and $1 \leq l \leq n$,

$$\lambda(c_s^i) = \begin{cases} \frac{1}{2}(n+1)(rk+s-1)+2 & for \;\; i=1, \\ \frac{1}{2}(n+1)(rk+ri-2i-s+1)+mi+i+1 & \\ & for \;\; 2 \leq i \leq k. \end{cases}$$

and

$$\lambda(x_{is}^l) = \begin{cases} \frac{1}{2}(n+1)(s-1)+l \;\; for \;\; i=1, & \\ \frac{1}{2}(n+1)(ir+1-s)+2-l & for \; 2 \leq i \leq k. \end{cases}$$

For $s = 2, 4, ..., r$

$$\lambda(c_s^i) = \begin{cases} \frac{1}{2}(n+1)s & for \;\; i=1, \\ \frac{1}{2}(n+1)(ir-s+2)-n+1 & for \;\; 2 \leq i \leq k. \end{cases}$$

For $s = 2, 4, ..., r-2$ and $1 \leq l \leq n$,

$$\lambda(x_{is}^l) = \begin{cases} \frac{1}{2}(n+1)(rk+s-2)+2+l & for \;\; i=1, \\ \frac{1}{2}(n+1)(rk+ri-2i-s+2)+(m+1)i+1-l & \\ & for \;\; 2 \leq i \leq k. \end{cases}$$

For $1 \leq i \leq 2$

$$\lambda(y_i^p) = \begin{cases} \frac{1}{2}(n+1)(rk+r-2)+m+1+i & for \;\; p=m, \\ \frac{1}{2}(n+1)(rk+r-2)+mi+i+1-p & \\ & for \;\; 1 \leq p \leq m-1. \end{cases}$$

For $3 \leq i \leq k \;\; and \;\; \alpha^i = \frac{r}{2}(n+1)(i-2)$;

$$\lambda(y_i^p) = \begin{cases} \frac{1}{2}(n+1)(2ri+rk-3r-2i+2)+mi-m+i+1 & \\ & for \;\; p=m. \end{cases}$$

and

$$\lambda(y_i^p) = \begin{cases} \frac{1}{2}(n+1)(rk+ir-r-2i+2)+mi-m+i+p & \\ & for \;\; 1 \leq p \leq \alpha^i, \\ \frac{1}{2}(n+1)(rk+ir-r-2i+2)+mi-m+i+1+p & \\ & for \;\; \alpha^i+1 \leq p \leq m-1. \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $[\frac{r}{2}(n+1)k+2]+1, [\frac{r}{2}(n+1)k+2]+2, \cdots, [\frac{r}{2}(n+1)k+2]+e$, where $s = [\frac{r}{2}(n+1)k+2]+1$. Consequently, λ is a $(s, 1)$-EAV labeling. Now, by Proposition 1.1, λ can be extended to a super $(a, 0)$-EAT labeling with magic constant $a = v + e + s = v + v - 1 + \frac{r}{2}(n + 1)k + 3 = 2v + \frac{r}{2}(n + 1)k + 2 = \frac{k}{2}[5r(n + 1) + 4(m - n)] + 4$. Similarly, λ can be extended to a super $(a', 2)$-EAT labeling with minimum edge-weight $a = v + 1 + s = v + \frac{r}{2}(n+1)k + 4 = \frac{k}{2}[3r(n+1) + 2(m-n)] + 5$. ∎

Theorem 3.2. If $n \geq 1$, $k \geq 3$, r even, $m \geq \frac{r}{2}(n+1)k+1$ and v even then $G \cong GEwt(n, m, r, k)$ admits a super $(a'', 1)$-EAT labeling.

Proof: Define $V(G)$, $E(G)$ and $\lambda : V(G) \rightarrow \{1, 2, ..., v\}$ as in Theorem 3.1. It follows that the set of edge-sums $A = \{a_i; 1 \leq i \leq e\}$, where $a_i = [\frac{r}{2}(n+1)k+2]+i$ constitutes an arithmetic sequence with common difference 1. Consequently, the set of edge-labels is $B = \{b_j\ ;\ 1 \leq j \leq e\}$, where $b_j = v_j + 1$. Now, the set of edge-weights is defined as $C = \{a_{2i-1} + b_{e-i+1}\ ;\ 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j} + b_{\frac{e-1}{2}-j+1}\ ;\ 1 \leq j \leq \frac{e+1}{2} - 1\}$. It is easy to see that C constitute an arithmetic sequence with $d = 1$ and $a = s + \frac{3}{2}v = \frac{1}{2}[4kr(n+1) + 3k(m-n) + 3] = \frac{k}{2}[4r(n+1) + 3(m-n)] + \frac{9}{2}$. Since, all the vertices receive the smallest labels, λ is a super $(a'', 1)$-EAT labeling. ∎

An illustration of the labeling schemes presented in Theorem 3.1 and 3.2 is given in Figure 2.

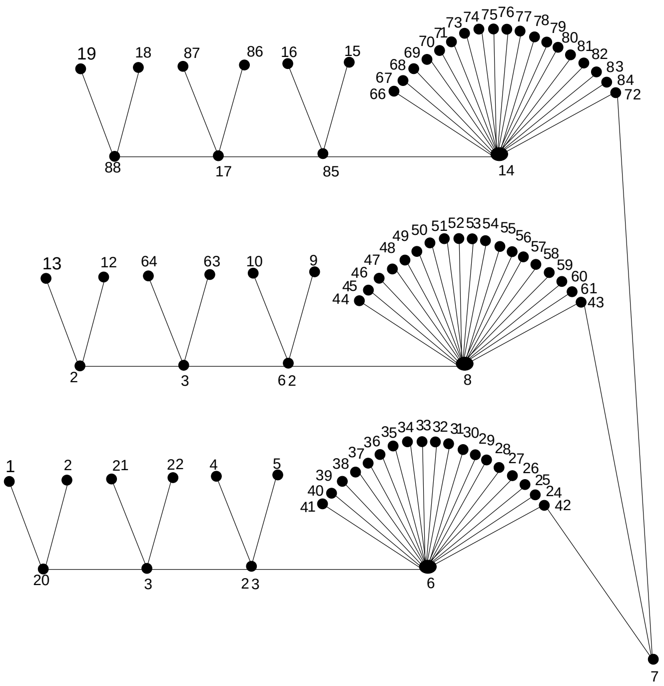

Figure 2: GEwt(2,19,4,3)

Here, for $n = 2$, $r = 4$, and $k = 3$, we have $m \geq \frac{r}{2}(n+1)k+1$=$\frac{4}{2}(2+1)3+1$=19, that is $m \geq 19$. So for $m = 19$, the generalized extended w-tree is $GEwt(2, 19, 4, 3)$ with $v = k(rn-n+m+r)+1 = 88$. As a consequence of the vertex-labeling which is formulated in Theorem 3.1, Figure 2 gives the set of edge-sums $\{21, 22, 23, ..., 107\}$ as a sequence of consecutive integers starting from $s = 21$. Thus, the generalized extended w-tree $GEwt(2, 19, 4, 3)$ admits a $(21, 1)$-EAV labeling. Consequently, we have a super $(a, 0)$-EAT labeling with $a = v + e + s = 88 + 87 + 21 = 196$, a super $(a', 2)$-EAT labeling with $a' = v + 1 + s = 88 + 1 + 21 = 110$ and a super $(a'', 1)$-EAT labeling with $a'' = s + \frac{3}{2}v = 153$ of the generalized extended w-tree $GEwt(2, 19, 4, 3)$. The values of a, a' and a'' also can be verified by $a = \frac{k}{2}[5r(n + 1) + 4(m - n)] + 4 = 196$, $a' = \frac{k}{2}[3r(n + 1) + 2(m - n)] + 5 = 110$ and $a'' = \frac{k}{2}[4r(n + 1) + 3(m - n)] + \frac{9}{2} = 153$.

Theorem 3.3. If $n \geq 1$, $k \geq 3$, r odd and $m \geq \frac{r-1}{2}(n + 1)k + k + 1$ then $G \cong GEwt(n, m, r, k)$ admits super $(a, 0)$-EAT and super $(a', 2)$-EAT labelings.

Proof: Define the vertex-labeling $\lambda : V \to \{1, 2, ..., v\}$ as follows:

$$\lambda(a) = \frac{r-1}{2}(n + 1) + 2.$$

For $s = 1, 3, ..., r$

$$\lambda(c_s^i) = \begin{cases} \frac{1}{2}(n + 1)(s - 1) + 1 & for\ \ i = 1, \\ \frac{1}{2}(n + 1)(ri - i - s + 1) + i + 1 & for\ \ 2 \leq i \leq k. \end{cases}$$

For $s = 1, 3, ..., r - 2$ and $1 \leq l \leq n$,

$$\lambda(x_{is}^l) = \begin{cases} \frac{1}{2}(n + 1)(kr - k + s - 1) + k + 1 + l & for\ \ i = 1, \\ \frac{1}{2}(n + 1)(rk + ir - k - i - s + 1) + mi + k + 2 - l & for\ \ 2 \leq i \leq k. \end{cases}$$

For $s = 2, 4, ..., r - 1$ and $1 \leq l \leq n$,

$$\lambda(c_s^i) = \begin{cases} \frac{1}{2}(n + 1)(kr - k + s - 2) + k + n + 2 & for\ \ i = 1, \\ \frac{1}{2}(n + 1)(rk + ir - k - i - s + 2) + mi - n + k + 1 & for\ \ 2 \leq i \leq k, \end{cases}$$

and

$$\lambda(x_{is}^l) = \begin{cases} \frac{1}{2}(n + 1)(s - 2) + 1 + l & for\ \ i = 1, \\ \frac{1}{2}(n + 1)(ri - i - s + 2) + i + 1 - l & for\ \ 2 \leq i \leq k. \end{cases}$$

For $1 \leq i \leq 2$

$$\lambda(y_i^p) = \begin{cases} \frac{1}{2}(n+1)(rk+r-k-1)+m+k+i & for \;\; p=m, \\ \\ \frac{1}{2}(n+1)(rk+r-k-1)+mi+i+k-p & \\ & for \;\; 1 \leq p \leq m-1. \end{cases}$$

For $3 \leq i \leq k \;\; and \;\; \alpha^i = \frac{r-1}{2}(n+1)(i-2)-2+i;$

$$\lambda(y_i^p) = \begin{cases} \frac{1}{2}(n+1)(2ri+rk-3r-2i-k+3)+mi-m+i+k \\ \qquad\qquad for \;\; p=m. \end{cases}$$

and

$$\lambda(y_i^p) = \begin{cases} \frac{1}{2}(n+1)(rk+ir-r-k-i+1)+mi-m+k+1+p \\ \qquad\qquad for \;\; 1 \leq p \leq \alpha^i, \\ \\ \frac{1}{2}(n+1)(rk+ir-r-k-i+1)+mi-m+k+2+p \\ \qquad\qquad for \;\; \alpha^i+1 \leq p \leq m-1. \end{cases}$$

The set of all edge-sums generated by the above formulae forms a consecutive integer sequence $[\frac{r-1}{2}(n+1)k+k+2]+1, [\frac{r-1}{2}(n+1)k+k+2]+2, \cdots, [\frac{r-1}{2}(n+1)k+k+2]+e$, where $s = [\frac{r-1}{2}(n+1)k+k+2]+1$. Consequently, λ is a $(s,1)$-EAV labeling. Now, by Proposition 1.1, λ can be extended to a super $(a,0)$-EAT labeling with magic constant $a = v+e+s = 2v-1+\frac{r-1}{2}(n+1)k+k+3 = 2v+\frac{r-1}{2}(n+1)k+k+2 = \frac{k}{2}[5(rn-n+r)+4m+1]+4$. Similarly, λ can be extended to a super $(a',2)$-EAT labeling with minimum edge-weight $a' = v+1+s = v+\frac{r-1}{2}(n+1)k+4 = \frac{k}{2}[3(rn-n+r)+2m+1]+4$. ■

Theorem 3.4. For $n \geq 1$, $k \geq 3$, r odd, $m \geq \frac{r-1}{2}(n+1)k+k+1$ and v even then $G \cong GEwt(n,m,r,k)$ admits a super $(a'',1)$-EAT labeling.

Proof: Define $V(G)$, $E(G)$ and $\lambda : V(G) \to \{1,2,...,v\}$ as in Theorem 3.3. It follows that the set of edge-sums $A = \{a_i; 1 \leq i \leq e\}$, where $a_i = \frac{r-1}{2}(n+1)k+k+2]+i$ constitute an arithmetic sequence with common difference 1. Consequently, the set of edge-labels is $B = \{b_j \; ; \; 1 \leq j \leq e\}$, where $b_j = v_j+1$. Now, the set of edge-weights is define as $C = \{a_{2i-1}+b_{e-i+1} \; ; \; 1 \leq i \leq \frac{e+1}{2}\} \cup \{a_{2j}+b_{\frac{e-1}{2}-j+1} \; ; \; 1 \leq j \leq \frac{e+1}{2}-1\}$. It is easy to see that C constitute an arithmetic sequence with $d=1$ and $a'' = s+\frac{3}{2}v = \frac{k}{2}[4(rn-n+r)+3m+1]+\frac{9}{2}$. Since, all the vertices receive the smallest labels, λ is a super $(a'',1)$-EAT labeling. ■

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