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Distance k-domination of some path related graphs

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Abstract

In this paper we determine the distance k-domination number for splitting graph of path as well as the graphs obtained by duplication of a vertex by an edge and duplication of an edge by a vertex.

Keywords: Dominating set, domination number, distance domination, splitting graph.
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1 Introduction

The graph $G = (V(G), E(G))$ we mean simple, finite, connected and undirected graph. A set $D \subseteq V(G)$ is called a dominating set if every vertex in $V(G) - D$ is adjacent to at least one vertex in D. For every vertex $v \in V(G)$, the open neighbourhood set $N(v)$ is the set of all vertices adjacent to v in G. That is, $N(v) = \{u \in V(G)/uv \in E(G)\}$. The closed neighbourhood set $N[v]$ of v is defined as $N[v] = N(v) \cup \{v\}$. The distance $d(u, v)$ between two vertices u and v is the length of the shortest path between u and v in G. If there is no path between u and v in G then $d(u, v) = \infty$. The open k-neighbourhood set $N_k(v)$ of vertex $v \in V(G)$ is the set of all vertices of G which are different from v and at distance at most k from v in G. That is, $N_k(v) = \{u \in V(G)/d(u, v) \leq k\}$. The closed k-neighbourhood set $N_k[v]$ of v is defined as $N_k[v] = N_k(v) \cup \{v\}$. Note that $N(v) = N_1(v)$. For terminology and notation not defined here, we follow West [7] and Haynes *et al*. [2]. The problem of finding a minimal distance k-dominating set (call k-basis) was considered by Slater [5] with special reference to communication networks while the distance k-dominating set was defined by Henning *et al*. [4]. For an integer $k \geq 1$, a set $D \subseteq V(G)$ is a distance k-dominating set of G if every vertex in $V(G) - D$ is within distance k from some vertex $v \in D$. That is, $N_k[D] = V(G)$. The minimum cardinality among all distance k-dominating sets of G is called the distance k-domination number of G and is denoted by $\gamma_k(G)$. It is obvious that $\gamma(G) = \gamma_1(G)$. A distance k-dominating set of cardinality $\gamma_k(G)$ is called a γ_k-set. The distance domination in the context of spanning tree is discussed in Griggs and Hutchinson [1] while bounds on the distance two-domination number and the classes of graphs attaining these bounds are reported in Sridharan *et al*. [6]. For more bibliographic references on distance k-domination readers are advised to refer the survey by Henning [3].

In general four types of problems are dealt in the study of domination in graphs.

(1) Introduction of new type of dominating sets.

(2) Study of bounds in terms of various graph theoretic parameters.
(3) Obtaining exact domination number for some graphs or family of graphs.
(4) Study of algorithmic and complexity results.

Our present work is intended to discuss the problem of the third kind. We compute $\gamma_k(G)$ for some path related graphs.

2 Some Definitions and Results on Distance k-domination

Definition 2.1. For a graph G, the splitting graph $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 2.2. Duplication of a vertex v_i by a new edge $e = v_i'v_i''$ in graph G produces a new graph G' such that $N(v_i') \cap N(v_i'') = \{v_i\}$.

Definition 2.3. Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.

Propostion 2.4. [3] Let $k \geq 1$ and D be a distance k-dominating set of a graph G. Then D is a minimal distance k-dominating set of G if and only if each $d \in D$ has at least one of the following two properties hold.
(1) There exist a vertex $v \in V(G) - D$ such that $N_k(v) \cap D = \{d\}$.
(2) The vertex d is at distance at least $k+1$ from every other vertex d of D in G.

Theorem 2.5. If $n \leq 2k+1$, $k \neq 1$, then $\gamma_k(S'(P_n)) = 1$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ which are added to obtain $S'(P_n)$. Then $D = \left\{v_{\lceil \frac{n}{2} \rceil}\right\}$ is distance k-dominating set of $S'(P_n)$ and hence $\gamma_k(S'(P_n)) = 1$. ■

Theorem 2.6. For $n > 2k+1$, $\gamma_k(S'(P_n)) = \begin{cases} \left\lfloor \frac{n}{2k+1} \right\rfloor + 1 & \text{for } n \equiv 1, 2, ..., 2k \,(\mathrm{mod}\,(2k+1)) \\ \frac{n}{2k+1} & \text{for } n \equiv 0 \,(\mathrm{mod}\,(2k+1)) \end{cases}$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ which are added to obtain $S'(P_n)$. Now every vertex from $\{v_{k+1}, v_{k+2}, \ldots, v_{n-k}\}$ dominates $2k+1$ vertices of v_i's and $2k+1$ vertices of u_i's at a distance k, while every vertex from $\{u_{k+1}, u_{k+2}, \ldots, u_{n-k}\}$ dominates $2k+1$ vertices of v_i's and $2k+1$ vertices of u_i's at a distance k. Therefore, at least one of the vertices from $\{v_{1+(2k+1)j}, v_{2+(2k+1)j}, \ldots, v_{(k+1)+(2k+1)j}, v_{(k+2)+(2k+1)j}, \ldots, v_{(2k+1)+(2k+1)j}\}$ must belong to every distance k-dominating set D of $S'(P_n)$.

Hence, $\gamma_k\left(S'(P_n)\right) \geq \left\lfloor \dfrac{n}{2k+1} \right\rfloor$. ...(1)

Now depending upon the number of vertices of P_n, we consider the following subsets.

For $n \equiv 1, 2, \ldots, k \,(\mathrm{mod}(2k+1))$, $D = \left\{v_{(k+1)+(2k+1)j}, v_n / 0 \leq j < \left\lfloor \frac{n}{2k+1} \right\rfloor\right\}$. Hence, $|D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1$.

For $n \equiv k+1, k+2, \ldots, 2k \,(\mathrm{mod}(2k+1))$, $D = \left\{v_{(k+1)+(2k+1)j} / 0 \leq j \leq \left\lfloor \frac{n}{2k+1} \right\rfloor\right\}$. Hence, $|D| =$

$\left\lfloor \frac{n}{2k+1} \right\rfloor + 1$.
For $n \equiv 0\,(\mathrm{mod}(2k+1))$, $D = \left\{ v_{(k+1)+(2k+1)j} / 0 \le j < \frac{n}{2k+1} \right\}$. Hence, $|D| = \frac{n}{2k+1}$.

We claim that each D is a distance k-dominating set as $d(v_{(k+1)+(2k+1)j}, v_{i+(2k+1)j}) \le k$, $d(v_{(k+1)+(2k+1)j}, u_{i+(2k+1)j}) \le k$ for $1 \le i \le 2k+1$, $d(v_n, v_{n-l}) \le k$ and $d(v_n, u_{n-l}) \le k$ for $1 \le l \le k$. Therefore, $N_k(v_{(k+1)+(2k+1)j}) = \{v_{1+(2k+1)j}, v_{2+(2k+1)j}, \dots, v_{k+(2k+1)j}, v_{(k+2)+(2k+1)j}, \dots, v_{(2k+1)+(2k+1)j}, u_{1+(2k+1)j}, u_{2+(2k+1)j}, \dots, u_{k+(2k+1)j}, u_{(k+2)+(2k+1)j}, \dots, u_{(2k+1)+(2k+1)j}\}$ and $N_k(v_n) = \{v_{n-1}, v_{n-2}, \dots, v_{n-k}, u_n, u_{n-1}, \dots, u_{n-k}\}$. Hence we have, $N_k[D] = V(S'(P_n))$ for $n \equiv k+1, k+2, \dots, 2k+1\,(\mathrm{mod}(2k+1))$ and $N_k[D] = V(S'(P_n))$ for $n \equiv 1, 2, \dots, k\,(\mathrm{mod}(2k+1))$. Now from the nature of $S'(P_n)$, one can observe that every vertex $d \in D$ is at a distance at least $k+1$ from every other vertex of D in $S'(P_n)$.
Thus by Proposition 2.4, above defined D is a minimal distance k-dominating set of $S'(P_n)$ and by (1) it is also of minimum cardinality for $n > 2k+1$.
Hence $\gamma_k(S'(P_n)) = \begin{cases} \left\lfloor \frac{n}{2k+1} \right\rfloor + 1 & \text{for } n \equiv 1, 2, ..., 2k\,(\mathrm{mod}\,(2k+1)) \\ \frac{n}{2k+1} & \text{for } n \equiv 0\,(\mathrm{mod}\,(2k+1)) \end{cases}$. ■

Theorem 2.7. Let G be a graph obtained by duplication of each vertex of path P_n, $n \le 2k-1$, by an edge then $\gamma_k(G) = 1$.

Proof: Let G be a graph obtained by duplication of vertices $v_1, v_2, \dots, v_n$ of path P_n by an edge $u_{2i-1}u_{2i} (1 \le i \le n)$. Then $D = \left\{ v_{\lfloor \frac{n}{2} \rfloor} \right\}$ is a distance k-dominating set as $n \le 2k-1$. Hence $\gamma_k(G) = 1$. ■

Theorem 2.8. Let G be a graph obtained by duplication of each vertex of path P_n, $n > 2k-1$, by an edge then $\gamma_k(G) = \begin{cases} \left\lfloor \frac{n}{2k-1} \right\rfloor + 1 & \text{for } n \equiv 1, 2, ..., 2k-2\,(\mathrm{mod}\,(2k-1)) \\ \frac{n}{2k-1} & \text{for } n \equiv 0\,(\mathrm{mod}\,(2k-1)) \end{cases}$.

Proof: Let G be a graph obtained by duplication of vertices $v_1, v_2, \dots, v_n$ of path $P_n = (v_1, v_2, \dots, v_n)$ by an edge $u_{2i-1}u_{2i}\,(1 \le i \le n)$. Now every vertex from the set $\{v_k, v_{k+1}, \dots, v_{n-(k-2)}\}$ dominates $2k$ vertices of v_i's and $4k-2$ vertices of u_i's at a distance k, while every vertex from $\{u_{2k-2}, u_{2k-1}, \dots, u_{n-(2k-4)}\}$ dominates $2k+1$ vertices of v_i's and $4k-6$ vertices of u_i's at a distance k. Therefore, at least one of the vertices from $\{v_{1+(2k-1)j}, v_{2+(2k-1)j}, \dots, v_{k+(2k-1)j}, \dots, v_{(2k-1)+(2k-1)j}\}$ must belong to every distance k-dominating set D of G.
Hence $\gamma_k(G) \ge \left\lfloor \frac{n}{2k-1} \right\rfloor$. . . . (1)
Now depending upon the number of vertices of P_n, we consider the following subsets.
For $n \equiv 1, 2, \dots, k-1\,(mod\,(2k-1))$, $D = \{v_{k+(2k-1)j}, v_n / 0 \le j < \left\lfloor \frac{n}{2k-1} \right\rfloor\}$. Hence, $|D| = \left\lfloor \frac{n}{2k-1} \right\rfloor + 1$.
For $n \equiv k, k+1, \dots, 2k-2\,(mod\,(2k-1))$, $D = \{v_{k+(2k-1)j} / 0 \le j \le \left\lfloor \frac{n}{2k-1} \right\rfloor\}$. Hence, $|D| = \left\lfloor \frac{n}{2k-1} \right\rfloor + 1$.
For $n \equiv 0\,(mod\,(2k-1))$, $D = \{v_{k+(2k-1)j} / 0 \le j < \frac{n}{2k-1}\}$. Hence, $|D| = \frac{n}{2k-1}$.
We claim that each D is a distance k-dominating set as $d(v_{k+(2k-1)j}, v_{i+(2k-1)j}) \le k$ for $1 \le i \le 2k-1$, $d(v_{k+(2k-1)j}, u_{l+(2k-1)j}) \le k$ for $0 \le l \le 4k-2$, $d(v_n, v_{n-r}) \le k$ for $0 \le r \le k$ and

$d(v_n, u_{2n-s}) \leq k$ for $0 \leq s \leq 2k$.
Therefore, $N_k(v_{k+(2k-1)j}) = \{v_{1+(2k-1)j}, v_{2+(2k-1)j}, \ldots, v_{k+(2k-1)j}, \ldots, v_{(2k-1)+(2k-1)j}, u_{1+(2k-1)j}, u_{2+(2k-1)j}, \ldots, u_{2k+(2k-1)j}, \ldots, u_{(4k-2)+(2k-1)j}\}$ and $N_k(v_n) = \{v_{n-k}, \ldots, v_{n-1}, v_n, u_{2n-(2k+1)}, u_{2n-2k}, \ldots, u_{2n-1}, u_{2n}\}$.
Hence we have $N_k[D] = V(G)$ for $n \equiv 0, k, k+1, \ldots, 2k-2\,(mod\,(2k-1))$ and $N_k[D] = V(G)$ for $n \equiv 1, 2, \ldots, k-1$. Now from the nature of graph G, one can observe that every vertex $d \in D$ is at a distance at least $k+1$ from every other vertex of D in G.
Thus by Proposition 2.4, above defined D is a minimal distance k-dominating set of G and by (1) it is also of minimum cardinality for $n > 2k-1$.
Hence, $\gamma_k(G) = \begin{cases} \left\lfloor \frac{n}{2k-1} \right\rfloor + 1 & \text{for } n \equiv 1, 2, ..., 2k-2\,(\mathrm{mod}\,(2k-1)) \\ \frac{n}{2k-1} & \text{for } n \equiv 0\,(\,\mathrm{mod}\,(2k-1)) \end{cases}$. ■

Theorem 2.9. Let G be a graph obtained by duplication of each edge of path P_n, $n \leq 2k+1$, by a vertex then $\gamma_k(G) = 1$.

Proof: Let G be a graph obtained by duplication of each edge $v_i v_{i+1}$ of path P_n by a vertex u_i, $(1 \leq i < n)$ Then $\left\{v_{\lfloor \frac{n}{2} \rfloor}\right\}$ is a distance k-dominating set of G as $n \leq 2k+1$. Hence $\gamma_k(G) = 1$. ■

Theorem 2.10. Let G be a graph obtained by duplication of each edge of path P_n, $n > 2k+1$, by a vertex then $\gamma_k(G) = \begin{cases} \left\lfloor \frac{n}{2k+1} \right\rfloor + 1 & \text{for}\, n \equiv 1, 2, ..., 2k\,(\mathrm{mod}(2k+1)) \\ \frac{n}{2k+1} & \text{for}\, n \equiv 0(\mathrm{mod}\ (2k+1)) \end{cases}$.

Proof: Let G be a graph obtained by duplicating each edge $v_i v_{i+1}$ of the path P_n by inserting the vertex u_i $(1 \leq i < n)$. Now every vertex from $\{v_{k+1}, v_{k+2}, \ldots, v_{n-k}\}$ dominates $2k+1$ vertices of v_i's and $2k$ vertices of u_i's at a distance k, while every vertex from $\{u_{n-k}, u_{n-k+1}, \ldots, u_{n-l}\}$ dominates $2k$ vertices of v_i's and $2k-1$ vertices of u_i's at a distance k. Therefore, at least one of the vertices from $\{v_{1+(2k+1)j}, v_{2+(2k+1)j}, \ldots, v_{(k+1)+(2k+1)j}, v_{(k+2)+(2k+1)j}, \ldots, v_{(2k+1)+(2k+1)j}\}$ must belong to every distance k-dominating set D of G.
Hence $\gamma_k(G) \geq \left\lfloor \frac{n}{2k+1} \right\rfloor$. ...(1)
Now depending upon the number of vertices of P_n, consider the following subsets.
For $n \equiv 1, 2, \ldots, k\,(mod\,(2k+1))$, $D = \{v_{(k+1)+(2k+1)j}, v_n / 0 \leq j < \left\lfloor \frac{n}{2k+1} \right\rfloor\}$. Hence, $|D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1$.
For $n \equiv k+1, k+2, \ldots, 2k\,(mod\,(2k+1))$, $D = \{v_{(k+1)+(2k+1)j} / 0 \leq j \leq \left\lfloor \frac{n}{2k+1} \right\rfloor\}$. Hence, $|D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1$.
For $n \equiv 0\,(mod\,(2k+1))$, $D = \{v_{(k+1)+(2k+1)j} / 0 \leq j < \frac{n}{2k+1}\}$. Hence, $|D| = \frac{n}{2k+1}$.

Now we claim that each D is a distance k-dominating set as $d(v_{(k+1)+(2k+1)j}, v_{i+(2k+1)j}) \leq k$, $d(v_{(k+1)+(2k+1)j}, u_{i+(2k+1)j}) \leq k$ where $1 \leq i \leq 2k+1$, $d(v_n, v_{n-l}) \leq k$ and $d(v_n, u_{n-l}) \leq k$ where $1 \leq l \leq k$.
Therefore, $N_k(v_{(k+1)+(2k+1)j}) = \{v_{1+(2k+1)j}, v_{2+(2k+1)j}, \ldots, v_{k+(2k+1)j}, v_{(k+2)+(2k+1)j}, \ldots, v_{(2k+1)+(2k+1)j}, u_{1+(2k+1)j}, u_{2+(2k+1)j}, \ldots, u_{k+(2k+1)j}, u_{(k+2)+(2k+1)j}, \ldots, u_{(2k+1)+(2k+1)j}\}$ and $N_k(v_n) = \{v_{n-1}, v_{n-2}, \ldots, v_{n-k}, u_{n-1}, u_{n-2}, \ldots, u_{n-k}\}$.
Hence we have $N_k[D] = V(G)$ for $n \equiv 0, k+1, k+2, \ldots, 2k\,(mod(2k+1))$ and $N[D] = V(G)$

for $n \equiv 1, 2, \ldots, k$. Now from the nature of graph G, one can observe that every vertex $d \in D$ is at a distance at least $k+1$ from every other vertex of D in G.
Thus by Proposition 2.4, above defined D is a minimal distance k-dominating set of G and by (1) it is also of minimum cardinality for $n > 2k+1$.
Hence $\gamma_k(G) = \begin{cases} \left\lfloor \frac{n}{2k+1} \right\rfloor + 1 & \text{for}\, n \equiv 1, 2, ..., 2k\, (\text{mod}(2k+1)) \\ \frac{n}{2k+1} & \text{for}\, n \equiv 0 (\text{mod } (2k+1)) \end{cases}$. ∎

3 Concluding Remarks

This work throw some light on distance k-domination of a super graph obtained by means of some graph operations on the given graph and more exploration is possible with respect to other domination concepts.

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References

[1] J. R. Griggs and J. P. Hutchinson, *On the r-domination number of a graph*, Discrete Mathematics, 101, (1992), 65-72.

[2] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[3] M. A. Henning, *Distance Domination in Graphs*, in : T. W. Haynes, S. T. Hedetniemi and P. J. Slater Ed., Domination in Graphs: Advanced Topics, Marcel Dekker, New York, (1998), 321-349.

[4] M. A. Henning, O. R. Swart and H. C. Swart, *Bounds on distance domination parameters*, J. Combin. Inform. System Sci., 16, (1991), 11-18.

[5] P. J. Slater, *R-Domination in Graphs*, Journal of Association for Computer Machinery, 23(3), (1976), 446-450.

[6] N. Sridharan, V.S.A. Subramanian and M.D. Elias, *Bounds on the Distance Two-Domination Number of a Graph*, Graphs and Combinatorics, 18(3), (2002), 667-675.

[7] D. B. West, *Introduction to graph theory*, 2/e, Prentice-Hall, New Delhi, India, 2003.