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Total complementary tree domination in grid graphs

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Abstract

Let $G = (V, E)$ be a nontrivial, simple, finite and undirected graph. A dominating set D is called a complementary tree dominating set if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. A dominating set D_t is called a total complementary tree dominating set if every vertex $v \in V$ is adjacent to an element of D_t and $< V - D_t >$ is a tree. The minimum cardinality of a total complementary tree dominating set is called the total complementary tree domination number of G and is denoted by γ_{tctd}. In this paper, we determine the total complementary tree domination numbers of some grid graph.

Keywords: Total domination, total complementary tree domination.
AMS Subject Classification(2010): 05C69.

1 Introduction

The graphs considered here are nontrivial, simple, finite and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The concept of domination was first studied by Ore [8]. A set $D \subseteq V$ is said to a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. The concept of complementary tree domination was introduced by S. Muthammai, M. Bhanumathi and P. Vidhya in [6]. A dominating set $D \subseteq V$ is called a complementary tree dominating (ctd) set, if the subgraph $< V - D >$ induced by $V - D$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. The concept of total domination in graphs was introdued by Cockayne, Daves and Hedetnimi [1]. The total domination number of a graph G denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. A dominating set S is called a total dominating set if every vertex $v \in V$ is adjacent to an element of S. A dominating set D_t is called a total complementary tree dominating set if every vertex $v \in V$ is adjacent to an element of D_t and $< V - D_t >$ is a tree. The minimum cardinality of a total complementary tree dominating set is called the total complementary tree domintion number of G and is denoted by $\gamma_{tctd}(G)$.

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The cartesian product of two graphs G_1 and G_2 is the graph, denoted by $G_1 \times G_2$ with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ (where $\times$ denotes the cartesian product of sets) and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V(G_1 \times G_2)$ are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and $(u_2, v_2) \in E(G_2)]$ or $[u_2 = v_2$ and $(u_1, v_1) \in E(G_1)]$. If each G_1 and G_2 is a path P_m and P_n (respectively), then we will call $P_m \times P_n$, a $m \times n$ grid graph. For notational convenience we denote $P_m \times P_n$ by $P_{m,n}$. The reader is referred to [4] for the survey of results on domination.

In this paper, we determine the total complementary tree domination number of $P_{m,n}$ where $m = 4, 6, 8$. S. Muthammai and P. Vidhya [7] have established $\gamma_{ctd}(P_{m,n})$, $m = 2, 3, 4, 5, 6$. $P_{1,n}$ is nothing but the path P_n on n vertices. S. Muthammai, M. Bhanumathi and P. Vidhya [6] have established $\gamma_{ctd}(P_n) = n - 2, n \geq 4$.

Notation. Let $1, \ldots, m$ and $1, \ldots, n$ be the vertices of P_m and P_n, respectively. Then the vertices of $P_{m,n}$ are denoted by $x_{i,j}$ where $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

2 Total Complementary Tree Domination in Grid Graphs

Theorem 2.1. For $n \geq 5$,

$$\gamma_{tctd}(P_{4,n}) = \begin{cases} \left\lfloor \dfrac{8n+2}{5} \right\rfloor, & n = 0, 1, 4 \ (mod\ 5) \\ \left\lfloor \dfrac{8n+6}{5} \right\rfloor, & n = 2, 3 \ (mod\ 5) \end{cases}$$

Proof: We present a total complementary tree dominating set (tctd) D_t of $P_{4,n}$ as follows.
Let $n \geq 5$. We split the set of columns of $P_{4,n}$ into blocks B_i, $B_i \cong P_{4,5}$ for $i = 1, 2, \ldots, q$.
$P_i = \{x_{1,5i-4}, x_{2,5i-4}, x_{2,5i-2}, x_{2,5i-1}, x_{2,5i}, x_{3,5i}, x_{4,5i-3}, x_{4,5i-2}\}$ dominates the first 4 columns of the block B_i, $i = 1, 2, \ldots, q$ such that $< P_{4,n} - P_i >$ is a tree.
Let $D_t = \bigcup_{i=1}^{q} P_i$. (Figure 1.)

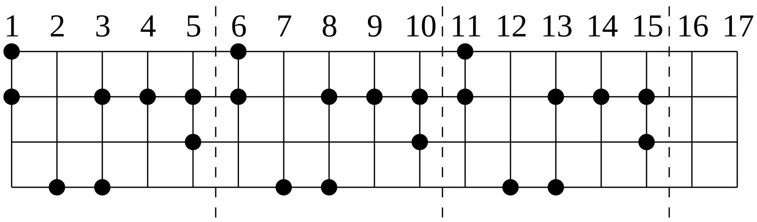

Figure 1.

We consider the following five cases.(Figure 2.)
Case (i): $n \equiv 0 \ (mod\ 5)$.
Let $n = 5q$. Clearly, $< V(P_{4,n}) - D_t >$ is a tree and D_t is a minimal total ctd set.
$|D_t| = 8q = \left\lfloor \dfrac{8n+2}{5} \right\rfloor$.

Case (ii): $n \equiv 1 \ (mod\ 5)$.
Let $D_1 = D_t \cup \{x_{2,n}, x_{3,n}\}$. This set is a total ctd set and $|D_1| = 8 \left\lfloor \dfrac{n}{5} \right\rfloor + 2 = \left\lfloor \dfrac{8n+2}{5} \right\rfloor$.

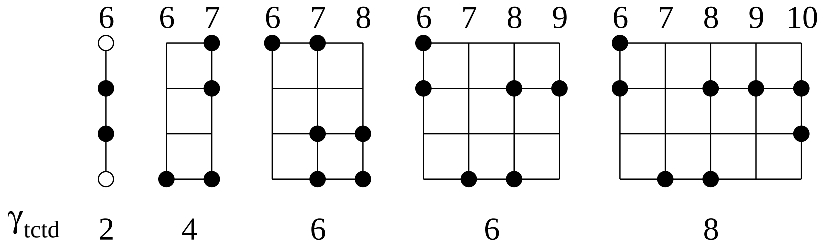

Figure 2.

Case (iii): $n \equiv 2\ (mod\ 5)$.

Let $D_2 = D_t \cup \{x_{4,n-1}, x_{1,n}, x_{2,n}, x_{4,n}\}$. This set is a tctd set and $|D_2| = 8\left\lfloor \frac{n}{5} \right\rfloor + 4 = \left\lfloor \frac{8n+4}{5} \right\rfloor$.

Case (iv): $n \equiv 3\ (mod\ 5)$.

Let $D_3 = D_t \cup \{x_{1,n-2}, x_{1,n-1}, x_{3,n-1}, x_{4,n-1}, x_{3,n}, x_{4,n}\}$. This set is a tctd set and $|D_3| = 8\left\lfloor \frac{n}{5} \right\rfloor + 6 = \left\lfloor \frac{8n+6}{5} \right\rfloor$.

Case (v): $n \equiv 4\ (mod\ 5)$.

Let $D_4 = D_t \cup \{x_{1,n-3}, x_{2,n-3}, x_{4,n-2}, x_{2,n-1}, x_{4,n-1}, x_{2,n}\}$. This set is a tctd set and $|D_4| = 8\left\lfloor \frac{n}{5} \right\rfloor + 6 = \left\lfloor \frac{8n+2}{5} \right\rfloor$.

Therefore, $\gamma_{tctd}(P_{4,n}) = \begin{cases} \left\lfloor \frac{8n+2}{5} \right\rfloor, & n \equiv 0, 1, 4\ (mod\ 5) \\ \left\lfloor \frac{8n+6}{5} \right\rfloor, & n = 2, 3\ (mod\ 5) \end{cases}$ for $n \geq 5$. ■

Remark 2.2. $\gamma_{tctd}(4, 2n) = 2n + 2$, $n = 1, 2$ and $\gamma_{tctd}(4, n) = 6$, $n = 3$.

Theorem 2.3. For $n \geq 7$,

$$\gamma_{tctd}(P_{6,n}) = \begin{cases} \left\lfloor \frac{18n+4}{7} \right\rfloor, & n = 0, 4, 6\ (mod\ 7) \\ \left\lfloor \frac{18n+12}{7} \right\rfloor, & n = 1, 2\ (mod\ 7) \\ \left\lfloor \frac{18n-6}{7} \right\rfloor, & n = 2, 5\ (mod\ 7) \end{cases}$$

Proof: We present a total complementary tree dominating set (tctd) D_t of $P_{6,n}$ as follows.

Let $n \geq 7$. We split the set of columns of $P_{6,n}$ into blocks B_i, $B_i \cong P_{6,7}$ for $i = 1, 2, \ldots, q$.

$P_i = \{x_{1,7i-6}, x_{1,7i-5}, x_{2,7i-3}, x_{2,7i-2}, x_{2,7i-1}, x_{2,7i}, x_{3,7i-5}, x_{3,7i-1}, x_{3,7i}, x_{4,7i-6}, x_{4,7i-5}, x_{4,7i-3}, x_{5,7i-6}, x_{5,7i-3}, x_{5,7i-1}, x_{6,7i-4}, x_{6,7i-3}, x_{6,7i-1}\}$ dominates the first 6 columns of the block B_i, $i = 1, 2, \ldots, q$ such that $< P_{6,n} - P_i >$ is a tree.

Let $D_t = \bigcup_{i=1}^{q} P_i$. (Figure 3.)

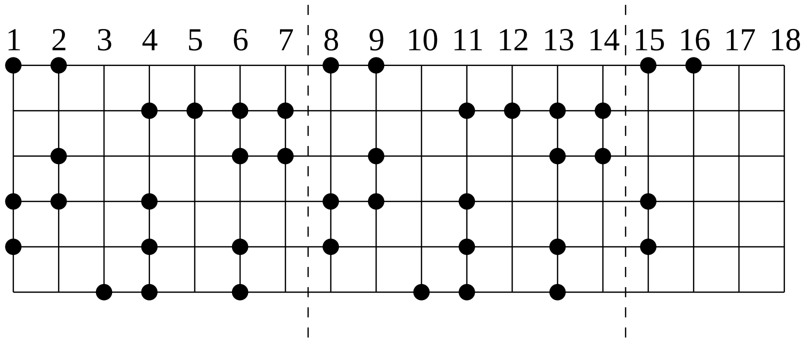

Figure 3.

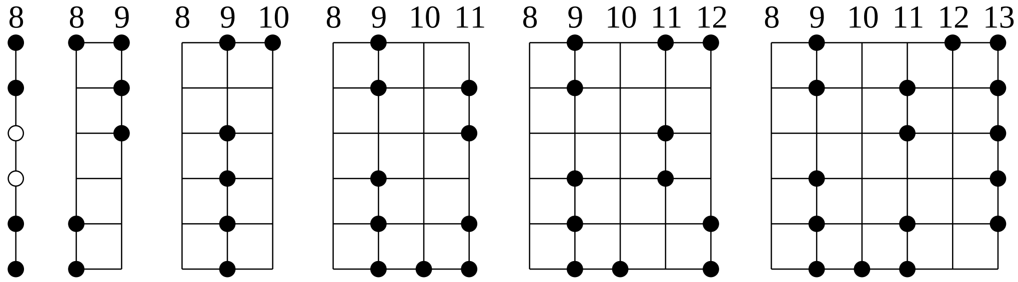

Figure 4.

We consider the following seven cases.(Figure 4.)

Case (i): $n \equiv 0\ (mod\ 7)$.

Let $n = 7q$. Clearly, $< V(P_{6,n}) - D_t >$ is a tree and D_t is a minimal total ctd set.

$|D_t| = 18q = \left\lfloor \frac{18n+4}{7} \right\rfloor$.

Case (ii): $n \equiv 1\ (mod\ 7)$.

Let $D_1 = D_t \cup \{x_{1,n}, x_{2,n}, x_{5,n}, x_{6,n}\}$. This set is a tctd set and $|D_1| = 18\left\lfloor \frac{n}{7} \right\rfloor + 4 = \left\lfloor \frac{18n+10}{7} \right\rfloor$.

Case (iii): $n \equiv 2\ (mod\ 7)$.

Let $D_2 = D_t \cup \{x_{1,n-1}, x_{5,n-1}, x_{5,n-1}, x_{6,n-1}, x_{1,n}, x_{2,n}, x_{3,n}\}$. $|D_2|$ is a tctd set and $|D_2| = 18\left\lfloor \frac{n}{7} \right\rfloor + 6 = \left\lfloor \frac{18n+12}{7} \right\rfloor$.

Case (iv): $n \equiv 3\ (mod\ 7)$.

Let $D_3 = D_t \cup \{x_{1,n-1}, x_{3,n-1}, x_{4,n-1}, x_{5,n-1}, x_{6,n-1}, x_{1,n}\}$. D_3 is a tctd set and $|D_3| = 18\left\lfloor \frac{n}{7} \right\rfloor + 6 = \left\lfloor \frac{18n-6}{7} \right\rfloor$.

Case (v): $n \equiv 4\ (mod\ 7)$.

Let $D_4 = D_t \cup \{x_{1,n-2}, x_{2,n-2}, x_{4,n-2}, x_{5,n-2}, x_{6,n-2}, x_{6,n-1}, x_{2,n}, x_{3,n}, x_{5,n}, x_{6,n}\}$. This set is a total ctd set and $|D_4| = 18\left\lfloor \frac{n}{7} \right\rfloor + 10 = \left\lfloor \frac{18n-2}{7} \right\rfloor = \left\lfloor \frac{18n+4}{7} \right\rfloor$.

Case (vi): $n \equiv 5\ (mod\ 7)$.

Let $D_5 = D_t \cup \{x_{1,n-3}, x_{2,n-3}, x_{4,n-3}, x_{5,n-3}, x_{6,n-3}, x_{6,n-2}, x_{1,n-1}, x_{3,n-1}, x_{2,n-1}, x_{1,n}, x_{5,n}, x_{6,n}\}$.

This set is a total ctd set and $|D_5| = 18\left\lfloor \frac{n}{7} \right\rfloor + 12 = \left\lfloor \frac{18n-6}{7} \right\rfloor$.

Case (vii): $n \equiv 6\ (mod\ 7)$.

Let $D_6 = D_t \cup \{x_{1,n-4}, x_{2,n-4}, x_{4,n-4}, x_{5,n-4}, x_{6,n-4}, x_{6,n-3}, x_{2,n-2}, x_{3,n-2}, x_{5,n-2}, x_{6,n-2}, x_{1,n-1}, x_{1,n}, x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n}, x_{5,n}\}$. D_6 is a tctd set and $|D_6| = 18\left\lfloor \frac{n}{7} \right\rfloor + 16 = \left\lfloor \frac{18n+4}{7} \right\rfloor$.

$$\text{Therefore, } \gamma_{tctd}(P_{6,n}) = \begin{cases} \left\lfloor \frac{18n+4}{7} \right\rfloor, & n = 0, 4, 6\ (mod\ 7) \\ \left\lfloor \frac{18n+12}{7} \right\rfloor, & n = 1, 2\ (mod\ 7) \quad \text{for } n \geq 7. \\ \left\lfloor \frac{18n-6}{7} \right\rfloor, & n = 3, 5\ (mod\ 7) \end{cases}$$ ∎

Remark 2.4. $\gamma_{tctd}(P_{6,n+1}) = 2n + 4$, $n = 1, 2, 3, 4$ and $\gamma_{ctd}(P_{6,n}) = 16$, $n = 6$.

Theorem 2.5. For $n \geq 9$,

$$\gamma_{tctd}(P_{8,n}) = \begin{cases} \left\lfloor \frac{28n+8}{9} \right\rfloor, & n \equiv 0, 1\ (mod\ 9) \\ \left\lfloor \frac{28n+16}{9} \right\rfloor, & n = 2, 4, 8\ (mod\ 9) \\ \left\lfloor \frac{28n+24}{9} \right\rfloor, & n \equiv 3, 5, 6, 7\ (mod\ 9) \end{cases}$$

Proof: We present a total complementary tree dominating set (tctd) D_t of $P_{8,n}$ as follows.

Let $n \geq 9$. We split the set of columns of $P_{8,n}$ into blocks B_i, $B_i \cong P_{8,9}$ for $i = 1, 2, \ldots, q$.

$P_i = \{x_{1,9i-8}, x_{1,9i-7}, x_{1,9i-1}, x_{2,9i-5}, x_{2,9i-4}, x_{2,9i-3}, x_{2,9i-1}, x_{3,9i-8}, x_{3,9i-6}, x_{3,9i-3}, x_{4,9i-8}, x_{4,9i-4}, x_{4,9i-2}, x_{4,9i}, x_{5,9i-7}, x_{5,9i-4}, x_{5,9i-2}, x_{5,9i}, x_{6,9i-7}, x_{6,9i-5}, x_{7,9i-5}, x_{7,9i-3}, x_{7,9i-8}, x_{7,9i}, x_{8,9i-8}, x_{8,9i-7}, x_{8,9i-4}\}$ dominates the first 8 columns of the block B_i, $i = 1, 2, \ldots, q$ such that $< P_{8,n} - P_i >$ is a tree.

Let $D_t = \bigcup_{i=1}^{q} P_i$. (Figure 5.)

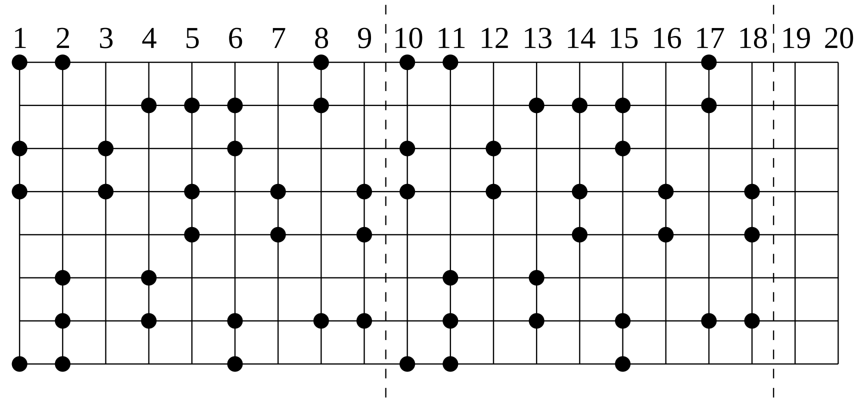

Figure 5.

We consider the following nine cases.(Figure 6.)

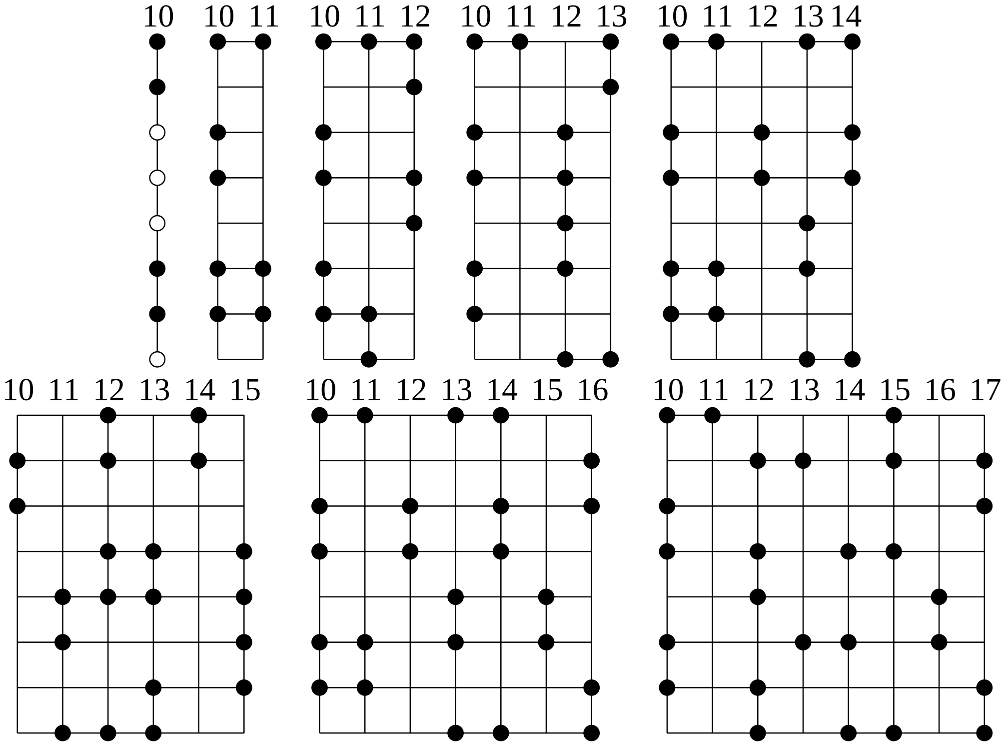

Figure 6.

Case (i): $n \equiv 0\ (mod\ 9)$.

Let $n = 9q$. Clearly, $< V(P_{8,n}) - D_t >$ is a tree and D_t is a minimal total ctd set.
$|D_t| = 28q = \left\lfloor \frac{28n+8}{9} \right\rfloor$.

Case (ii): $n \equiv 1\ (mod\ 9)$.

Let $D_1 = D_t \cup \{x_{1,n}, x_{2,n}, x_{6,n}, x_{7,n}\}$. Then D_1 is a tctd set and $|D_1| = 28 \left\lfloor \frac{n}{9} \right\rfloor + 4 = \left\lfloor \frac{28n+8}{9} \right\rfloor$.

Case (iii): $n \equiv 2\ (mod\ 9)$.

Let $D_2 = D_t \cup \{x_{1,n-1}, x_{3,n-1}, x_{4,n-1}, x_{6,n-1}, x_{7,n-1}, x_{1,n}, x_{6,n}, x_{7,n}\}$.
This is a tctd set and $|D_2| = 28 \left\lfloor \frac{n}{9} \right\rfloor + 8 = \left\lfloor \frac{28n+16}{9} \right\rfloor$.

Case (iv): $n \equiv 3\ (mod\ 9)$.

Let $D_3 = D_t \cup \{x_{1,n-2}, x_{3,n-2}, x_{4,n-2}, x_{6,n-2}, x_{7,n-2}, x_{1,n-1}, x_{7,n-1}, x_{8,n-1}, x_{1,n}, x_{2,n}, x_{4,n}, x_{5,n}\}$.
D_3 is a minimal tctd set and $|D_3| = 28 \left\lfloor \frac{n}{9} \right\rfloor + 12 = \left\lfloor \frac{28n+24}{9} \right\rfloor$.

Case (v): $n \equiv 4\ (mod\ 9)$.

Let $D_4 = D_t \cup \{x_{1,n-3}, x_{3,n-3}, x_{4,n-3}, x_{6,n-3}, x_{7,n-3}, x_{1,n-2}, x_{3,n-3}, x_{4,n-3}, x_{5,n-3}, x_{6,n-3}, x_{1,n}, x_{2,n}, x_{8,n-3}, x_{8,n}\}$.
This set is a minimal tctd set and $|D_4| = 28 \left\lfloor \frac{n}{9} \right\rfloor + 14 = \left\lfloor \frac{28n+16}{9} \right\rfloor$.

Case (vi): $n \equiv 5\ (mod\ 9)$.

Let $D_5 = D_t \cup \{x_{1,n-4}, x_{3,n-4}, x_{4,n-4}, x_{6,n-4}, x_{7,n-4}, x_{1,n-3}, x_{6,n-3}, x_{7,n-3}, x_{3,n-2}, x_{4,n-2}, x_{1,n-1},$

$x_{5,n-1}, x_{6,n-1}, x_{8,n-1}, x_{1,n}, x_{3,n}, x_{4,n}, x_{8,n}\}$.
This set is a minimal tctd set and $|D_5| = 28\left\lfloor\frac{n}{9}\right\rfloor + 18 = \left\lfloor\frac{28n+24}{9}\right\rfloor$.
Case (vii): $n \equiv 6\ (mod\ 9)$.
Let $D_6 = D_t \cup \{x_{1,n-5}, x_{3,n-5}, x_{4,n-5}, x_{6,n-5}, x_{7,n-5}, x_{1,n-4}, x_{6,n-4}, x_{7,n-4}, x_{3,n-3}, x_{4,n-3}, x_{1,n-2}, x_{5,n-2}, x_{6,n-2}, x_{8,n-2}, x_{1,n-1}, x_{3,n-1}, x_{4,n-1}, x_{8,n-1}, x_{3,n}, x_{4,n}, x_{5,n}, x_{6,n}\}$.
D_6 is a minimal tctd set and $|D_6| = 28\left\lfloor\frac{n}{9}\right\rfloor + 20 = \left\lfloor\frac{28n+16}{9}\right\rfloor$.
Case (viii): $n \equiv 7\ (mod\ 9)$.
Let $D_7 = D_t \cup \{x_{1,n-6}, x_{3,n-6}, x_{4,n-6}, x_{6,n-6}, x_{7,n-6}, x_{1,n-5}, x_{6,n-5}, x_{7,n-5}, x_{3,n-4}, x_{4,n-4}, x_{1,n-3}, x_{5,n-3}, x_{6,n-3}, x_{8,n-3}, x_{1,n-2}, x_{3,n-2}, x_{4,n-2}, x_{8,n-2}, x_{5,n-1}, x_{6,n-1}, x_{2,n}, x_{3,n}, x_{7,n}, x_{8,n}\}$.
This set is a minimal tctd set and $|D_7| = 28\left\lfloor\frac{n}{9}\right\rfloor + 24 = \left\lfloor\frac{28n+24}{9}\right\rfloor$.
Case (ix): $n \equiv 8\ (mod\ 9)$.
Let $D_8 = D_t \cup \{x_{1,n-7}, x_{3,n-7}, x_{4,n-7}, x_{6,n-7}, x_{7,n-7}, x_{1,n-6}, x_{2,n-5}, x_{4,n-5}, x_{5,n-5}, x_{7,n-5}, x_{8,n-5}, x_{2,n-4}, x_{6,n-4}, x_{4,n-3}, x_{6,n-3}, x_{8,n-3}, x_{1,n-2}, x_{2,n-2}, x_{4,n-2}, x_{8,n-2}, x_{5,n-1}, x_{6,n-1}, x_{2,n}, x_{3,n}, x_{7,n}, x_{8,n}\}$.
This set is a minimal tctd set and $|D_8| = 28\left\lfloor\frac{n}{9}\right\rfloor + 26 = \left\lfloor\frac{28n+16}{9}\right\rfloor$.

$$\text{Therefore, } \gamma_{tctd}(P_{8,n}) = \begin{cases} \left\lfloor\frac{28n+8}{9}\right\rfloor, & n \equiv 0,1\ (mod\ 9) \\ \left\lfloor\frac{28n+16}{9}\right\rfloor, & n = 2,4,8\ (mod\ 9) \\ \left\lfloor\frac{28n+24}{9}\right\rfloor, & n \equiv 3,5,6,7\ (mod\ 9) \end{cases} \quad \blacksquare$$

Remark 2.6. $\gamma_{tctd}(P_{8,n}) = 6$ if $n = 2$, $\gamma_{tctd}(P_{8,2n+1}) = 6n+4$ if $n = 1,2,3$ and $\gamma_{tctd}(P_{8,2n}) = 6n+8$ if $n = 2,3,4$.

Acknowledgements

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