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On near felicitous labelings of graphs

V. Lakshmi Alias Gomathi, A. Nagarajan, A. Nellai Murugan
Department of Mathematics,
V.O.C. College, Tuticorin - 628 001, INDIA.
E-mail: lakshmi10674@gmail.com, anellai.vocc@gmail.com

Abstract

A simple graph G is called near felicitous if there exists a 1 – 1 function $f:V(G) \rightarrow \{0, 1, 2, . . . , q - 1, q + 1\}$ such that the set of induced edge labels $f^*(uv) = (f(u) + f(v)) \pmod q$ are all distinct when the addition is taken modulo q with residues 1, 2, 3, . . . , q. It is shown that an even subgraph of a near felicitous graph with an even number of edges contain an even number of odd labeled edges. As a consequence, some families of graphs are shown to be non – near felicitous.

Keywords: Near felicitous, cartesian product, wreath product, odd edge labeling, near odd edge labeling, even edge labeling, near even edge labeling, even graph.

AMS Subject Classification (2010): 05C78.

1 Introduction

In 1966, Rosa [7] introduced α - valuation of a graph and subsequently Golomb introduced graceful labeling. In 1980, Graham and Slonae [5] introduced the harmonious labeling of a graph. Several graph labelings have been found in Gallian Survey [4]. Lee, Schmeichel and Shee [6] introduced the concept of felicitous graph as a generalization of a harmonious graph. A *felicitous* labeling of a graph G, with q edges is an injection $f:V(G) \rightarrow \{0, 1, 2,. . . , q\}$ so that the induced edge labels $f^*(xy) = (f(x) + f(y)) \pmod q$ are distinct.

Near Graceful lebeling was introduced by Frucht [3] and near α - labeling was introduced by S. El – Zanati, M. Kenig and C. Vanden Eynden [8]. It motivates us to define the concept of near felicitous labeling as follows.

A near felicitous labeling of a graph G, with q edges is an injection $f:V(G) \rightarrow \{0, 1, 2, . . . , q - 1, q + 1\}$ so that the induced edge labels $f^*(xy) = (f(x) + f(y)) \pmod q$ are distinct and $f^*(E(G)) = \{1, 2, 3, . . . , q\}$.

The graphs we consider are simple. For notation and terminology, we refer to [1]. Throughout this paper, f denotes a 1 – 1 function from $V(G)$ to a subset of the set of non- negative integers and, for any edge $e = xy \in E(G)$, $f^*(e) = f(x) + f(y)$. Let f be a near felicitous labeling of G. Then an edge e of G is called an odd edge under f, if $f^*(e)$ is odd. For any two graphs G and H, the cartesian product $G \times H$ is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(u,v) \quad (u', v')$. either $uu' \in E(G)$ and $v = v'$ or $u = u'$ and $vv' \in E(H)\}$. The wreath product $G * H$ has vertex set $V(G) \times V(H)$ in which (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Let G_1 and G_2 be any two graphs and ϕ. $V(G_2) \to V(G_1)$ be any mapping. Then $G_2[G_1]$ is the graph by attaching at each vertex v of G_2 a copy of G_1 rooted at $\phi(v)$. $V(G)$ x $V(H)$ in which (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. For any two graphs G and H, by $(G \circ H)(v)$, we mean the graph obtained by fusing a copy of H at the vertex v of G. The ***Middle graph*** M(G) of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it. The ***Total graph*** T(G) of a graph G has the vertex $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in G.

Let G be a (p, q) graph. A 1 – 1 function $f{:}V(G) \to \{0, 1, 2, \ldots, q\}$ is said to be an odd – edge labeling of G, if for every edge $e = uv \in E(G)$, $f(u) + f(v)$ is odd and $f^*(E(G)) = \{1, 3, 5, \ldots, 2q - 1\}$. We observe that if G admits an odd – edge labeling, then G is bipartite. However, the converse is not true. For example, C_6 is bipartite, but it has no odd – edge labeling. In [6], it has been proved that $P_2 \cup C_{2k+1}$ and $P_3 \cup C_{2k+1}$ are felicitous and conjectured that $P_n \cup C_{2k+1}$ is felicitous for all $n \geq 4$. But $P_5 \cup C_5$ is near felicitous as shown in Figure 1.

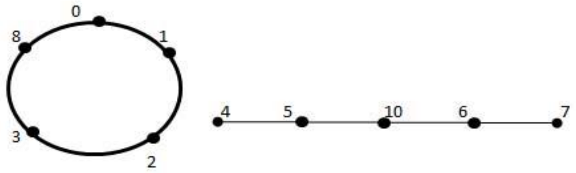


Figure 1: *A near felicitous labeling of* $P_5 \cup C_5$.

2 Main Results

Definition 2.1. Let G be a graph. A 1 – 1 function $f{:}V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ is said to be a near even edge labeling of G, if for every edge $e = uv \in E(G)$, $f(u) + f(v)$ is even and $f^*(E(G)) = \{2, 4, 6, \ldots, 2q\}$.

Similarly, a 1 – 1 function $f{:}V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ is said to be a near odd edge labeling if for every edge $e = uv \in E(G)$, $f(u) + f(v)$ is odd and $f^*(E(G)) = \{1, 3, 5, \ldots, 2q-1\}$.

A subgraph H of a graph G is said to be an even subgraph of G, if the degree of every vertex of H is even in H.

Theorem 2.2. Let G be a near felicitous graph with even number of edges. Then every even subgraph G' of G contains an even number of odd edges.

Proof: Let g be a near felicitous labeling of G. Clearly, for any even subgraph G' of G, we have

$$\sum_{e \in E(G')} g^*(e) = \sum_{v \in V(G')} \deg(v) g(v) (\text{mod} q) \qquad (1)$$

Since q is even and also $\deg(v)$ is even for every $v \in V(G')$, the L.H.S. of (1) is even. So we conclude that the number of edges with odd labels is even. ■

Corollary 2.3. No even graph with $4n + 2$ edges is near felicitous.

Proof: Suppose an even graph G with 4n + 2 edges is near felicitous, then the number of odd edges of G is $\frac{4n+2}{2}$ = $2n + 1$, which is odd, a contradiction. Hence, no even graph with $4n + 2$ edges is near felicitous. ■

Corollary 2.4. If G is an r – regular graph on p vertices, then M(G) is not near felicitous in the following cases.

(i) $p \equiv 2 \pmod 4$ and $r \equiv 2 \pmod 8$.

(ii) $p \equiv 2 \pmod 4$ and $r \equiv 6 \pmod 8$.

(iii) $p \equiv 1 \pmod 2$ and $r \equiv 4 \pmod 8$.

Proof: In the above cases, $|E(M(G))| = \binom{pr(r+1)}{2} \equiv 2 (\mathrm{mod} 4)$ M(G) is an even graph. The result follows from Corollary 2.3. ■

Corollary 2.5. If G is a d – regular graph, then T(G) is not near felicitous in the following case. $|E(G)| \equiv 1 \pmod 2$ and $d \equiv 0 \pmod 4$

Proof: Since T(G) is even and $|E(T(G))| = (2 + d)\ |E(G)| \equiv 2 \pmod 4$, the result follows from Corollary 2.3. ■

Let $C_{n,m}^{\uparrow}$ stand for the graph obtained from $C_n \times P_m$ by taking two new distinct vertices, say, u, v and joining u to all the vertices of C_n^1 and v to all the vertices of C_n^m.

Corollary 2.6. The following graphs are not near felicitous:

(a) $C_{n,m}$, when $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 2$.

(b) $C_{n,m}^{\uparrow}$ when $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 2$.

Proof: The above graphs are all even. Further,

(a) $|E(C_{n,m})| = n(2m + 1) \equiv 2 \pmod 4$.

(b) $|E(C_{n,m}^{\uparrow})| = n(2m + 1) \equiv 2 \pmod 4$. ■

Corollary 2.7. The graph $C_n \vee K_m^c$ when $n \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 2$ is not near felicitous.

Proof: The above graph is even. Further, $|E(C_n \vee K_m^c)| = n(m + 1) \equiv 2 \pmod 4$. ■

Corollary 2.8. Suppose G_1 and G_2 are any two even graphs, then $G_2[G_1]$ is not near felicitous in the following cases.

(i) $|E(G_1)| \equiv 1 \pmod 4$, $|V(G_2)| \equiv 1 \pmod 4$ and $|E(G_2)| \equiv 1 \pmod 4$.

(ii) $|E(G_1)| \equiv 1 \pmod 4$, $|V(G_2)| \equiv 3 \pmod 4$ and $|E(G_2)| \equiv 3 \pmod 4$.

Proof: Since $G_2[G_1]$ is even and $|E(G_2[G_1])| = |V(G_2)|\ |E(G_1)| + |E(G_2)| \equiv 2 \pmod 4$ in both the cases, $G_2[G_1]$ is not near felicitous. The result follows from Corollary 2.3. ■

Theorem 2.9. Let G be a graph with odd number of edges and let f: $V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ be a near – even edge labeling of G. Then f is a near felicitous labeling of G.

Proof: Let $V(G) = \{0, 1, 2, \ldots, q-1, q+1\}$, then $f^*(E(G)) = \{2, 4, 6, \ldots, 2q\}$. Since q is odd, $f^*(E(G)) = \{2, 4, 6, \ldots, q-1, q+1, \ldots, 2q-2, 2q\}$. After taking (mod q), $f^*(E(G)) = \{2, 4, 6, \ldots, q-1, 1, 3, 5, \ldots, q-2, q\} = \{1, 2, 3, \ldots, q-1, q\}$. Then f is a near felicitous labeling of G. ■

Theorem 2.10. Let G be a graph with odd number of edges and $f: V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ be a near odd edge labeling of G. Then f is a near felicitous labeling of G.

Proof: Let $V(G) = \{0, 1, 2, \ldots, q-1, q+1\}$ and let $f^*(E(G)) = \{1, 3, 5, \ldots, 2q-1\}$ be a near odd edge labeling of G. Since q is odd, then $f^*(E(G)) = \{1, 3, 5, \ldots. q, q+2, q+4, \ldots, 2q-1\}$ After taking (mod q), $f^*(E(G)) = \{1, 3, 5, \ldots. q, 2, 4, \ldots, q-1\} = \{1, 2, 3, \ldots, q-1, q\}$. Then, f is a near felicitous labeling of G. ■

Observation 2.11. Let G be a graph with even number of edges and $f: V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ be a near even edge labeling of G. Then f is not a near felicitous labeling of G.

Proof: Let $f: V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ be a near even edge labeling of G. Then $f^*(E(G)) = \{2, 4, 6, \ldots, 2q\}$. Since q is even, $f^*(E(G)) = \{2, 4, 6, \ldots, q-2, q, q+2, q+4, \ldots, 2q-2, 2q\}$. After taking (mod q), $f^*(E(G)) = \{2, 4, 6, \ldots, q-2, q, 2, 4, \ldots, q-2, q\}$. Hence, the edge labels are not distinct. Hence, f is not a near felicitous labeling of G. ■

Observation 2.12. Let G be a graph with even number of edges and $f: V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ be a near odd edge labeling of G. Then f is not a near felicitous labeling of G.

Proof: Let $f: V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ be a near odd edge labeling of G. Then $f^*(E(G)) = \{1, 3, 5, \ldots, 2q-1\}$. Since q is even, $f^*(E(G)) = \{1, 3, 5, \ldots, q-1, q+1, q+3, \ldots, 2q-3, 2q-1\}$. After taking (mod q), $f^*(E(G)) = \{1, 3, 5, \ldots, q-1, 1, 3, 5, \ldots, q-1\}$. Hence, the edge labels are not distinct. Hence, f is not a near felicitous labeling of G. ■

Observation 2.13. If G and H are near felicitous then their cartesian product $G \times H$ need not be near felicitous. For example, $K_{1,4l+3}$ and K_2 are near felicitous, but $K_{1,4l+3} \times K_2$ is not near felicitous, since $E(K_{1,4l+3} \times K_2) \equiv 2 \pmod 4$ and $K_{1,4l+3} \times K_2$ is even.

Observation 2.14. If G and H are near felicitous then their wreath product $G * H$ need not be near felicitous. For example, let $G = K_m$ and $H = K_n$ with $m \neq n$ and $1 < m, n < 5$. We know that $K_m * K_n \cong K_{mn}$. Clearly as $mn > 5$, $K_m * K_n$ is not near felicitous. Similarly, $G * H$ need not be near felicitous, when one of them, say G is near felicitous and the other namely, H is not near felicitous. For example, let $G = K_4$ and $H = K_m$, $m \geq 5$.

Observation 2.15. If G and H are near felicitous, then $(G \circ H)(v)$ need not be near felicitous. For example, C_3 and C_{4k+3} are near felicitous. But $C_3 \circ C_{4k+3}(v)$ is not near felicitous, where v is any vertex of C_3, since $C_3 \circ C_{4k+3}(v)$ is even and $|E(C_3 \circ C_{4k+3}(v)| = 4k + 6 \equiv 2 \pmod 4$.

Observation 2.16. Let G be a near felicitous graph and a $1-1$ function $f: V(G) \to \{0, 1, 2, \ldots, q-1, q+1\}$ is said to be a near even edge labeling of G, if for every edge $e = uv \in E(G)$, $f(u) + f(v)$ is even and $f^*(E(G)) = \{2, 4, 6, \ldots, 2q\}$.

Observation 2.17. Suppose a connected graph G admits near even edge labeling, then

(i) All the vertices of G possess either even labels or odd labels.

(ii) $p \le \left\lfloor \frac{q+3}{2} \right\rfloor$.

Observation 2.18. If G is disconnected, then the vertex labeling of each component is with the same parity.

Observation 2.19. It follows from the above observation that if G admits near even edge labeling and if $q < 2p - 3$, then G is a disconnected graph.

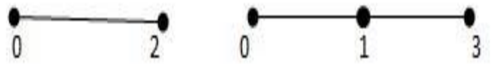
Remark 2.20. Let G be a (p, q) graph. Let f be a near felicitous labeling. Define $f_1(uv) = f(u) + f(v)$ for every $uv \in E(G)$. Then $f^*(uv) = f_1(uv)(\text{mod } q)$.

Theorem 2.21. P_n is a near felicitous graph.

Proof: Let $V(P_n) = \{u_i.\ 1 \le i \le n\}$ and $E(P_n) = \{(u_i\ u_{i+1}).\ 1 \le i \le n-1\}$.

Case (i): $n = 2, 3$.

The labelings of P_2 and P_3 are shown in the Figure 2.



P_2 P_3

Figure 2: *Near felicitous labelings of P_2 and P_3.*

Case (ii): $n \ge 5$ and $n \equiv 1 \pmod 2$.

Define a function $f: V \rightarrow \{0, 1, 2, \ldots q-1, q+1 = n\}$ by

$$f(u_i) = \frac{i+1}{2}, \quad 1 \le i \le n \text{ and } i \equiv 1 \pmod 2$$

For $i \equiv 0 \pmod 2$,

$$f(u_i) = \begin{cases} \frac{n+i+1}{2}, & 1 \le i \le n-5 \\ 0, & i = n-3 \\ n, & i = n-1 \end{cases}$$

The edge labels are as follows.

$$f_1(u_i\, u_{i+1}) = \begin{cases} \frac{n+3}{2} + i, & 2 \le i \le n-5 \\ \frac{n-3}{2} + (i-n+4), & n-4 \le i \le n-2 \\ \frac{3n+1}{2}, & i = n-1 \end{cases}$$

Now, $f_1(E(G)) = \{\frac{n+3}{2} + 1, \frac{n+3}{2} + 2, \ldots, \frac{n+3}{2} + n - 5\} \cup \{\frac{n-3}{2} + (n-4-n+4), \frac{n-3}{2} + (n-3-n+4), \frac{n-3}{2} + (n-2-n+4)\} \cup \{\frac{3n+1}{2}\}$.

$= \{\frac{n+5}{2}, \frac{n+7}{2}, \ldots, \frac{3n-7}{2}\} \cup \{\frac{n-3}{2}, \frac{n-1}{2}, \frac{n+1}{2}\} \cup \{\frac{3n+1}{2}\}.$

$= \{\frac{n-3}{2}, \frac{n-1}{2}, \frac{n+1}{2}, \frac{3n+1}{2}, \frac{n+5}{2}, \frac{n+7}{2}, \ldots, \frac{3n-7}{2}\}.$

After taking (mod q), $f^*(E(G)) = f_1(E(G)$ (mod q) = $\{\frac{n-3}{2}, \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}, \ldots, \frac{n-5}{2}\}$.

Hence, P_n, $n \geq 5$ and $n \equiv 1$ (mod 2) is a near felicitous graph.

Case (iii): $n \geq 4$ and $n \equiv 0$ (mod 2)

Define a function $f: V \rightarrow \{0, 1, 2, \ldots, q-1, q+1 = n\}$ by

$f(u_1) = 0$

$f(u_i) = i + 1, \quad 3 \leq i \leq n-1$ and $i \equiv 1$ (mod 2)

For $i \equiv 0$ (mod 2),

$$f(u_i) = \begin{cases} i-1, & 2 \leq i \leq n-2 \\ 2, & i = n \end{cases}$$

The edge labels are as follows.

$f_1(u_1 u_2) = 1$

$$f_1(u_i u_{i+1}) = \begin{cases} 2i+1, & 2 \leq i \leq n-2 \\ 3, & i = n-1 \end{cases}$$

Now, $f_1(E(G)) = \{1\} \cup \{5, 7, \ldots, 2(n-3)+1, 2(n-2)+1\} \cup \{3\} = \{1\} \cup \{5, 7, \ldots, 2n-5, 2n-3\} \cup \{3\} = \{1, 3, 5, \ldots, 2n-5, 2n-3\} = \{1, 3, 5, \ldots, 2q-1\}$.

Hence, by Theorem 2.10, P_n, $n \geq 4$ and $n \equiv 0$ (mod 2) is a near felicitous graph. ■

Illustration 2.22. Near felicitous labelings of P_{11} and P_{10} are given below.

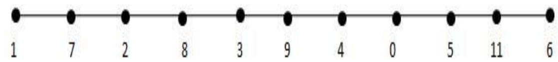


Figure 3: *A near felicitous labeling of* P_{11}.

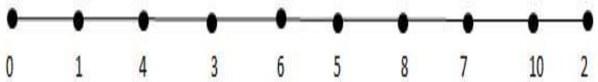


Figure 4: *A near felicitous labeling of* P_{10}.

Theorem 2.23. C_n is a near felicitous graph for $n \not\equiv 2$ (mod 4).

Proof: Let $V(C_n) = \{u_i.\ 1 \leq i \leq n\}$ and $E(C_n) = \{(u_i u_{i+1}).\ 1 \leq i \leq n-1\} \cup \{(u_n u_1)\}$.

Case (i): $n \equiv 0$ (mod 4).

Define a function $f: V \rightarrow \{0, 1, 2, \ldots q-1, q+1 = n+1\}$ by

$f(u_i) = \frac{i+1}{2}, 1 \leq i \leq n-1$ and $i \equiv 1$ (mod 2).

For $i \equiv 0 \pmod 2$,

$$f(u_i) = \begin{cases} \frac{n}{2} + \frac{i}{2}, & 2 \le i \le \frac{n}{2} \\ \frac{n}{2} + \frac{i}{2} + 1, & \frac{n}{2} + 1 \le i \le n - 4 \\ 0, & i = n - 2 \\ n + 1, & i = n \end{cases}$$

The edge labels are as follows.

$$f^*(u_i\, u_{i+1}) = \begin{cases} \frac{n}{2} + (i + 1), & 1 \le i \le \frac{n}{2} - 1 \\ 1, & i = \frac{n}{2} \\ i - \frac{n}{2} + 2, & \frac{n}{2} + 1 \le i \le n - 1 \end{cases}$$

$$f^*(u_n\, u_1) = \quad 2.$$

$$\text{Now, } f^*(E(G)) = \{\tfrac{n}{2} + 2,\ \tfrac{n}{2} + 3,\ \ldots,\ \tfrac{n}{2} + \tfrac{n}{2} - 1 + 1\} \cup \{1\} \cup \{\tfrac{n}{2} + 1 - \tfrac{n}{2} + 2,\ \tfrac{n}{2} + 2 - \tfrac{n}{2} + 2,\ \ldots,\ n - 1 - \tfrac{n}{2} + 2\} \cup \{2\}.$$

$$= \{\tfrac{n}{2} + 2,\ \tfrac{n}{2} + 3,\ \ldots,\ n\} \cup \{1\} \cup \{3, 4,\ \ldots,\ \tfrac{n}{2} + 1\} \cup \{2\}.$$

$$= \{1, 2, 3, 4,\ \ldots,\ \tfrac{n}{2} + 1,\ \tfrac{n}{2} + 2,\ \tfrac{n}{2} + 3,\ \ldots,\ n\}.$$

Clearly, the edge labels are distinct and hence, C_n, $n \equiv 0 \pmod 4$ is a near felicitous graph.

Case (ii): $n \equiv 1, 3 \pmod 4$.

Define a function $f: V \rightarrow \{0, 1, 2, \ldots, q - 1, q + 1 = n + 1\}$ by

$$f(u_i) = \begin{cases} 2(i - 1), & 1 \le i \le \frac{n+3}{2} \\ 2(i - \frac{n+3}{2} - 1) + 3, & \frac{n+3}{2} + 1 \le i \le n \end{cases}$$

The edge labels are as follows.

$$f_1(u_i\, u_{i+1}) = \begin{cases} 4i - 2, & 1 \le i \le \frac{n+3}{2} - 1 \\ 4(i - \frac{n+3}{2} + 1), & \frac{n+3}{2} \le i \le n - 1 \end{cases}$$

$$f(u_n\, u_1) = \quad n - 2.$$

Now, $f_1(E(G)) = \{2, 6, \ldots, 4(\frac{n+3}{2} - 2) - 2, 4(\frac{n+3}{2} - 1) - 2\} \cup \{4(\frac{n+3}{2} - \frac{n+3}{2} + 1),$

$4(\frac{n+3}{2} + 1 - \frac{n+3}{2} + 1), \ldots, 4(n - 2 - \frac{n+3}{2} + 1), 4(n - 1 - \frac{n+3}{2} + 1)\} \cup \{n-2\}.$

$= \{2, 6, \ldots, 2n-4, 2n\} \cup \{4, 8, \ldots, 2n-6\} \cup \{n-2\}.$

$= \{2, 4, 6, \ldots, 2n-6, 2n-4, n-2, 2n\}.$

After taking (mod q), $f^*(E(G)) = f_1(E(G)$ (mod q) $= \{2, 4, 6, \ldots, n-6, n-4, n-2, n\}$

$= \{1, 2, 3, \ldots, n\}.$

Hence, C_n, when $n \equiv 1, 3 \pmod 4$ is a near felicitous graph. ■

Illustration 2.24. Near felicitous labeling of C_8 and C_9 are shown in Figures 5 and 6 below.

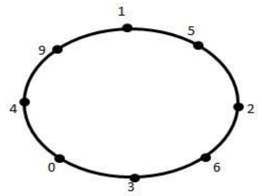


Figure 5: *A near felicitous labeling of C_8.*

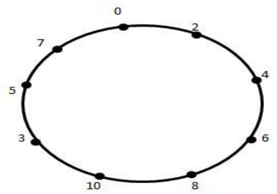


Figure 6: *A near felicitous labeling of C_9.*

Remark 2.25. C_n is not a near felicitous graph when $n \equiv 2 \pmod 4$.

Proof: Since C_n is an even graph with $4n + 2$ edges, by Corollary 2.3, C_n is not a near felicitous graph when $n \equiv 2 \pmod 4$. ■

Theorem 2.23. $K_{m,n}$ is a near felicitous graph for all m and n.

Proof: **Case (i):** $m = 1$.

Let $V(K_{1,n}) = \{u_i.\ 0 \le i \le n\}$ and $E(K_{1,n}) = \{(u_o u_i) \ / \ 1 \le i \le n\}$.

Define a function $f: V \rightarrow \{0, 1, 2, 3, \ldots, q-1, q+1 = n+1\}$ by

$$f(u_o) = 1$$

$$f(u_i) = \begin{cases} 0, & i = 1 \\ i & 2 \le i \le n-1 \\ n+1, & i = n \end{cases}$$

The edge labels are as follows.

$$f^*(u_0 u_i) = \begin{cases} 1, & i = 1 \\ i+1, & 2 \le i \le n-1 \\ 2, & i = n \end{cases}$$

Now $f^*(E(G)) = \{1\} \cup \{3, 4, \ldots, n\} \cup \{2\} = \{1, 2, 3, 4, \ldots, n\}$.

Hence, $K_{1,n}$ is a near felicitous graph.

Case (ii): $m > 1$ and $n \geq 1$.

Let $V(K_{m,n}) = \{u_i, v_j, 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and $E(K_{m,n}) = \{(u_i v_j) / 1 \leq i \leq m$ and $1 \leq j \leq n\}$.

Define a function $f: V \rightarrow \{0, 1, 2, \ldots, q-1, q+1 = mn+1\}$ by

$$f(u_i) = i, \quad 1 \leq i \leq m$$
$$f(v_j) = mj + 1, \quad 1 \leq j \leq n$$

The edge labels are as follows.

For $1 \leq j \leq n$, $f_1(u_i v_j) = i + mj + 1$, $1 \leq i \leq m$

Now $f_1(E(G) = \{m+2, m+3, \ldots, 2m+1, 2m+2, 2m+3, \ldots, 3m+1, \ldots, mn, mn+1, mn+2, mn+3, \ldots, mn+m+1\}$.

After taking (mod q), $f^*(E(G)) = f_1(E(G)$ (mod q) $= \{m+2, m+3, \ldots, 2m+1, 2m+2, 2m+3, \ldots, 3m+1, \ldots, mn, 1, 2, 3, \ldots, m+1\}$.

$= \{1, 2, 3, \ldots, m+1, m+2, m+3, \ldots, 2m+1, 2m+2, 2m+3, \ldots, 3m+1, \ldots, mn\}$.

Clearly, all the edge labels are distinct and hence, $K_{m,n}$ is a near felicitous graph for all m and n. ■

Illustration 2.27. Near felicitous labelings of $K_{1,7}$ and $K_{4,3}$ are shown in Figures 7 and 8 respectively.

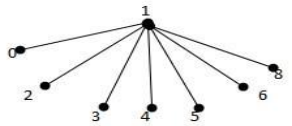


Figure 7: *A near felicitous labeling of* $K_{1,7}$.

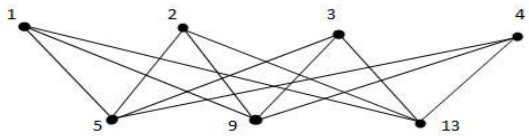


Figure 8: *A near felicitous labeling of* $K_{4,3}$.

Theorem 2.28. The graph $H_{n,n}$ is a near felicitous graph for $n \geq 2$.

Proof: Let $V(H_{n,n}) = \{u_i / 1 \leq i \leq n\} \cup \{v_j / 1 \leq j \leq n\}$ and $E(H_{n,n}) = \{(v_j u_i).\ 1 \leq i \leq n,\ i \leq j \leq n\}$.

Case (i): $n = 2$.

The graph $H_{2,2}$ is near felicitous as shown in the Figure 9.

Figure 9: *A near felicitous labeling of* $H_{2,2}$.

Case (ii): $n \geq 3$.

Let u_i and v_j be the vertices of $H_{n,n}$ where $1 \leq i \leq n$ and $1 \leq j \leq n$.

Define a function $f: V \rightarrow \{0, 1, 2, \ldots, q-1, q+1 = \frac{n(n+1)}{2} + 1\}$ by

$$f(u_i) = i - 1, \quad 1 \leq i \leq n$$

$$f(v_j) = \begin{cases} n + 1, & j = 1, \\ (n - j + 1) + f(v_{j-1}), & 2 \leq j \leq n-2 \\ \frac{n(n+1)}{2} + 1, & j = n-1, \\ \frac{n(n+1)}{2} - 1, & j = n. \end{cases}$$

The edge labels are as follows:

For $1 \leq j \leq n$, $f_1(u_i v_j) = f(v_j) + (i-1)$, $j \leq i \leq n$.

Now, $f_1(E(G)) = \{n + 1, n + 2, n + 3, \ldots, n + 1 + n - 1, n - 1 + n + 1 + 1, n - 1 + n + 1 + 2, \ldots,$

$$n - 1 + n + 1 + n - 1, \frac{n(n+1)}{2} - 1 + n - 1, \frac{n(n+1)}{2} + 1 + n - 1 - 1, \frac{n(n+1)}{2} + 1 + n - 1\}.$$

$$= \{n + 1, n + 2, n + 3, \ldots, 2n, 2n + 1, 2n + 2, \ldots, 3n - 1, \ldots, \frac{n(n+1)}{2} + n - 2, \frac{n(n+1)}{2} + n - 1, \frac{n(n+1)}{2} + n\}.$$

After taking (mod q), $f^*(E(G)) = f_1(E(G)$ (mod q) $= \{n + 1, n + 2, n + 3, \ldots, 2n, 2n + 1, 2n + 2, \ldots, 3n - 1, \ldots, n - 2, n - 1, n\}$. ■

Illustration 2.29. A near felicitous labeling of $H_{6,6}$ is given below.

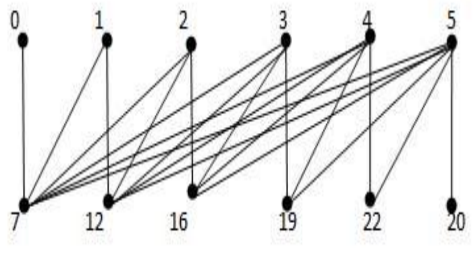


Figure 10: *A near felicitous labeling of* $H_{6,6}$.

Theorem 2.30. The graph $B_{m,n}$ is a near felicitous graph for all m and n.

Proof: Let $V(B_{m,n}) = \{u, v\} \cup \{u_i, v_j.\ 1 \leq i \leq m$ and $1 \leq j \leq n\}$ and $E(B_{m,n}) = \{(uv)\} \cup \{(uu_i).\ 1 \leq i \leq m\} \cup \{(vv_j).\ 1 \leq j \leq n\}$.

Case (i): $m < n$.

Define $f: V \rightarrow \{0, 1, 2, \ldots, q-1, q+1 = m+n+2\}$ by

$f(u) = 1$, $f(v) = m+n+2$,

$f(u_i) = 2(i-1), 1 \leq i \leq m$

$$f(v_j) = \begin{cases} 2j+1, & 1 \leq j \leq m-1 \\ m+j, & m \leq j \leq n \end{cases}$$

The edge labels are as follows.

$f_1(uv) = m+n+3$

$f_1(uu_i) = 2i-1, \quad 1 \leq i \leq m$

$$f(vv_j) = \begin{cases} 2j+2, & 1 \leq j \leq m-1 \\ m+j+1, & m \leq j \leq n \end{cases}$$

Now, $f_1(E(G)) = \{m+n+3\} \cup \{1, 3, 5, \ldots, 2m-1\} \cup \{4, 6, \ldots, 2(m-1)+2\} \cup \{m+m+1, m+m+2, \ldots, m+n+1\}$

$= \{m+n+3\} \cup \{1, 3, 5, \ldots, 2m-1\} \cup \{4, 6, \ldots, 2m\} \cup \{2m+1, 2m+2, \ldots, m+n+1\}$

$= \{1, 3, 4, \ldots, 2m, 2m+1, 2m+2, \ldots, m+n+1, m+n+3\}$.

After taking (mod q), $f^*(E(G)) = \{f_1(E(G)) \pmod q\} = \{1, 3, 4, \ldots, 2m, 2m+1, 2m+2, \ldots, m+n+1, 2\} = \{1, 2, 3, \ldots, 2m, 2m+1, 2m+2, \ldots, m+n+1\}$.

Hence, $B_{m,n}$ is a near felicitous graph.

Case (ii): $m = n$.

Let $V(B_{n,n}) = \{u, v\} \cup \{u_i, v_i. 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{(uv) \cup (uu_i) \cup (vv_i). 1 \leq i \leq n\}$.

Define $f: V \rightarrow \{0, 1, 2, \ldots, q-1, q+1 = 2n+2\}$ by

$f(u) = 0$,

$f(v) = 2$,

$f(u_i) = 2(i+1), \quad 1 \leq i \leq n$

$f(v_i) = 2i-1, \quad 1 \leq i \leq n$

The edge labels are as follows.

$f_1(uv) = 2$

$f_1(uu_i) = 2(i+1), 1 \leq i \leq n$

$f_1(vv_i) = 2i+1, \quad 1 \leq i \leq n$

Now, $f_1(E(G)) = \{2\} \cup \{4, 6, \ldots, 2n+2\} \cup \{3, 5, \ldots, 2n+1\}$

$= \{2, 3, 4, \ldots, 2n+1, 2n+2\}$.

After taking (mod q), $f^*(E(G)) = \{f_1(E(G)) \pmod q\} = \{2, 3, 4, \ldots, 2n+1, 1\}$

$= \{1, 2, 3, \ldots, 2n+1\}$.

Hence, $B_{n,n}$ is a near felicitous graph for all n. ■

Illustration 2.31. Near felicitous labelings of $B_{4,6}$ and $B_{4,4}$ are given below.

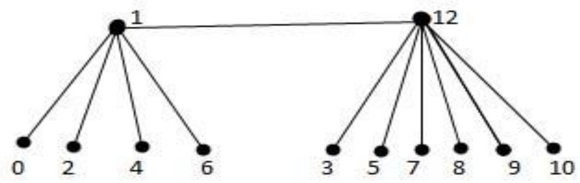


Figure 11: *A near felicitous labeling of* $B_{4,6}$.

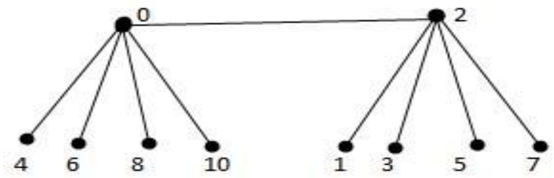


Figure 12: *A near felicitous labeling of* $B_{4,4}$.

Theorem 2.32. The graph $B(m, n, u_o)$ is a near felicitous graph.

Proof: Let $V(B(m, n, u_o)) = \{u, v, u_o\} \cup \{u_i.\ 1 \le i \le m\} \cup \{v_j.\ 1 \le j \le n\}$ and $E(B(m, n, u_o)) = \{(uu_o), (vu_o)\} \cup \{(uu_i).\ 1 \le i \le m\} \cup \{(vv_j).\ 1 \le j \le n\}$.

Let $m > n$. Define $f: V \to \{0, 1, 2, \ldots, q-1, q+1 = m+n+3\}$ by

$$f(u) = 0, \quad f(u_o) = m+1,$$
$$f(v) = m+n+3$$
$$f(u_i) = i, \quad 1 \le i \le m$$
$$f(v_j) = m+1+j, \quad 1 \le j \le n$$

The edge labels are as follows.

$$f_1(uu_o) = m+1, \quad f_1(vu_o) = 2m+n+4$$
$$f(uu_i) = i, \quad 1 \le i \le m$$
$$f(vv_j) = 2m+n+4+j, \quad 1 \le j \le n$$

Now, $f_1(E(G)) = \{m+1\} \cup \{2m+n+4\} \cup \{1, 2, 3, \ldots m\} \cup \{2m+n+5, 2m+n+6, \ldots, 2m+2n+4\}$

$= \{1, 2, 3, \ldots m, m+1, 2m+n+4, 2m+n+5, 2m+n+6, \ldots, 2m+2n+4\}$.

After taking (mod q), $f^*(E(G)) = \{f_1(E(G)) \pmod q\}$

$= \{1, 2, 3, \ldots, m, m+1, m+2, m+3, \ldots, m+n+2\}$.

Let $m = n$. Define $f: V \to \{0, 1, 2, \ldots, q-1, q+1 = 2m+3\}$ by

$$f(u) = 0, \quad f(u_o) = m+1,$$
$$f(v) = 2m+3$$
$$f(u_i) = i, \quad 1 \le i \le m$$
$$f(v_i) = m+1+i, \quad 1 \le i \le m$$

The labels of the edges are as follows.

$$f_1(u\,u_o) = m+1, \quad f_1(vu_o) = 3m+4$$

$$f(uu_i) = i, \quad 1 \le i \le m$$
$$f(vv_i) = 3m + 4 + i, \quad 1 \le i \le m$$

Now, $f_1(E(G)) = \{m + 1\} \cup \{3m + 4\} \cup \{1, 2, 3, \ldots m\} \cup \{3m + 5, 3m + 6, \ldots, 4m + 4\}$
$= \{1, 2, 3, \ldots m, m + 1, 3m + 4, + 5, 3m + 6, \ldots, 4m + 4\}$.

After taking (mod q), $f^*(E(G)) = f_1(E(G)) \pmod q$
$= \{1, 2, 3, \ldots, m, m + 1, m + 2, m + 3, \ldots, 2m + 2\}$.

*H*ence, B(m, n, u_o) is a near felicitous graph. ■

Illustration 3.33. A near felicitous labeling of B(4,3, u_o) is shown in Figure 13.

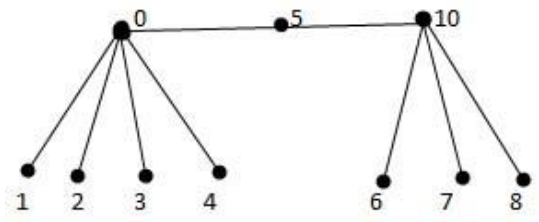


Figure 13: *A near felicitous labeling of* $B(4,3, u_o)$.

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