

Soft semi connected and Soft locally semi connected properties in Soft topological spaces

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Abstract

In this paper we study the soft semi separation of soft sets, soft semi connected sets and soft locally semi connected sets and prove some properties related to these topics.

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1 Introduction

Theories such as theory of vague sets and theory of rough sets are considered as mathematical tools for dealing with uncertainties. But these theories have their own difficulties. Molodtsov [3] introduced the concept of soft sets in order to solve complicated problems in economics, engineering and the like. In 2011, Shabir and Naz [6] introduced and studied the concepts soft topological space and their related concepts such as soft interior, soft closed, soft subspace and soft separation axioms. In 2012, E. Peyghan, B. Samadi and A. Taybi [4] introduced and studied the notions of soft connected topological spaces after a review of preliminary definitions. In this paper, we introduce some concepts such as soft semi connectedness, soft locally semi connectedness and exhibit some results related to these concepts.

2 Preliminaries

For basic notations and definitions not given here, the readers can refer [1, 6]. Hereafter, U refers to an initial universe, E is a set of parameters, $\wp(U)$ is the power set U , and A is a nonempty subset of E .

Definition 2.1. A soft set F_A on the universe U is defined by the set of ordered pairs $F_A = \{(x, f_A(x)) / x \in E, f_A(x) \in \wp(U)\}$, where $f_A : E \rightarrow \wp(U)$ such that $f_A(x) = \phi$ if $x \notin A$. Here, f_A is called an approximate function of the soft set F_A . The value of $f_A(x)$ may be arbitrary. Some of them may be empty and some may have nonempty intersection.

As an illustration, let us consider the following example.

A soft set F_E describes the "attractiveness of the houses" which Mr. X going to buy. Let U be the set of houses under consideration, E be the set of parameters. Each parameter is a word or a sentence. Assume

$E = \{\text{expensive; beautiful; wooden; cheap; in the green surroundings; modern}\}$. In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on.

Note that the sets F_E may be arbitrary. Some of them may be empty, some may have nonempty intersection.

A soft point is an element of a soft set F_A . The class of all soft sets over U is denoted by $S(U)$.

Example 2.2. Let $U = \{u_1, u_2, u_3\}$, $E = \{x_1, x_2, x_3\}$, $A = \{x_1, x_2\}$ and $F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2, u_3\})\}$. Then $F_{A_1} = \{(x_1, \{u_1\})\}$, $F_{A_2} = \{(x_1, \{u_2\})\}$, $F_{A_3} = \{(x_1, \{u_1, u_2\})\}$, $F_{A_4} = \{(x_2, \{u_2\})\}$, $F_{A_5} = \{(x_2, \{u_3\})\}$, $F_{A_6} = \{(x_2, \{u_2, u_3\})\}$, $F_{A_7} = \{(x_1, \{u_1\}), (x_2, \{u_2\})\}$, $F_{A_8} = \{(x_1, \{u_1\}), (x_2, \{u_3\})\}$, $F_{A_9} = \{(x_1, \{u_1\}), (x_2, \{u_2, u_3\})\}$, $F_{A_{10}} = \{(x_1, \{u_2\}), (x_2, \{u_2\})\}$, $F_{A_{11}} = \{(x_1, \{u_2\}), (x_2, \{u_3\})\}$, $F_{A_{12}} = \{(x_1, \{u_2\}), (x_2, \{u_2, u_3\})\}$, $F_{A_{13}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}$, $F_{A_{14}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\})\}$, $F_{A_{15}} = F_A$, $F_{A_{16}} = F_\phi$ are all soft subsets of F_A .

Here $F_{A_i} \tilde{\subseteq} F_A$ means that F_{A_i} is soft subset of F_A .

Definition 2.3. Let $F_A \in S(U)$. A soft topology on F_A , denoted by $\tilde{\tau}$ is a collection of soft subsets of F_A having the following properties:

1. $F_A, F_\phi \in \tilde{\tau}$
2. $\{F_{A_i} \tilde{\subseteq} F_A : i \in I \subseteq N\} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcup}_{i \in I} F_{A_i} \in \tilde{\tau}$
3. $\{F_{A_i} \tilde{\subseteq} F_A : 1 \leq i \leq n, n \in N\} \subseteq \tilde{\tau} \Rightarrow \tilde{\bigcap}_{i=1}^n F_{A_i} \in \tilde{\tau}$

The pair $(F_A, \tilde{\tau})$ is called a soft topological space.

Example 2.4. Let us consider the soft subsets of F_A that are given in Example 2.2. Then $\tilde{\tau}_1 = \{F_\phi, F_A\}$ and $\tilde{\tau}_2 = \{F_\phi, F_A, F_{A_2}, F_{A_{11}}, F_{A_{13}}\}$ are soft topologies on F_A .

Definition 2.5. Let $(F_A, \tilde{\tau})$ be a soft topological space. Then every element of $\tilde{\tau}$ is called a soft open set. Clearly, F_ϕ and F_A are soft open sets.

Definition 2.6. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then F_B is said to be soft closed if the soft set $F_B^{\tilde{c}}$ is soft open. The complement of a soft set F_B is denoted by $F_B^{\tilde{c}}$ is defined by $F_B^{\tilde{c}} = \{(x, f_B^c(x)) / x \in B, f_B^c(x) \in \wp(U)\}$, where $f_B^c : B \rightarrow \wp(U)$ is given by $f_B^c(x) = U - f_B(x)$ for all $x \in B$.

Definition 2.7. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then the soft interior of a soft set F_B is denoted by $F_B^{\tilde{o}}$ and is defined as the soft union of all soft open subsets of F_B . Thus, $F_B^{\tilde{o}}$ is the largest soft open set contained in F_B .

Definition 2.8. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then the soft closure of F_B , denoted by $\overline{F_B}$ is defined as the soft intersection of all soft closed supersets of F_B . Note that $\overline{F_B}$ is the smallest soft closed set containing F_B .

Definition 2.9. A soft set F_B in a soft topological space $(F_A, \tilde{\tau})$ is called soft semi open (written S.S.O) if and only if there exists a soft open set F_O such that $F_O \tilde{\subset} F_B \tilde{\subset} \overline{F_O}$.

Definition 2.10. A soft set F_B in a soft topological space $(F_A, \tilde{\tau})$ is said to be soft semi closed if and only if there exists a soft closed set F_C such that $F_C^{\tilde{\circ}} \tilde{\subset} F_B \tilde{\subset} F_C$, where $F_C^{\tilde{\circ}}$ denotes the soft interior of F_C in F_A .

Definition 2.11. The intersection of all soft semi closed sets containing a soft subset F_B of a soft topological space $(F_A, \tilde{\tau})$ is said to be the soft semi closure of F_B and is denoted by \mathbf{F}_B .

3 Soft Semi Connected

Definition 3.1. The two non null soft subsets F_A, F_B of a soft topological space $(F_A, \tilde{\tau})$ are said to be soft semi separated if and only if $\mathbf{F}_A \tilde{\cap} F_B = F_A \tilde{\cap} \mathbf{F}_B = F_\phi$, where F_ϕ denotes the soft null set.

Definition 3.2. In a soft topological space $(F_A, \tilde{\tau})$ a soft set which cannot be expressed as the union of two soft semi separated sets is said to be soft semi connected set. The soft topological space $(F_A, \tilde{\tau})$ is said to be soft semi connected if and only if F_A is soft semi connected.

Theorem 3.3. A soft topological space $(F_A, \tilde{\tau})$ is soft semi connected if and only if the only soft sets in $S(U)$ that are both soft semi open and soft semi closed over F_A are F_ϕ and F_A .

Proof: Let $(F_A, \tilde{\tau})$ be soft semi connected. Let F_O be any soft semi open and soft semi closed subset of $(F_A, \tilde{\tau})$. Then $F_O^{\tilde{\circ}}$ is both soft semi open and soft semi closed. Since $(F_A, \tilde{\tau})$ is disjoint union of the soft semi open sets F_O and $F_O^{\tilde{\circ}}$, the hypothesis implies that either $F_O = F_\phi$ or $F_O = F_A$.

Conversely, suppose that $F_A = F_B \tilde{\cup} F_C$, where F_B and F_C are disjoint non-empty soft semi open subsets of F_A . Then F_B is both soft semi open and soft semi closed. By assumption $F_B = F_\phi$ or F_A . Therefore, F_A is soft semi connected. ■

Theorem 3.4. If the soft semi open sets F_C and F_D form a soft semi separation of F_A and if $(F_B, \tilde{\tau})$ is a soft semi connected subspace of $(F_A, \tilde{\tau})$, then $F_B \tilde{\subset} F_C$ or $F_B \tilde{\subset} F_D$.

Proof: Since F_C and F_D are disjoint soft semi open sets in F_A , $F_C \tilde{\cap} F_B$ and $F_D \tilde{\cap} F_B$ are soft semi open in F_B . These two soft sets are disjoint and their union is F_B . If they were both non-empty, they would constitute a separation of F_B . Therefore one of them is empty. Hence F_B must lie entirely in F_C or in F_D . ■

Theorem 3.5. Let F_B be a soft semi connected subspace of $(F_A, \tilde{\tau})$. If $F_B \tilde{\subset} F_C \tilde{\subset} \mathbf{F}_B$ then F_C is also soft semi connected.

Proof: Let F_B be soft semi connected and let $F_B \tilde{\subset} F_C \tilde{\subset} F_B$. Suppose that $F_C = F_D \tilde{\cup} F_F$ is a soft semi separation of F_C by soft semi open sets. Then by Theorem 3.4 above, F_B must lie entirely in F_D or in F_F . Suppose that $F_B \tilde{\subset} F_D$, then $\mathbf{F}_B \tilde{\subset} \mathbf{F}_D$. Since F_D and F_F are disjoint, F_C cannot intersect F_F . This contradicts the fact that F_F is a non-empty soft subset of F_C . So $F_F = F_\phi$ which implies F_C is soft semi connected. ■

Theorem 3.6. The union of a collection of soft semi connected subspaces of $(F_A, \tilde{\tau})$ that have non-null intersection is soft semi-connected.

Proof: Let $\{(F_{B_\alpha}, \tilde{\tau}_{B_\alpha})\}_{\alpha \in J}$ be an arbitrary collection of soft semi connected subspaces of $(F_A, \tilde{\tau})$. Suppose that $F_B = \bigcup_{\alpha \in J} F_{B_\alpha} = F_C \tilde{\cup} F_D$, where F_C and F_D form a soft semi separation of F_B . By hypothesis, we may choose a soft point $(x, f_{B_\alpha}(x))$ such that $(x, f_{B_\alpha}(x)) \in \bigcap_{\alpha \in J} F_{B_\alpha}$ and it must belong to either a soft subset F_C or a soft subset F_D . Since F_C, F_D are disjoint, we must have $F_{B_\alpha} \tilde{\subset} F_C$ for all $\alpha \in J$, and so $F_B \tilde{\subset} F_C$. From this we obtain that $F_D = F_\phi$, which is a contradiction. This proves the theorem. ■

Definition 3.7. Let $(F_A, \tilde{\tau})$ be a soft topological space and $(x, f_A(x)) \in F_A$. The soft semi component of $(x, f_A(x))$, denoted by $S.S.C((x, f_A(x)))$, is the union of all soft semi connected subsets of F_A containing $(x, f_A(x))$. The sets like $S.S.C((x, f_A(x)))$ are called soft semi components of F_A .

From Theorem 3.7, we see that the soft set $S.S.C((x, f_A(x)))$ is soft semi connected.

Theorem 3.8. In a soft topological space $(F_A, \tilde{\tau})$,

- (i) each soft semi component $S.S.C((x, f_A(x)))$ is a maximal soft semi connected set in F_A ,
- (ii) the set of all distinct soft semi components of soft points of F_A form a partition of F_A and
- (iii) each $S.S.C((x, f_A(x)))$ is soft semi closed in F_A .

Proof: (i) The proof follows from the definition.

(ii) Let $S.S.C((x, f_A(x)))$ and $S.S.C((x_1, f_A(x_1)))$ be two soft semi components of distinct soft points $(x, f_A(x))$ and $(x_1, f_A(x_1))$ in F_A .

If $S.S.C((x, f_A(x))) \tilde{\cap} S.S.C((x_1, f_A(x_1))) \neq F_\phi$, then by Theorem 3.7, $S.S.C((x, f_A(x))) \tilde{\cup} S.S.C((x_1, f_A(x_1)))$ is soft semi connected.

But $S.S.C((x, f_A(x))) \tilde{\subset} S.S.C((x, f_A(x))) \tilde{\cup} S.S.C((x_1, f_A(x_1)))$ which contradicts the maximality of $S.S.C((x, f_A(x)))$. Now for any soft point $(x, f_A(x)) \in F_A$, $(x, f_A(x)) \in S.S.C((x, f_A(x)))$ and $\tilde{\bigcup}_{(x, f_A(x)) \in F_A} (x, f_A(x)) \tilde{\subset} \tilde{\bigcup}_{(x, f_A(x)) \in F_A} S.S.C((x, f_A(x)))$.

This implies that $(x, f_A(x)) \tilde{\subset} \tilde{\bigcup}_{(x, f_A(x)) \in F_A} S.S.C((x, f_A(x))) \tilde{\subset} F_A$.

(iii) Let $(x, f_A(x))$ be any soft point in F_A . Then $\mathbf{S.S.C}(\mathbf{x}, \mathbf{f}_A(\mathbf{x}))$ is a soft semi connected set containing $(x, f_A(x))$ by Theorem 3.6. But $S.S.C((x, f_A(x)))$ is the maximal soft semi connected set containing $(x, f_A(x))$. So $\mathbf{S.S.C}(\mathbf{x}, \mathbf{f}_A(\mathbf{x})) \tilde{\subset} S.S.C((x, f_A(x)))$. Hence $S.S.C((x, f_A(x)))$ is soft semi closed. ■

4 Soft Locally Semi Connected

Definition 4.1. A soft topological space $(F_A, \tilde{\tau})$ is called soft locally semi connected at $(x, f_A(x)) \in F_A$ if and only if for every soft semi open set F_U containing $(x, f_A(x))$, there exists a soft semi connected open set F_C such that $(x, f_A(x)) \in F_C \tilde{\subset} F_U$. $(F_A, \tilde{\tau})$ is called soft locally semi connected if and only if it is a soft locally semi connected at every soft point of F_A .

Remark 4.2. If a soft topological space $(F_A, \tilde{\tau})$ is locally semi connected then it is locally connected but the converse is not true as shown by the following example:

Let $(F_A, \tilde{\tau})$ be the soft topological space, where F_A and its soft subsets are considered as in Example 2.2. Consider $\tilde{\tau} = \{F_A, F_\phi, F_{A_2}, F_{A_{11}}, F_{A_{13}}\}$. Here $\tilde{\tau}^c = \text{Class of all soft closed sets} = \{F_\phi, F_A, F_{A_9}, F_{A_7}, F_{A_5}\}$. $SSO(\tilde{\tau}) = \text{Class of all soft semi open sets} = \{F_A, F_\phi, F_{A_2}, F_{A_3}, F_{A_{10}}, F_{A_{11}}, F_{A_{12}}, F_{A_{13}}, F_{A_{14}}\}$. $SSC(\tilde{\tau}) = \text{Class of all soft semi closed sets} = \{F_\phi, F_A, F_{A_9}, F_{A_6}, F_{A_8}, F_{A_7}, F_{A_1}, F_{A_5}, F_{A_4}\}$. It is verified that $(F_A, \tilde{\tau})$ is soft locally connected. But we show that it is not soft locally semi connected. Here F_{A_3} is a soft semi open set containing $(x_1, f_A(x_1))$, but there is no soft open subset of F_{A_3} containing $(x_1, f_A(x_1))$ and so $(F_A, \tilde{\tau})$ is not soft locally semi connected at $(x_1, f_A(x_1))$. Therefore, F_A is not soft locally semi connected.

Remark 4.3. Soft local semi connectedness does not imply soft semi connectedness as shown by the following example:

Let $(F_A, \tilde{\tau})$ be the soft topological space, where F_A and its soft subsets are considered as in Example 2.2. Consider $\tilde{\tau} = \{F_A, F_\phi, F_{A_2}, F_{A_3}, F_{A_{11}}, F_{A_{12}}, F_{A_{14}}\}$. $\tilde{\tau}^c = \{F_\phi, F_A, F_{A_9}, F_{A_6}, F_{A_7}, F_{A_1}, F_{A_4}\}$. $SSO(\tilde{\tau}) = \{F_A, F_\phi, F_{A_2}, F_{A_3}, F_{A_{10}}, F_{A_{11}}, F_{A_{12}}, F_{A_{13}}, F_{A_{14}}\}$. $SSC(\tilde{\tau}) = \{F_\phi, F_A, F_{A_9}, F_{A_6}, F_{A_8}, F_{A_7}, F_{A_1}, F_{A_5}, F_{A_4}\}$. The soft semi open sets containing $(x_1, f_A(x_1))$ are $F_{A_3}, F_{A_{14}}$ and F_A . Clearly the soft set F_{A_3} is soft semi connected and open. Therefore, F_A is soft locally semi connected at $(x_1, f_A(x_1))$. The soft semi open sets containing $(x_2, f_A(x_2))$ are $F_{A_{12}}$ and F_A . We show that $F_{A_{12}}$ is soft semi connected. Let $A = F_{A_2}, B = F_{A_6}$. Then soft semi closure of A , $\mathbf{A} = F_A$ and so $B \tilde{\cap} \mathbf{A} \neq \phi$. Therefore, $F_{A_{12}}$ is soft semi connected and soft open and so F_A is soft locally semi connected at $(x_2, f_A(x_2))$. Therefore, F_A is soft locally semi connected. Now we show that F_A is not soft semi connected. Let $A = F_{A_4}, B = F_{A_9}$. Then $\mathbf{A} = \mathbf{F}_{A_4}$ and $\mathbf{B} = \mathbf{F}_{A_9}$, $A \tilde{\cap} \mathbf{B} = \phi$ and $B \tilde{\cap} \mathbf{A} = \phi$ and so A and B are two soft semi separated sets. Hence F_A can be expressed as the soft union of two soft semi separated sets and so F_A is not soft semi connected.

Remark 4.4. Soft semi connectedness does not imply soft local semi connectedness as shown by the following example:

We consider the soft topological space in Remark 4.2. Here we show that F_A is soft semi connected. We first choose $A = F_{A_1}, B = F_{A_{12}}$. Then $\mathbf{B} = F_A$ and so $A \tilde{\cap} \mathbf{B} \neq \phi$. We next choose $A = F_{A_2}, B = F_{A_9}$. Then $\mathbf{A} = F_A$ and so $\mathbf{A} \tilde{\cap} B \neq \phi$. We next choose $A = F_{A_3}, B = F_{A_6}$. Then $\mathbf{A} = F_A$ and so $\mathbf{A} \tilde{\cap} B \neq \phi$. We next choose $A = F_{A_4}, B = F_{A_{14}}$. Then $\mathbf{B} = F_A$ and so $A \tilde{\cap} \mathbf{B} \neq \phi$. Finally, we choose $A = F_{A_5}, B = F_{A_{13}}$. Then $\mathbf{B} = F_A$ and so $A \tilde{\cap} \mathbf{B} \neq \phi$. Thus we see that F_A cannot be expressed as the soft union of two soft semi separated sets and hence F_A is soft semi connected. But in Remark 4.2, we have F_A is not soft locally semi connected.

Theorem 4.5. A soft topological space $(F_A, \tilde{\tau})$ is soft locally semi connected if and only if the soft semi components of soft semi open set are soft open sets.

Proof: Suppose that $(F_A, \tilde{\tau})$ is soft locally semi connected. Let $F_B \tilde{C} F_A$ be soft semi open and F_C be a soft semi component of F_B . If $(x, f_A(x)) \in F_C$, then because $(x, f_A(x)) \in F_B$, there is a soft semi connected soft open set F_U such that $(x, f_A(x)) \in F_U \tilde{C} F_B$. Since F_C is the soft semi component of

$(x, f_A(x))$ and F_U is soft semi connected, we have $(x, f_A(x)) \in F_U \tilde{C} F_C$. This shows that F_C is soft open.

Conversely, let $(x, f_A(x)) \in F_A$ be arbitrary and let F_B be a soft semi open set containing $(x, f_A(x))$. Let F_C be the soft semi component of F_B such that $(x, f_A(x)) \in F_C$. Now F_C is a soft semi connected soft open set such that $(x, f_A(x)) \in F_C \tilde{C} F_B$. This proves the theorem. ■

Theorem 4.6. A soft topological space $(F_A, \tilde{\tau})$ is soft locally semi connected if and only if given any soft point $(x, f_A(x)) \in F_A$ and a soft semi-open set F_U containing $(x, f_A(x))$, there is a soft open set F_C containing $(x, f_A(x))$ such that F_C is contained in a single soft semi component of F_U .

Proof: Let F_A be soft locally semi-connected, $(x, f_A(x)) \in F_A$ and F_U be a soft semi-open set containing $(x, f_A(x))$. Let F_V be a soft semi component of F_U that contains $(x, f_A(x))$. Since F_A is soft locally semi connected and F_U is soft semi open, there is a soft semi connected open set F_C such that $(x, f_A(x)) \in F_C \tilde{C} F_U$. By Theorem 3.9, F_V is the maximal soft semi connected set containing $(x, f_A(x))$ and so $(x, f_A(x)) \in F_C \tilde{C} F_V \tilde{C} F_U$. Since soft semi components are disjoint sets, it follows that F_C is not contained in any other soft semi component of F_U .

Conversely, we suppose that given any soft point $(x, f_A(x)) \in F_A$ and any soft semi open set F_U containing $(x, f_A(x))$, there is a soft open set F_C containing $(x, f_A(x))$ which is contained in a single soft semi component F_F of F_U . Then $(x, f_A(x)) \in F_C \tilde{C} F_F \tilde{C} F_U$. Let $(x_1, f_A(x_1)) \in F_F$, then $(x_1, f_A(x_1)) \in F_U$. Thus there is a soft open set F_O such that $(x_1, f_A(x_1)) \in F_O$ and F_O is contained in a single soft semi component of F_U . As the soft semi components are disjoint soft sets and $(x_1, f_A(x_1)) \in F_F$, $(x_1, f_A(x_1)) \in F_O \tilde{C} F_F$. Thus F_F is soft open. Thus for every $(x, f_A(x)) \in F_A$ and for every soft semi open set F_U containing $(x, f_A(x))$, there is a soft semi connected open set F_F such that $(x, f_A(x)) \in F_F \tilde{C} F_U$. Thus $(F_A, \tilde{\tau})$ is soft locally semi connected at $(x, f_A(x))$. Since $(x, f_A(x)) \in F_A$ is arbitrary, $(F_A, \tilde{\tau})$ is soft locally semi connected. This proves the theorem. ■

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