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Isolate domination in Unicyclic graphs

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Abstract

A subset D of the vertex set $V(G)$ of a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D. A dominating set D such that $< D >$ has an isolated vertex is called an isolate dominating set and the minimum cardinality of an isolate dominating set is called the isolate domination number of G and is denoted by $\gamma_0(G)$. In this paper we characterize the unicyclic graphs in which the order equals the sum of the isolate domination number and its maximum degree.

Keywords: Dominating set, isolate dominating set, isolate domination number.
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1 Introduction

One of the fastest developing areas in graph theory is the study of domination and related subset problems such as independence, covering and matching. In fact, there are scores of graph theoretic concepts involving domination, covering and independence. The bibliography in domination maintained by Haynes et. al.[4] currently has over 1200 entries in which one can find an appendix listing some 75 different types of domination and domination related parameters which have been studied in the literature. Hedetniemi and Laskar [5] edited a recent issue of Discrete Mathematics devoted entirely to domination, and a survey of advanced topics in domination is given in the book by Hatynes et.al.[3]. In 1978, Cockayne et.al.[2] defined what has now become a well-known inequality chain of domination related parameters of a graph G

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G) \tag{1}$$

where $ir(G)$ and $IR(G)$ denote the lower and upper irredundance numbers, $\gamma(G)$ and $\Gamma(G)$ denote the lower and upper domination numbers and $i(G)$ and $\beta_0(G)$ denote the independent domination number and independence number of a graph G. Since then more number of papers have been published in which this inequality chain is the focus of study. Researchers have considered the following areas in their studies.

(i) Conditions under which two or more of these parameters are equal.

(ii) Parameters whose value lie between two consecutive parameters in (1).

(iii) Extensions of the inequality chain(1) in either direction.

(iv) Ratios of consecutive pairs of parameters in (1).

(v) Inequality chains similar to (1) for other parameters. For example, similar inequality chains exist for edge domination and for mixed domination parameters.

These are studied in chapter 3 of the recent book Fundamentals of Domination in Graphs by Haynes et.al. [4].

In Chapter 6 of [4] it is suggested that a type of domination is fundamental if,

(i) Every connected, non-trivial graph has a dominating set of this type and

(ii) This type of domination set is defined in terms of some 'natural' property of the subgraph $< D >$ induced by D.

For any dominating set D, some of the fundamental types of domination include:

(a) Domination: $< D >$ can be any graph.

(b) Total domination: $< D >$ has minimum degree at least 1.

(c) Independent domination: $< D >$ has no edges.

(d) Connected domination: $< D >$ is connected.

(e) Paired domination: $< D >$ has a perfect matching.

In [6] we introduced the study of a new fundamental type of domination namely *Isolate domination*. A dominating set D is said to be an isolate dominating set if $< D >$ has an isolated vertex.

We also defined in [6] the isolate domination number $\gamma_0(G)$ to be the minimum cardinality of an isolate dominating set in a graph G. This invariant is particularly interesting in that it provides more examples whose value lie between two consecutive parameters γ and i in (1). Further, for any graph G on n vertices a vertex together with its non-neighbours form an isolate dominating set and therefore the value of $\gamma_0(G)$ is at most $n - \Delta$. In [6] we investigated some some properties of graphs acheiving this bound. In this paper we characterize the unicyclic graphs attaining this bound.

2 Definitions and Notations

By a graph $G = (V, E)$, we mean a connected, finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer toChartrand and Lesniak [1]. All graphs in this paper are assumed to be non-trivial. The *open neighbourhood* of a vertex $v \in V$ is $N(v) = \{x \in V : vx \in E\}$, that is the set of vertices adjacent to v. The *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. If $S \subseteq V$ and $v \in S$, then a vertex u is said to be a *private neighbour* of v with respect to S if $N(u) \cap S = \{v\}$. The subgraph induced by a set $S \subseteq V$ is denoted by $< S >$.

A set $S \subseteq V$ is a *dominating set* if every vertex in $V - S$ is adjacent to a vertex of S. A dominating set $S \subset V$ is called an *isolate dominating set* if $< S >$ has an isolated vertex. The minimum cardinality of an (isolate) dominating set is called the *(isolate) domination number* of G and is denoted by $(\gamma_0(G))\gamma(G)$. A minimum (isolate) dominating set of a graph G is called a $(\gamma_0 - set)\gamma - set$.

Also, for a vertex v in a graph G, we define N_v to be the set of all vertices in G which are not adjacent to v, that is N_v = V - $N[v]$. A vertex x in $N(v)$ is said to be a *major vertex* with respect to v if x is adjacent to all the vertices of N_v. The set of all major and non-major vertices with respect to v are denoted by $M(v)$ and $M'(v)$ repectively.

For a vertex v in G, we define the properties $C_1(v), C_2(v)$ and $C_3(v)$ as follows:

$C_1(v)$: N_v is independent.

$C_2(v)$: A vertex of $N(v)$ having more than one neighbour in N_v belongs to $M(v)$.

$C_3(v)$: If $N(v)$ has $l \geq 1$ pendant vertices, then no subset of N_v with fewer than $|N_v|$ - l vertices dominate $M'(v)$.

Theorem 2.1. ([6]) Let G be a graph on n vertices with $\gamma_0(G) = n - \Delta$. Then $\Delta \geq n/2$ and the conditions $C_1(v)$ to $C_3(v)$ hold for any vertex v of degree Δ.

3 Unicyclic graphs with $\gamma_0(G) + \Delta(G) = |V(G)|$

In this section we obtain a structural characterization of the class of unicyclic graphs G for which $\gamma_0(G) + \Delta(G) = |V(G)|$. If G is a cycle then it is either C_3 or C_4. In order to characterize such unicyclic graphs G when it is not a cycle we describe the following families of unicyclic graphs.

For a given positive integer l, let H_1 and H_2 be the graphs obtained by attaching l pendant edges at a vertex v of C_3 and C_4 respectively. Also H is a graph obtained from H_1 by attaching a pendant edge each at $l+1$ neighbours of v except at a neighbour of v in C_3. We now define the families of unicyclic graphs as follows:

(i) $\mathcal{G}_1$ is the class of unicyclic graphs obtianed from H_1 by attaching at most l pendant edges at exactly one neighbour of v. (Note that the graph H_1 itself belongs to $\mathcal{G}_1$).

(ii) $\mathcal{G}_2$ is the class of unicyclic graphs obtained from H_1 by attaching a pendant edge in at most $l+1$ neigbours of v and it does not contain the graph H.

(iii) $\mathcal{G}_3$ is the class of all unicyclic graphs obtained from H_2 by attaching at most l pendant edges at exactly one neighbour of v on the cycle C_4 (H_2 itself belongs to $\mathcal{G}_3$).

(iv) $\mathcal{G}_4$ is the class of unicyclic graphs obtaned from H_2 by attaching a pendant edge in at most $l-1$ pendant vertices of H_2.

Theorem 3.1. Let G be a unicyclic graph which is not a cycle. Then $\gamma_0(G) + \Delta(G) = |V(G)|$ if and only if $G \in \bigcup_{i=1}^{4} \mathcal{G}_i$.

Proof: Let G be a unicyclic graph with $\gamma_0(G) + \Delta(G) = |V(G)|$ and $l = \Delta(G) - 2$. By Theorem 2.1, the conditions $C_1(v)$ to $C_3(v)$ are satisfied for any vertex v of maximum degree. Now, let C be the cycle in G. Then every vertex x in $V - C$ will have at most one neighbour in C so that the set N_x is not independent and hence by the condition $C_1(v)$, x is not a vertex of maximum degree and therefore the vertices of maximum degree lie only on the cycle C. Further, by the condition $C_1(v)$, as N_v is independent for a vertex v of maximum degree, the cycle C is either C_3 or C_4 and also $|N_v| \leq \Delta - 1$ as $\Delta(G) \geq \frac{n}{2}$. Now we consider the following two cases.

Case 1: $C = C_3$.

If $N_v = \phi$ then $G \cong H_1 \in \mathcal{G}_1$. Let us assume that $N_v \neq \phi$ then there is at most one major vertex in $N(v)$, for otherwise we have a cycle other than C.

Suppose $N(v)$ has exactly one major vertex, say x. That is, x is adjacent to all the vertices in N_v. If x lies on C, obviously $|N_v| \leq \Delta - 2$. If x lies outside C, then x cannot be of maximum degree so that $|N_v| \leq \Delta - 2$. Thus in either case G is isomorphic to a graph obtained from H_1 by attaching at most l pendant edges at exactly one neighbour of v and so G belongs to $\mathcal{G}_1$.

Suppose there is no major vertex in $N(v)$. Then by the condition $C_2(v)$, every vertex in $N(v)$ has at most one neighbour in N_v, where $|N_v| \leq \Delta - 1$. Further in this case if $N(v) = \Delta - 1$ then both the neighbours of v in C_3 have adjacent vertices in N_v, otherwise the set of all supports in $N(v)$ will form an isolate dominating set of cardinality less than $n - \Delta$ and so G is not isomorphic to H. Hence G belongs to the family $\mathcal{G}_2$

Case 2: $C = C_4$.

In this case the vertex v has a non-neighbour, say y, in C and so the set N_v is non-empty. If y is the only vertex in N_v then G is isomorphic to the graph H_2 which belongs to the family $\mathcal{G}_3$. Assume that N_v has a vertex other than y and x is a major vertex in $N(v)$. If x lies outside C, then the vertices x and y along with v and one of its neighbours in C form a cycle (of length 4) other than C and therefore x must lie on C. Also G being unicyclic x is the only major vertex in $N(v)$ and also the vertices in $N(v)$ which do not lie on C have no neighbours in N_v. Therefore G is isomorphic to a graph obtained from H_2 by attaching at most l pendant edges at exactly one neighbour of v in C and hence it belongs to $\mathcal{G}_3$.

Now let us consider the case that $N(v)$ has no major vertices. Then by the condition $C_2(v)$ both the neighbours of v in C are not adjacent to any of the vertices in $N_v - \{y\}$ and of course each of the neigbhbours of v in $V - C$ has at most one adjacent in $N_v - \{y\}$. Further, If all the vertices in $N(v) - V(C_3)$ have neighbours in N_v then the set of all supports in $N(v)$ together with y will form an isolate dominating set of cardinality less than $n - \Delta$. Therefore at most $l - 1$ vertices of $N(v) - V(C_3)$ have neighbours in N_v and so $G \in \mathcal{G}_4$.

Conversely, suppose $G \in \mathcal{G}_1$. Then G is obtained from H_1 by attaching $r \leq l$ pendant edges at a neighbour w of v. Hence $n = r + l + 3$ and $\Delta = l + 2$. Now, an isolate dominating set D of G can not contain both v and w simultaneously. If D contains neither v nor w, then D must contain all the pendant vertices along with a vertex on the cycle and so $|D| \geq r + l + 1 \geq 2r + 1 \geq n - \Delta$. If D contains the vertex v, then D should contain all the r non-neigbours of v and so $|D| \geq r + 1 = n - \Delta$; and the simillar argument works when D contains the vertex w.

Suppose $G \in \mathcal{G}_2$. Now consider an isolate dominating set D of G. If $v \in D$ then D must contain all the pendant vertices so that $|D| \geq n - \Delta$ and also D will have at least $l+1$ vertices when $v \notin D$ so that $|D| \geq n - \Delta$.

Simillar argument can be applied for the graphs belonging to either $\mathcal{G}_3$ or $\mathcal{G}_4$. ■

Remark 3.2. If G is a unicyclic graph on n vertices which belongs to either $\mathcal{G}_1$ or $\mathcal{G}_4$ then $n \leq 2\Delta - 1$ and if G is either in $\mathcal{G}_2$ or $\mathcal{G}_3$ then $n \leq 2\Delta$. Thus, once Δ is fixed the above theorem completly characterizes all the unicyclic graphs for which $\gamma_0(G) + \Delta(G) = |V(G)|$. The list of all unicyclic graphs with $\Delta = 4$ for which $\gamma_0(G) + \Delta(G) = |V(G)|$ is given below.

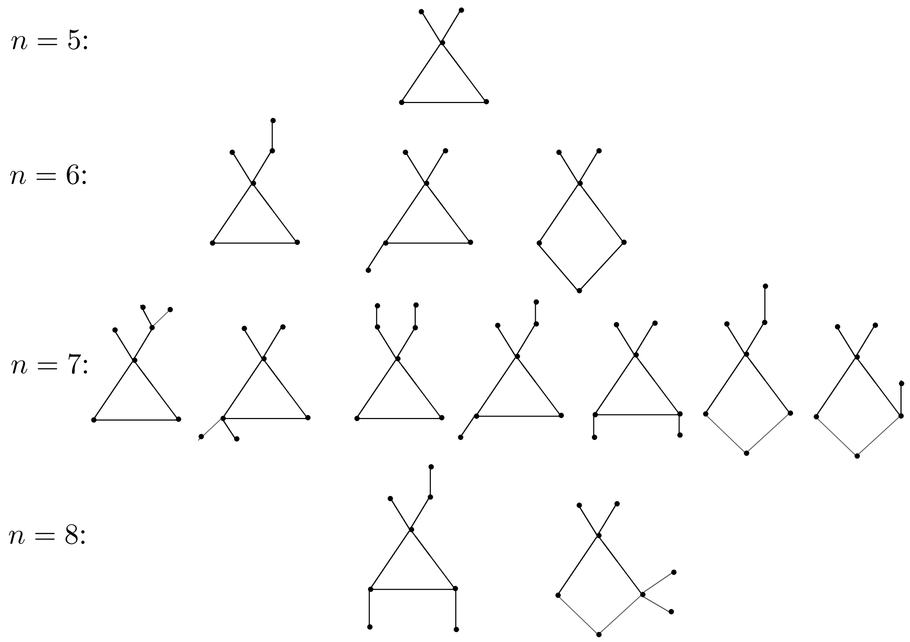

Figure 1: *Unicyclic graphs G with* $\Delta(G) = 4$ *and* $\gamma_0(G) + \Delta(G) = |V(G)|$.

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