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Boolean filters of distributive lattices

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Abstract

In this paper we introduce the notion of Boolean filters in a pseudo-complemented distributive lattice and characterize the class of all Boolean filters. Further a set of equivalent conditions are derived for a proper filter to become a prime Boolean filter. Also a set of equivalent conditions is derived for a pseudo-complemented distributive lattice to become a Boolean algebra. Finally, a Boolean filter is characterized in terms of congruences.

Keywords: Pseudo-complemented distributive lattice, Boolean algebra, Boolean filter, maximal filter, congruence.
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1 Introduction

The theory of pseudo-complements in lattices, and particularly in distributive lattices was developed by M.H. Stone [10], O. Frink [3] and George Gratzer [4]. Later many authors like R. Balbes [1], O. Frink [3] extended the study of pseudo-complements to characterize Stone lattices. The concept of Boolean deductive systems was introduced in Residual lattices by E. Turunen [11]. Later, M. Haveshki, A. B. Saeid and E. Eslami [5] studied the properties of Boolean deductive systems in BL-algebras.

The aim of this paper is to introduce the notion of Boolean filters in pseudo-complemented distributive lattices as a generalization of Boolean deductive systems in BL-algebras and prove some of the properties of these Boolean filters. It is observed that every maximal filter is a Boolean filter and the converse is not true. However, a set of equivalent conditions is established for every proper filter to become a maximal filter which leads to an equivalency between maximal filters and prime Boolean filters. The class of all Boolean filters are characterized. Some properties of the homomorphic images and direct products of Boolean filters are observed. A set of equivalent conditions are also derived for every pseudo-complemented distributive lattice to become a Boolean algebra. Finally, the Boolean filters are characterized in terms of known filter congruence.

The reader is referred to [4] for the notions and notations. However, we present some of the preliminary definitions and results for the ready reference.

Definition 1.1. [4] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a lattice if for all $x, y, z \in L$, it satisfies the following properties.

(1) $x \wedge x = x$, $x \vee x = x$
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$

Definition 1.2. [4] A lattice L is called distributive if for all $x, y, z \in L$ it satisfies the following properties.

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Definition 1.3. [4] Let $(L, \wedge, \vee)$ be a lattice. A partial ordering relation $\leq$ is defined on L by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

The pseudo-complement b^* of an element b is the greatest element disjoint from b, if such an element exists. The defining property of b^* is:

$$a \wedge b = 0 \;\Leftrightarrow\; a \wedge b^* = a \;\Leftrightarrow\; a \leq b^*$$

where $\leq$ is a partial ordering relation on the lattice L.

A distributive lattice L in which every element has a pseudo-complement is called a pseudo-complemented distributive lattice. For any two elements a, b of a pseudo-complemented lattice, we have the following.

(1). $a \leq b$ implies $b^* \leq a^*$
(2). $a \leq a^{**}$
(3). $a^{***} = a^*$
(4). $(a \vee b)^* = a^* \wedge b^*$
(5). $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

An element a of L is called a dense element if $a^* = 0$ and the set D of all dense elements of L forms a filter in L.

A proper filter P of a lattice L is called a prime filter if $x \vee y \in P$ implies $x \in P$ or $y \in P$ for all $x, y \in L$. A proper filter M of L is called maximal if there exists no proper filter Q such that $M \subset Q$. In a distributive lattice, every maximal filter is a prime filter but not the converse. However, in a relatively complemented lattice, every prime filter is maximal. It is noted that prime filters have also been used to classify the 0-distributivity of semilattices [6]. For distributive lattices, we have the following theorem related to prime filters.

Theorem 1.4. *[4] Let L be a distributive lattice and $x, y \in L$ such that $x \neq y$. Then there exists a prime filter P such that $x \in P$ and $y \notin P$.*

Throughout this note, unless otherwise mentioned, all lattices are bounded and pseudo-complemented distributive lattices.

2 Boolean filters and their properties

In this section, the concept of Boolean filters is introduced in a pseudo-complemented distributive lattice. Further the direct products and the homomorphic images of Boolean filters are studied. Finally, a set of equivalent conditions are derived for every filter of L to become a Boolean filter.

Definition 2.1. Let L be a pseudo-complemented distributive lattice. A filter F of L is called a Boolean filter if $x \vee x^* \in F$ for each $x \in L$.

Since $x \vee x^* \in D$ for all $x \in L$, it is evident that D is a Boolean filter of L. In fact it is the smallest Boolean filter of L.

Example 2.2. *Let $L = \{0, a, b, c, d, 1\}$ be a distributive lattice whose Hasse diagram is given in the following figure.*

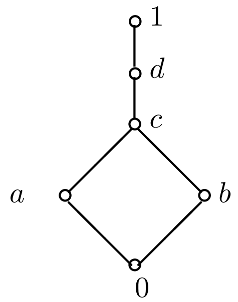

Figure 1: *Hasse diagram of the distributive lattice $L = \{0, a, b, c, d, 1\}$.*

Consider the filters $F_1 = \{a, c, d, 1\}; F_2 = \{b, c, d, 1\}; F_3 = \{c, d, 1\}; F_4 = \{d, 1\}$ and $F_5 = \{1\}$. Then clearly F_1, F_2 and F_3 are Boolean filters where as F_4 and F_5 are not Boolean, because of $a \vee a^* = a \vee b = c \notin F_4 \cup F_5$.

Proposition 2.3. *Every maximal filter of L is a Boolean filter.*

Proof. Let M be a maximal filter of L. Suppose $x \vee x^* \notin M$ for some $x \in L$. Then $M \vee [x \vee x^*) = L$. Hence $0 = a \wedge b$ for some $a \in M$ and $b \in [x \vee x^*)$. Then we have the following consequence.

$$\begin{aligned} a \wedge b = 0 \quad &\Rightarrow \quad a \wedge (x \vee x^*) = 0 \\ &\Rightarrow \quad a \wedge x = 0 \text{ and } a \wedge x^* = 0 \\ &\Rightarrow \quad a \leq x^* \text{ and } a \leq x^{**} \\ &\Rightarrow \quad a \leq x^* \wedge x^{**} = 0 \end{aligned}$$

which is a contradiction to the fact that $0 \in M$. Hence $x \vee x^* \in M$ for all $x \in L$. Therefore, M is a Boolean filter of L. ∎

Corollary 2.4. *A proper filter of a pseudo-complemented lattice L which contains either x or x^* for all $x \in L$ is a Boolean filter.*

Proof. Let F be a proper filter of L satisfying the given condition. We show that F is maximal. Suppose G is a proper filter of L such that $F \subset G$. Choose $a \in G - F$. Since $a \notin F$, by the condition, we get $a^* \in F \subset G$. Since $a \in G$ and $a^* \in G$, we get $0 = a \wedge a^* \in G$, which is a contradiction. Therefore, F is a maximal filter. Thus by Proposition 2.3, F is a Boolean filter. ∎

Corollary 2.5. *In a relatively complemented lattice, every prime filter is a Boolean filter.*

The converse of Proposition 2.3 is not true in general. For, in Example 2.2, the filter F_3 is a Boolean filter but not a maximal filter.

A set of equivalent conditions are derived for a Boolean filter to become a maximal filter.

Theorem 2.6. *Let F be a proper filter of a pseudo-complemented lattice L. Then the following conditions are equivalent.*

(1) F is maximal.

(2) $x \notin F$ implies $x^* \in F$ for all $x \in L$.

(3) F is prime Boolean.

Proof. $(1) \Rightarrow (2)$: Assume that F is a maximal filter of L. Suppose $x \in L - F$, then $F \vee [x) = L$ which yields that $a \wedge x = 0$ for some $a \in F$. Hence $a \leq x^*$, which implies that $x^* \in F$.

$(2) \Rightarrow (3)$: Let $x \in L$. Suppose $x \vee x^* \notin F$. Then it is clear that $x \notin F$ and $x^* \notin F$, which is a contradiction to the condition (2). Hence F is a Boolean filter of L. Suppose $x \vee y \in F$ and $x \notin F$. Then by condition (2), we get $x^* \in F$. Hence $x^* \wedge y = 0 \vee (x^* \wedge y) = (x^* \wedge x) \vee (x^* \wedge y) = x^* \wedge (x \vee y) \in F$. Since $x^* \wedge y \leq y$, we get that $y \in F$. Therefore, F is a prime Boolean filter of L.

$(3) \Rightarrow (1)$: Assume that F is a prime Boolean filter of L. Suppose F is not maximal. There exists a proper filter F' of L such that $F \subset F'$. Choose $x \in F' - F$. Since F is Boolean, we get $x \vee x^* \in F$. Since F is prime and $x \notin F$, we get $x^* \in F \subset F'$. Hence it concludes that $0 = x \wedge x^* \in F'$, which is a contradiction. Therefore, F is a maximal filter. ∎

The following proposition is obvious from Definition 2.1.

Proposition 2.7. *Let F, G be two filters of a pseudo-complemented lattice such that $F \subseteq G$. If F is a Boolean filter then so is G.*

We now characterize the Boolean filters in the following:

Theorem 2.8. *Let F be a proper filter of a pseudo-complemented lattice L. Then the following conditions are equivalent.*

(1) F is a Boolean filter.

(2) $x^{**} \in F$ implies $x \in F$.

(3) For $x, y \in L, x^* = y^*$ and $x \in F$ imply $y \in F$.

Proof. $(1) \Rightarrow (2)$: Assume that F is a Boolean filter of L. Suppose $x^{**} \in F$. Since F is a Boolean filter, we get $x \vee x^* \in F$. Hence $x = x \vee 0 = (x \wedge x^{**}) \vee (x^* \wedge x^{**}) = (x \vee x^*) \wedge x^{**} \in F$. Therefore, condition (2) holds.

$(2) \Rightarrow (3)$: Let $x, y \in L$ and $x^* = y^*$. Suppose $x \in F$, then $y^{**} = x^{**} \in F$. Hence by the condition (2), it follows that $y \in F$.

$(3) \Rightarrow (1)$: Let $x \in D$. Then $x^* = 0 \leq a^*$ for any $a \in F$. Hence $a^{**} \leq x^{**}$ and $a^{**} \in F$. Hence $x^{**} \in F$. Since $x^* = x^{***}$ and $x^{**} \in F$, by the condition (3), we get $x \in F$. Hence $D \subseteq F$. Since D is a Boolean filter, by Proposition2.7, we get that F is a Boolean filter of L. ■

Now we discuss about the homomorphic images of Boolean filters of pseudo-complemented distributive lattices. By a homomorphism on a pseudo-complemented lattice, we mean a bounded homomorphism which also preserves the pseudo-complementation, that is, $f(x^*) = f(x)^*$ for all $x \in L$.

Theorem 2.9. *Let $(L, \vee, \wedge, ^*, 0, 1)$ and $(L', \vee, \wedge, ^*, 0', 1')$ be two pseudo-complemented lattices and ψ a homomorphism from L onto L'. Then we have the following conditions.*

(1) $\psi(F)$ is a Boolean filter of L' whenever F is a Boolean filter of L.
(2) $\psi^{-1}(G)$ is a Boolean filter of L whenever G is a Boolean filter of L'.

Proof. (1). Suppose F is a Boolean filter of L. It is known that $\psi(F)$ is a filter of L'. Let $y \in L'$. Since ψ is onto, there exists $x \in L$ such that $\psi(x) = y$. Since F is a Boolean filter of L, we get $x \vee x^* \in F$. Now $y \vee y^* = \psi(x) \vee \psi(x)^* = \psi(x) \vee \psi(x^*) = \psi(x \vee x^*) \in \psi(F)$. Therefore, $\psi(F)$ is a Boolean filter of L'.

(2). Let G be a Boolean filter of L'. Clearly $\psi^{-1}(G)$ is a filter of L. Let $x \in L$. Then $\psi(x \vee x^*) = \psi(x) \vee \psi(x^*) = \psi(x) \vee \psi(x)^* \in G$, since $\psi(x) \in L'$. Hence we get $x \vee x^* \in \psi^{-1}(G)$. Therefore, $\psi^{-1}(G)$ is a Boolean filter of L. ■

Let L_1 and L_2 be two pseudo-complemented distributive lattices with $*$ as their pseudo-complementation. Then $L_1 \times L_2$ is also a pseudo-complemented distributive lattice with respect to the point-wise operations in which the pseudo-complementation is given as follows:

$$(a,\ b)^* = (a^*,\ b^*)$$

Now we discuss about the direct products of Boolean filters of a pseudo-complemented distributive lattice.

Theorem 2.10. *If F_1 and F_2 are Boolean filters of L_1 and L_2 respectively, then $F_1 \times F_2$ is a normal filter of the product lattice $L_1 \times L_2$. Conversely, every Boolean filter F of $L_1 \times L_2$ can be expressed as $F = F_1 \times F_2$ where F_1 and F_2 are Boolean filters of L_1 and L_2 respectively.*

Proof. Let F_1 and F_2 be Boolean filters of L_1 and L_2 respectively. Since $1 \in F_1$ and $1 \in F_2$, we get $(1, 1) \in F_1 \times F_2$. Clearly $F_1 \times F_2$ is a filter of $L_1 \times L_2$. Let $x \in L_1$ and $y \in L_2$. Since F_1 and F_2 are Boolean filters of L_1 and L_2 respectively, we get $x \vee x^* \in F_1$ and $y \vee y^* \in F_2$. Hence $(x, y) \vee (x, y)^* = (x \vee x^*, y \vee y^*) \in F_1 \times F_2$. Therefore, $F_1 \times F_2$ is a Boolean filter of $L_1 \times L_2$.

Conversely, let F be any Boolean filter of $L_1 \times L_2$. Consider the projections $\Pi_i : L_1 \times L_2 \longrightarrow L_i$ for $i = 1, 2$. Let F_1 and F_2 be the projections of F on L_1 and L_2 respectively. That is $\Pi_i(F) = F_i$ for $i = 1, 2$. We prove that F_1 and F_2 are Boolean filters of L_1 and L_2 respectively. Since $(1, 1) \in F$, we get $1 = \Pi_1(1, 1) \in F_1$. Clearly F_1 is a filter of L_1. Let $x \in L_1$ and $x^{**} \in F_1$. Then $(x, 1)^{**} = (x^{**}, 1^{**}) = (x^{**}, 1) \in F$. Since F is a Boolean filter, we get $(x, 1) \in F$. Thus $x = \Pi_1(x, 1) \in \Pi_1(F) = F_1$. Therefore, F_1 is a Boolean filter of L_1. Similarly, we get F_2 is a Boolean filter of L_2.

Next we prove that $F = F_1 \times F_2$. Clearly $F \subseteq F_1 \times F_2$. Let $(x, y) \in F_1 \times F_2$. Then $x^{**} \in F_1 = \Pi_1(F)$ and $y^{**} \in F_2 = \Pi_2(F)$. Hence $(x^{**}, 1) \in F$ and $(1, y^{**}) \in F$. Since F is a filter, we have $(x, y)^{**} = (x^{**}, y^{**}) = (x^{**} \wedge 1, 1 \wedge y^{**}) = (x^{**}, 1) \wedge (1, y^{**}) \in F$. Since F is a Boolean filter, we get that $(x, y) \in F$. Thus we have $F_1 \times F_2 \subseteq F$ and hence $F = F_1 \times F_2$. ∎

We recall the well known Glivinko type congruence ψ defined on L such that $(x, y) \in \psi$ if and only if $x^* = y^*$ for all $x, y \in L$. We derive a set of equivalent conditions for every filter of L to become a Boolean filter.

Theorem 2.11. *Let L be a pseudo-complemented distributive lattice. Then the following conditions are equivalent.*

(1) L is a Boolean algebra.
(2) Every filter is a Boolean filter.
(3) Every principal filter is a Boolean filter.
(4) Every prime filter is a Boolean filter.
(5) ψ is the smallest congruence.

Proof. $(1) \Rightarrow (2)$: It is a fact that L is a Boolean algebra if and only if it has a unique dense element. Assume that L has a unique dense element, precisely 1. Let F be a filter of L. Then $x \vee x^* = 1 \in F$ for all $x \in L$. Therefore, F is a Boolean filter of L.
$(2) \Rightarrow (3)$: It is obvious.
$(3) \Rightarrow (4)$: Assume that every principal filter of L is a Boolean filter. Then clearly $[1)$ is a Boolean filter of L. Since $[1) \subseteq P$, by Proposition 2.7, we get that P is also a Boolean filter of L.
$(4) \Rightarrow (5)$: Assume that every prime filter is a Boolean filter. Let $x, y \in L$ be such that $(x, y) \in \psi$. Suppose $x \neq y$. Then there exists a prime filter P such that $x \in P$ and $y \notin P$. Hence $y^{**} = x^{**} \in P$. Since P is Boolean, we get $y \in P$, which is a contradiction. Hence $x = y$. Therefore, ψ is the smallest congruence.
$(5) \Rightarrow (1)$: Assume the condition (5). Suppose L has two dense elements, say x, y. Then we get $x^* = 0 = y^*$. Hence $(x, y) \in \psi$.

Therefore, by condition (5), we get $x = y$. Thus L has a unique dense element and hence is a Boolean algebra. ∎

For any filter F of a distributive lattice, a congruence relation Ψ_F is defined by $(x, y) \in \Psi_F$ if there exist $f \in F$ such that $x \wedge f = y \wedge f$. The associated quotient lattice is denoted by $L_{/\Psi(F)}$ and Ψ denotes the canonical epimorphism of L onto the quotient lattice. For $x \in L, \Psi(x) = \widehat{x} =$ the congruence class of x modulo Ψ_F. It is well-known that the elements of F are all congruent under Ψ_F and the equivalence class of F is the largest element in $L_{/\Psi_F}$. It is also clear that $L_{/\Psi_F}$ is a distributive lattice. This congruence was studied in detail by T.P. Speed [6]. Now, Boolean filters are characterized in terms of congruence Ψ_F.

Theorem 2.12. *Let F be a filter of a pseudo-complemented distributive lattice L. Then the following conditions are equivalent.*

(1) F is a Boolean filter.
(2) $L_{/\Psi_F}$ is a Boolean algebra.

Proof. $(1) \Rightarrow (2)$: Assume that F is a Boolean filter of L. Let $\widehat{x} \in L_{/\Psi_F}$. We have always $x \wedge x^* = 0$ and hence $\widehat{x} \wedge \widehat{x^*} = \widehat{x \wedge x^*} = \widehat{0}$. Since F is a Boolean filter, we get that $x \vee x^* \in F$. Hence we have $\widehat{x} \vee \widehat{x^*} = \widehat{x \vee x^*} = F$. Therefore, $L_{/\Psi_F}$ is a Boolean algebra.
$(2) \Rightarrow (1)$: Assume that $L_{/\Psi_F}$ is a Boolean algebra. Let $x \in L$. Then $\widehat{x} \in L_{/\Psi F}$. Since $L_{/\Psi F}$ is a Boolean algebra, there exists $y \in L$ such that $\widehat{x \wedge y} = \widehat{x} \wedge \widehat{y} = \widehat{0}$ and $\widehat{x \vee y} = \widehat{x} \vee \widehat{y} = F$. Hence it follows that $(x \wedge y, 0) \in \Psi_F$ and $x \vee y \in F$. Since $(x \wedge y, 0) \in \Psi_F$, there exists $f \in F$ such that $x \wedge y \wedge f = 0$ and thus we get $y \wedge f \leq x^*$. Therefore, we get the following consequence.

$$\begin{aligned}
x \vee y \in F \text{ and } f \in F \quad &\Rightarrow \quad (x \vee y) \wedge f \in F \\
&\Rightarrow \quad (x \wedge f) \vee (y \wedge f) \in F \\
&\Rightarrow \quad (x \wedge f) \vee x^* \in F \qquad \text{since } y \wedge f \leq x^* \\
&\Rightarrow \quad (x \vee x^*) \wedge (f \vee x^*) \in F \\
&\Rightarrow \quad x \vee x^* \in F
\end{aligned}$$

Therefore, F is a Boolean filter of L. ∎

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