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# A new kind of Pál type $(0; 0, 1)$ interpolation

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**Abstract**

Consider the trignometric polynomials $r_n(x) = \frac{\cos(2n+1)\frac{\theta}{2}}{\cos\frac{\theta}{2}}$ and $s_n(x) = \frac{\sin(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$. These polynomials are such that their zeros are interscaled and neither of the polynomials is the extrema of the other polynomial. On these interscaled set of nodes we study an interpolation process when Lagrange data is prescribed on one set of nodes and Hermite data is prescribed on the other set of nodes.

**Keywords**: Interpolation, Pál Type interpolation.
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## 1 Introduction

Consider an interscaled set of points

$$-\infty < x_{n,n} < y_{n-1,n} < \cdots < x_{2,n} < y_{1,n} < x_{1,n} < \infty \tag{1.1}$$

where $\{x_{i,n}\}_{i=1}^{n}$ are the zeros of the polynomial $W_n(x)$, that is,

$$W_n(x) = \prod_{i=1}^{n}(x - x_{i,n}) \tag{1.2}$$

and $\{y_{i,n}\}_{i=1}^{n-1}$ are the zeros of $W'_n(x)$. In 1975, L. G. Pál [9] intiated the study of an interpolation process when the function values were prescribed at one set of nodes and first derivatives were prescribed on

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another set of nodes. Precisely, he proved that for given arbitrary numbers $\{\alpha^*_{i,n}\}_{i=1}^{n}$ and $\{\beta^*_{i,n}\}_{i=1}^{n-1}$ there exists a unique polynomial of degree $\leq 2n-1$ satisfying the conditions:

$$\begin{aligned} R_n(x_{i,n}) &= \alpha^*_{i,n}\ , \quad i=1,2,\cdots,n, && (1.3) \\ R'_n(y_{i,n}) &= \beta^*_{i,n}\ , \quad i=1,2,\cdots,n-1 && (1.4) \end{aligned}$$

and an initial condition $R_n(a)=0$, where $a$ is a given point, different from the nodal points $\{x_{i,n}\}_{i=1}^{n}$ and $\{y_{i,n}\}_{i=1}^{n}$. Since then the above interpolatory problem have been taken up by many mathematicians on different set of nodes viz. on finite interval [[1], [14] etc.], infinite interval [[3], [5]-[8], [11], [16], [17] etc.] and on unit circle, most of which are given in [4]. In almost all the problems on Pál type interpolations the two set of nodes considered are the zeros of a polynomial and its extrema.

In this paper we have considered a different type of Pál type $(0;0,1)$ interpolation in the sense that the two set of nodes considered are interscaled and they are the zeros of two different polynomials such that none is the extrema of the other.

Let $\{x_{n,k}\}_{k=0}^{n}$ and $\{y_{n,k}\}_{k=1}^{n}$ be two distinct point systems in the interval [-1,1] which are interscaled such that

$$-1 = y_{n,n+1} < x_{n,n} < y_{n,n} < \cdots < y_{n,2} < x_{n,1} < y_{n,1} < x_{n,0} = 1 \tag{1.5}$$

where $\{x_{n,k}\}_{k=1}^{n}$ and $\{y_{n,k}\}_{k=1}^{n}$ are the distinct zeros of

$$r_n(x) = \frac{\cos(2n+1)\frac{\theta}{2}}{\cos\frac{\theta}{2}} \tag{1.6}$$

and

$$s_n(x) = \frac{\sin(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}} \tag{1.7}$$

respectively, with $x=\cos\theta$, that* is

$$\begin{cases} r_n(x_{n,k}) = 0, & k=1,2,\cdots,n \\ s_n(y_{n,k}) = 0, & k=1,2,\cdots,n. \end{cases} \tag{1.8}$$

For arbitrarily given numbers $\{\alpha_k\}_{k=0}^{n}$, $\{\beta_k\}_{k=1}^{n+1}$ and $\{\gamma_k\}_{k=1}^{n}$ we have to find a polynomial $R_n(x)$ of lowest possible degree satisfying the conditions:

$$\begin{cases} R_n(x_k) = \alpha_k, \quad k=0,1,\cdots,n \\ R_n(y_k) = \beta_k, \quad k=1,\cdots,n,n+1 \\ \{R'_n(x)\}_{y_k} = \gamma_k, \quad k=1,2,...,n \end{cases} \tag{1.9}$$

The novelty in this communication comes from the inter-scaled set of nodes as none of them is the extrema of the other. Hence we call such an interpolation a new kind of Pál type $(0;0,1)$ interpolation.

In [12], [13] the authors have considered the problem when the Lagrange data is prescribed on one set of zeros of (1.5) and weighted first derivatives are prescribed on the other set of zeros of (1.5).

* In the sequel we will use $k$ instead of $k,n$ in the subscript.

In Section 2 preliminaries are given. Section 3 deals with the explicit representation of the fundamental polynomials when $R_n(x)$ satisfies the conditions (1.9) and also when $x_k$'s and $y_k$'s are interchanged in (1.9). Statement of convergence theorems in both the cases are given in Section 3.3. Section 4 is devoted to the estimation of the fundamental polynomials leading to the proof of convervegence theorems which are given in Section 5.

## 2 Preliminaries

Let $r_n(x)$ and $s_n(x)$ be given by (1.6) and (1.7) respectively. On differentiating with respect to $x$, we have

$$r_n(x) + 2(1+x)r_n'(x) = (2n+1)s_n(x) \tag{2.1}$$

and

$$s_n(x) - 2(1-x)s_n'(x) = (2n+1)r_n(x). \tag{2.2}$$

They also satisfy the recurrence relation [15]

$$2(1-x^2)s_n'(x) = \{2(n+1)x+1\}s_n(x) - 2(n+1)s_{n+1}(x) \tag{2.3}$$

and

$$2(1-x^2)r_n'(x) = \{2(n+1)x-1\}r_n(x) - 2(n+1)r_{n+1}(x). \tag{2.4}$$

We also have

$$O\left(r_n(x)\right) = O\left(s_n(x)\right) = c\sqrt{n} \tag{2.5}$$

where $c$ is independent of $n$ and $x$. Let $\{x_k\}_{k=1}^n$ and $\{y_k\}_{k=1}^n$ be the distinct zeros of (1.6) and (1.7) respectively then, we have

$$r_n(y_k) = \frac{(-1)^k\sqrt{2}}{\sqrt{1+y_k}}, \tag{2.6}$$

$$s_n(x_k) = \frac{(-1)^k\sqrt{2}}{\sqrt{1-x_k}}, \tag{2.7}$$

$$r_n'(x_k) = \frac{(-1)^k(2n+1)}{2(1+x_k)\sqrt{1-x_k}}, \tag{2.8}$$

$$r_n'(y_k) = -\frac{r_n(y_k)}{2(1+y_k)}, \tag{2.9}$$

$$s_n'(x_k) = \frac{s_n(x_k)}{2(1-x_k)}, \tag{2.10}$$

$$s_n'(y_k) = \frac{(-1)^k\sqrt{2}(2n+1)}{2(1-y_k)\sqrt{1+y_k}} \tag{2.11}$$

and

$$\begin{cases} r_n(-1) = (2n+1), \quad r_n(1) = 1 \\ s_n(-1) = (-1)^n, \quad s_n(1) = (2n+1). \end{cases} \tag{2.12}$$

Let

$$L_k(x) = \frac{s_n(x)}{(x-y_k)s'_n(y_k)}, \quad k = 1, 2, ...., n \tag{2.13}$$

and

$$l_k(x) = \frac{r_n(x)}{(x-x_k)r'_n(x_k)}, \quad k = 1, 2, ...., n. \tag{2.14}$$

For $L_k(x)$ and $l_k(x)$ defined above, we have [15]

$$\sum_{k=1}^{n} |L_k(x)| = \sum_{k=1}^{n} |l_k(x)| = O(n). \tag{2.15}$$

We need the following estimates

$$c_1 \frac{k}{n} \leq \sqrt{1-t_k^2} \leq c_2 \frac{k}{n}, \tag{2.16}$$

where $t_k$'s are either the zeros of $r_n(x)$ or $s_n(x)$.

## 3 Existence and Explicit Representation of the interpolatory polynomial

### 3.1 Pál Type $(0; 0, 1)$-Interpolation

Let $(2n+1)$ points in [-1,1] be given by (1.5). Then to the prescribed numbers $\{\alpha_k\}_{k=0}^{n}$, $\{\beta_k\}_{k=1}^{n+1}$ and $\{\gamma_k\}_{k=1}^{n}$, there exists a unique polynomial $R_n(x)$ of degree $\leq 3n+1$ satisfying the conditions (1.9).

The polynomial $R_n(x)$ satisfying the condition (1.9) can be explicitly represented as

$$R_n(x) = \sum_{k=0}^{n} \alpha_k A_k(x) + \sum_{k=1}^{n+1} \beta_k B_k(x) + \sum_{k=1}^{n} \gamma_k C_k(x) \tag{3.1}$$

where $\{A_k(x)\}_{k=0}^{n}$, $\{B_k(x)\}_{k=1}^{n+1}$ and $\{C_k(x)\}_{k=1}^{n}$ are uniquely determined polynomial each of degree $\leq 3n+1$ satisfying the following conditions: For $k = 0, 1, ....n$

$$\begin{cases} A_k(x_j) = \delta_{jk}, & j = 0, 1, \cdots, n \\ A_k(y_j) = 0, & j = 1, 2, \cdots, n, n+1 \\ A'_k(y_j) = 0, & j = 1, 2, \cdots, n \end{cases} \tag{3.2}$$

For $k = 1, 2, \cdots, n, n+1$

$$\begin{cases} B_k(x_j) = 0, & j = 0, 1, \cdots, n \\ B_k(y_j) = \delta_{jk}, & j = 1, 2, \cdots, n, n+1 \\ B'_k(y_j) = 0, & j = 1, 2, \cdots, n \end{cases} \tag{3.3}$$

and for $k = 1, 2, \cdots, n$

$$\begin{cases} C_k(x_j) = 0, & j = 0, 1, \cdots, n \\ C_k(y_j) = 0, & j = 1, 2, \cdots, n, n+1 \\ C'_k(y_j) = \delta_{jk}, & j = 1, 2, \cdots, n \end{cases} \tag{3.4}$$

The explicit forms of the $A_k(x), k = 0, 1, \cdots, n,$ $B_k(x), k = 1, \cdots, n, n+1$ and $C_k(x), k =$

$1, \cdots, n$ are given in the following theorems.

**Theorem 3.1.** *The fundamental polynomial* $\{C_k(x)\}_{k=1}^n$ *satisfying conditions* (3.4) *are given by*

$$C_k(x) = \frac{(1-x^2)r_n(x)s_n(x)}{(1-y_k^2)r_n(y_k)s_n'(y_k)}L_k(x) \tag{3.5}$$

*where* $L_k(x)$ *are given by* (2.13).

**Theorem 3.2.** *The fundamental polynomials* $\{B_k(x)\}_{k=1}^{n+1}$ *satisfying the conditions* (3.3) *are given by: for* $k = 1, 2, \cdots, n$

$$B_k(x) = \frac{(1-x^2)r_n(x)L_k^2(x)}{(1-y_k^2)r_n(y_k)} + c_{1k}C_k(x) \tag{3.6}$$

*where* $L_k(x)$ *are given by* (2.13),

$$c_{1k} = -\frac{-2y_k r_n(y_k) + (1-y_k^2)r_n'(y_k) + 2(1-y_k^2)r_n(y_k)L_k'(y_k)}{(1-y_k^2)r_n(y_k)} \tag{3.7}$$

*and* $C_k(x)$ *are given by* (3.5). *For* $k = n+1$, *we have*

$$B_{n+1}(x) = \frac{(1-x)r_n(x)s_n^2(x)}{2r_n(-1)s_n^2(-1)} \tag{3.8}$$

**Theorem 3.3.** *The fundamental polynomials* $\{A_k(x)\}_{k=0}^n$ *satisfying the conditions* (3.2) *are given by for* $k = 0$,

$$A_0(x) = \frac{(1+x)r_n(x)s_n^2(x)}{2r_n(1)s_n^2(1)} \tag{3.9}$$

*and for* $k = 1, 2, \cdots, n$

$$A_k(x) = \frac{(1-x^2)s_n^2(x)l_k(x)}{(1-x_k^2)s_n^2(x_k)} \tag{3.10}$$

*where* $l_k(x)$ *are given by* (2.14).

The proofs of the Theorems 3.1, 3.2 and 3.3 are quite similar to that of theorems in [3].

### 3.2 Pál Type $(0;0,1)$-Interpolation with interchanged nodes

In this section we consider the problem defined in the previous section when the nodes are interchanged, that is when Lagrange data is prescribed on the zeros of $s_n(x)$ given by (1.7) and Hermite data is prescribed on the zeros $r_n(x)$ given by (1.6). Precisely we have that, for $(2n+1)$ points given by (1.5) and to the prescribed numbers $\{\alpha_k^*\}_{k=1}^{n+1}$, $\{\beta_k^*\}_{k=0}^n$ and $\{\gamma_k^*\}_{k=1}^n$, there exists a unique polynomial

$R_n^*(x)$ of degree $\leq 3n+1$ satisfying the conditions

$$\begin{cases} R_n^*(y_k) = \alpha_k^*, \quad k = 1\cdots, n, n+1 \\ R_n^*(x_k) = \beta_k^*, \quad k = 0, 1, \cdots, n \\ (R_n^*)'(x_k) = \gamma_k^*, \quad k = 1, 2, \cdots, n \end{cases} \tag{3.11}$$

The polynomial $R_n^*(x)$ satisfying the above conditions can be explicitly represented as

$$R_n^*(x) = \sum_{k=1}^{n+1} \alpha_k^* A_k^*(x) + \sum_{k=0}^{n} \beta_k^* B_k^*(x) + \sum_{k=1}^{n} \gamma_k^* C_k^*(x) \tag{3.12}$$

where $\{A_k^*(x)\}_{k=1}^{n+1}, \{B_k^*(x)\}_{k=0}^{n}$ and $\{C_k^*(x)\}_{k=1}^{n}$ are uniquely determined polynomial each of degree $\leq 3n+1$ satisfying the following conditions: For $k = 1, 2, \cdots, n, n+1$

$$\begin{cases} A_k^*(y_j) = \delta_{jk}, \quad j = 1, 2, \cdots, n, n+1 \\ A_k^*(x_j) = 0, \quad j = 0, 1, 2\cdots, n \\ (A_k^*)'(x_j) = 0, \quad j = 1, 2\cdots, n \end{cases} \tag{3.13}$$

For $k = 0, 1, \cdots, n$

$$\begin{cases} B_k^*(y_j) = 0, \quad j = 1\cdots, n, n+1 \\ B_k^*(x_j) = \delta_{jk}, \quad j = 0, 1, 2\cdots, n \\ (B_k^*)'(x_j) = 0, \quad j = 1, 2\cdots, n \end{cases} \tag{3.14}$$

and for $k = 1, 2, \cdots, n$

$$\begin{cases} C_k^*(y_j) = 0, \quad j = 1\cdots, n, n+1 \\ C_k^*(x_j) = 0, \quad j = 0, 1, 2\cdots, n \\ (C_k^*)'(x_j) = \delta_{jk}, \quad j = 1, 2\cdots, n \end{cases} \tag{3.15}$$

The explicit forms of the $A_k^*(x), k = 1, \cdots, n, n+1$, $B_k^*(x), k = 0, 1, \cdots, n$ and $C_k^*(x), k = 1, \cdots, n$ are given in the following theorems.

**Theorem 3.4.** *The fundamental polynomial $\{A_k^*(x)\}_{k=1}^{n+1}$ satisfying conditions* (3.13) *are given by for $k = 1, 2, \cdots, n$*

$$A_k^*(x) = \frac{(1-x^2) r_n^2(x) L_k(x)}{(1-y_k^2) r_n^2(y_k)} \tag{3.16}$$

*$L_k(x)$ are given by* (2.13) *and for $k = n+1$*

$$A_{n+1}^*(x) = \frac{(1-x) r_n^2(x) s_n(x)}{2 r_n^2(-1) s_n(-1)} \tag{3.17}$$

**Theorem 3.5.** *The fundamental polynomials $\{C_k^*(x)\}_{k=1}^{n}$ satisfying the conditions* (3.15) *are given by*

$$C_k^*(x) = \frac{(1-x^2) s_n(x) r_n(x) l_k(x)}{(1-x_k^2) s_n(x_k) r_n'(x_k)} \tag{3.18}$$

*where* $l_k(x)$ *are given by* (2.14).

**Theorem 3.6.** *The fundamental polynomials* $\{B_k^*(x)\}_{k=0}^n$ *satisfying the conditions* (3.14) *are given by for* $k=0$

$$B_0^*(x) = \frac{(1+x)r_n^2(x)s_n(x)}{2r_n^2(1)s_n(1)} \tag{3.19}$$

*and for* $k = 1, 2, \cdots, n$

$$B_k^*(x) \quad = \quad \frac{(1-x^2)s_n(x)l_k^2(x)}{(1-x_k^2)s_n(x_k)} + c_{2k}C_k^*(x) \tag{3.20}$$

*where*

$$c_{2k} = -\frac{-2x_k s_n(x_k) + (1-x_k^2)s_n'(x_k) + 2(1-x_k^2)s_n(x_k)l_k'(x_k)}{(1-x_k^2)s_n(x_k)} \tag{3.21}$$

*and* $l_k(x)$ *are given by* (2.14).

### 3.3 Convergence Theorems

In this section we state the main theorems which deal with convergence of the interpolatory polynomials $R_n(x)$ and $R_n^*(x)$ defined by (3.1) and (3.12) respectively.

**Theorem 3.7.** *If* $f : \mathcal{R} \to \mathcal{R}$ *is a continuously differentiable function then*

$$R_n(f,x) = \sum_{i=0}^{n} f(x_i)A_i(x) + \sum_{i=1}^{n+1} f(y_i)B_i(x) + \sum_{i=1}^{n} f'(y_i)C_i(x) \tag{3.22}$$

*satisfies the relation*

$$\begin{aligned} |f(x) - R_n(f,x)| &\leq c\delta_n(x)\omega\left(f', \delta_n(x)\right) + \sum_{k=1}^{n}\left(\frac{2c_2}{k}\right)\omega\left(f', \delta_n(x_k)\right) \\ &+ \sum_{k=1}^{n}\sqrt{2}\left[\frac{c_4}{2n+1} + \frac{c_5}{k}\right]\omega\left(f', \delta_n(y_k)\right) \end{aligned} \tag{3.23}$$

*where* $c_r$ *are constants independent of n and x,* $w(f,.)$ *is the modulus of continuity and*

$$\delta_n(x) = \frac{\sqrt{1-x^2}}{n}. \tag{3.24}$$

**Theorem 3.8.** *If* $f : \mathcal{R} \to \mathcal{R}$ *is a continuously differentiable function then*

$$R_n^*(f,x) = \sum_{i=1}^{n+1} f(x_i)A_i^*(x) + \sum_{i=0}^{n} f(y_i)B_i^*(x) + \sum_{i=1}^{n} f'(y_i)C_i^*(x) \tag{3.25}$$

*satisfies the relation*

$$
\begin{aligned}
&|f(x) - R_n^*(f,x)| \le c_6\delta_n(x)\omega\left(f', \delta_n(x)\right) + \sum_{k=1}^{n}\left(\frac{c_7}{k}\right)\omega\left(f', \delta_n(y_k)\right) \\
&+ \sum_{k=1}^{n}\left[\frac{c_8}{k} + \frac{\sqrt{2}c_9}{2n+1}\right]\omega\left(f', \delta_n(x_k)\right)
\end{aligned}
\tag{3.26}
$$

*where $w(f,.)$ is the modulus of continuity, $c_r$ are constants independent of n and x and $\delta$ is given by* (3.24).

We prove only Theorem 3.7 as the proof of Theorem 3.8 is quite similar to that of Theorem 3.7. To prove Theorem 3.7 we need the estimates of the fundamental polynomials, which are discussed in the next section.

## 4 Estimation of the Fundamental Polynomials

**Lemma 4.1.** *Let $C_k(x)$ be given by* (3.5) *For $-1 \le x \le 1$, we have*

$$
|C_k(x)| = \frac{c}{2n+1} \tag{4.1}
$$

*where c is a contant independent of $n$ and $x$.*

**Proof.** Since for $x = \cos\theta$, we have

$$
\left|(1-x^2)r_n(x)s_n(x)\right| = \frac{1}{2}\left|\cos 2n\theta - \cos 2(n+1)\theta\right| \le 1.
$$

Hence by using (2.6), (2.11) and (2.15) in (3.5), (4.1) follows. ∎

**Lemma 4.2.** *Let $B_k(x), k = 1, 2, ....n$ be given by* (3.6) *then for $-1 \le x \le 1$, we have*

$$
|B_k(x)| \le \left|\frac{\sqrt{2}}{(1-y_k^2)}\right|\left[2 + \frac{1}{2(2n+1)}\right] \tag{4.2}
$$

**Proof.** Let $B_k(x)$ be given by (3.6), then

$$
|B_k(x)| \le \left|\frac{(1-x^2)r_n(x)L_k^2(x)}{(1-y_k^2)r_n(y_k)}\right| + |c_{1k}C_k(x)| \tag{4.3}
$$

where

$$
|c_{1k}| = \left|\frac{-2y_k}{(1-y_k^2)} + \frac{r_n'(y_k)}{r_n(y_k)} + 2L_k'(y_k)\right|
$$

For finding the value of $L_k'(x_k)$, differentiating (2.13) with respect to $x$, we have

$$
2L_k'(y_k) = \frac{s_n''(y_k)}{s_n'(y_k)}
$$

Also by differentiating (2.2) with respect to $x$ and using (2.6), (2.9) and (2.11), we have

$$s_n''(y_k) = \left(\frac{2y_k+1}{1-y_k^2}\right) s_n'(y_k)$$

which implies

$$2L_k'(y_k) \quad = \quad \frac{2y_k+1}{1-y_k^2} \tag{4.4}$$

Hence

$$|c_{1k}| = \left|\frac{-2y_k}{(1-y_k^2)} - \frac{1}{2(1+y_k)} + \frac{2y_k+1}{2(1-y_k^2)}\right| \leq \frac{1}{2(1-y_k^2)} \tag{4.5}$$

Hence by using $\left|\sqrt{(1+x)}r_n(x)\right| \leq 2$, (2.15), (4.5) and Lemma 4.1 in the equation (4.3), (4.2) follows. ∎

**Lemma 4.3.** *Let* $A_k(x), k = 1, \cdots, n$ *be given by* (3.10)*, then for* $-1 \leq x \leq 1$*, we have*

$$|A_k(x)| \leq \frac{2}{(1-x_k^2)} \tag{4.6}$$

**Proof.** By using $\left|(1+x)s_n^2(x)\right| \leq 1$, (2.7) and (2.15) in (3.10), (4.6) follows. ∎

## 5 Proof of theorem 3.7

In order to prove Theorem 3.7, we need the following result due to I.E. Gopengaus [2].

Let $f \in C^r[-1,1]$, then for $n \geq 4r+5$, there exists a polynomial $Q_n(x)$ of degree at most $n$ such that for all $x \in [-1,1]$ and for $k = 0, 1, \cdots, r$

$$\left|f^{(k)}(x) - Q_n^{(k)}(x)\right| \leq c_k\left(\delta_n(x)\right)^{r-k} \omega\left(f^{(r)}, \delta_n(x)\right) \tag{5.1}$$

where $\delta_n(x)$ are given by (3.24) and $c_k's$ are constants independent of f, n and x.

**Proof.** From the uniqueness of $R_n(x)$ in (3.1) it follows that every polynomial $Q_n(x)$ of degree $\leq$ $3n+1$ with the property (5.1) satisfies the relation

$$Q_n(x) = \sum_{k=0}^{n} Q_n(x_k)A_k(x) + \sum_{k=1}^{n+1} Q_n(y_k)B_k(x) + \sum_{k=1}^{n} Q_n'(y_k)C_k(x) \tag{5.2}$$

Therefore

$$\begin{aligned} |f(x)-R_n(f,x)| &\leq |f(x)-Q_n(x)|+|R_n(f,x)-Q_n(x)| \\ &\leq |f(x)-Q_n(x)|+\sum_{i=0}^{n}|f(x_i)-Q_n(x_i)|\,|A_i(x)| \\ &+\sum_{i=1}^{n+1}|f(y_i)-Q_n(y_i)|\,|B_i(x)|+\sum_{i=1}^{n}\left|f'(y_i)-Q_n'(y_i)\right|\,|C_i(x)| \end{aligned} \tag{5.3}$$

By (5.1) for $r=1$ and $k=0$ we have $f(x_0)-Q_n(x_0)=f(y_{n+1})-Q_n(y_{n+1})=0$ which implies that

$$\begin{aligned} |f(x)-R_n(f,x)| &\leq |f(x)-Q_n(x)|+\sum_{i=1}^{n}|f(x_i)-Q_n(x_i)|\,|A_i(x)| \\ &+\sum_{i=1}^{n}|f(y_i)-Q_n(y_i)|\,|B_i(x)|+\sum_{i=1}^{n}\left|f'(y_i)-Q_n'(y_i)\right|\,|C_i(x)| \\ &\equiv J_1+J_2+J_3+J_4 \end{aligned} \tag{5.4}$$

By (5.1) for $r=1$ and $k=0$ we have

$$J_1 \leq c\delta_n(x)\omega\left(f',\delta_n(x)\right) \tag{5.5}$$

Again by (5.1) for $r=1$ and $k=0$, (2.16) and (4.6) we have

$$\begin{aligned} J_2 &\leq \sum_{k=1}^{n} c_1\delta_n(x_k)\omega\left(f',\delta_n(x_k)\right)|A_k(x)| \\ &\leq \sum_{k=1}^{n}\left(\frac{2c_1}{n\sqrt{1-x_k^2}}\right)\omega\left(f',\delta_n(x_k)\right) \\ &\leq \sum_{k=1}^{n}\left(\frac{2c_2}{k}\right)\omega\left(f',\delta_n(x_k)\right) \end{aligned} \tag{5.6}$$

Again by (5.1) for $r=1$ and $k=0$, (2.16) and (4.2) we have

$$\begin{aligned} J_3 &\leq \sum_{k=1}^{n} c_3\delta_n(y_k)\omega\left(f',\delta_n(y_k)\right)|B_k(x)| \\ &\leq \sum_{k=1}^{n}\frac{\sqrt{2}c_3}{n\sqrt{1-y_k^2}}\left[2+\frac{1}{2(2n+1)}\right]\omega\left(f',\delta_n(y_k)\right) \\ &\leq \sum_{k=1}^{n}\frac{\sqrt{2}c_4}{k}\omega\left(f',\delta_n(y_k)\right) \end{aligned} \tag{5.7}$$

Again by (5.1) for $r = 1$ and $k = 1$, (2.16) and (4.1) we have

$$\begin{aligned} J_4 &\leq \sum_{k=1}^{n} c_5 \omega\left(f', \delta_n(y_k)\right) |C_k(x)| \\ &\leq \sum_{k=1}^{n} \frac{c_5\sqrt{2}}{2n+1} \omega\left(f', \delta_n(y_k)\right) \end{aligned} \tag{5.8}$$

By substituting the values of (5.5), (5.6), (5.7) and (5.8) in (5.4), the theorem follows. ■

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