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Total restrained domination subdivision number for Cartesian product graph

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Abstract

In this paper we determine the total restrained dominating set and the total restrained domination subdivision number for Cartesian product graph.

Keywords: Total restrained dominating set, Cartesian product graph, total restrained domination subdivision number.

AMS Subject Classification (2010): 05C69.

1 Introduction

Let $G = (V, E)$ be a simple graph on the vertex set V. In a graph G, A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number of a graph G [$\gamma(G)$] is the minimum size of a dominating set of vertices in G. The domination subdivision number of a graph G is the minimum number of edges that must be subdivided in order to increase the domination number of a graph and is denoted by $sd_\gamma(G)$. A set S of vertices in a graph $G(V,E)$ is called total restrained dominating set if every vertex $v \in V$ is adjacent to an element of S and every vertex of $V - S$ is adjacent to the vertex in $V - S$.

The Cartesian product of G and H [$G \times H$] is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (i) $u = u'$ and vv' belongs to $E(H)$, or (ii) $v = v'$ and uu' belongs to $E(G)$. Domination number of Cartesian products is further studied in [2], [3], [4], [8], [9] and [10]. The subdivision graph of G is a graph which is obtained by subdividing each edge of G exactly once and is denoted by $S(G)$. The domination subdivision number of a graph G is the minimum number of edges that must be subdivided in order to increase the domination number of a graph [14] and is denoted by $\mathrm{sd}\gamma(G)$.

The total restrained domination in graphs was introduced in [15]. A set S of vertices in a graph $G(V,E)$ is called a total restrained dominating set if every vertex $v \in V$ is adjacent to an element of S and every vertex of $V - S$ is adjacent to a vertex in $V - S$. The total restrained domination number of a graph $G[\gamma_{rt}(G)]$ is the minimum cardinality of a total restrained dominating set in G. For a detailed literature on domination one can refer [6, 7]. Haynes et al. [5] showed that $1 \le \mathrm{sd}\gamma_t(G) \le 4$. for any grid graph $P_{n,m}$. We use the following known results for the present work.

Theorem 1.1. [12] For $n \ge 4$, we have $\gamma_t(P_{n,n}) = \gamma_t(P_{n-4,n-4}) + \gamma_t(C_{4n-4})$ and

$$\gamma_t(P_{n,n}) = \begin{cases} \frac{n^2+2n}{4} & if\ n \equiv 0,2\ (mod\ 4) \\ \frac{n^2+2n+1}{4} & if\ n \equiv 1\ (mod\ 4) \\ \frac{n^2+2n-3}{4} & if\ n \equiv 3\ (mod\ 4). \end{cases}$$

Theorem 1.2. [12] For a graph G, $\gamma_t(G) = \gamma_{tr}(G)$ if and only if G has a $\gamma_t(G)$ set such that $G[V(G) - S]$ has no isolated vertex.

Theorem 1.3. [12]

1) For any $n \ge 2$, $\gamma_{tr}(P_{n,n}) = \gamma_t(P_{n,n})$.
2) For any $n \ge 2$, $\gamma_{tr}(P_{n,n+2}) = \gamma_t(P_{n,n+2})$.
3) For any $n \ge 2$, $\gamma_{tr}(P_{2n-1,n}) = \gamma_t(P_{2n-1,n})$.
4) For any $n \ge 2$, and $m \equiv n(mod\ n+1)$, $\gamma_{tr}(P_{n,m}) = \gamma_t(P_{n,m})$.
5) For any $n \ge 8$, $\gamma_{tr}(P_{8,n}) = \gamma_t(P_{8,n})$.
6) For any $n \ge 6$, $\gamma_{tr}(P_{6,n}) = \gamma_t(P_{6,n})$.
7) For any $n \ge 5$, $\gamma_{tr}(P_{5,n}) = \gamma_t(P_{5,n})$.
8) For any $n \ge 4$, $\gamma_{tr}(P_{4,n}) = \gamma_t(P_{4,n})$.

Theorem 1.4. [13] For $n \ge 2$,

$$sd\gamma_t(P_{2,n}) = \begin{cases} 1, & n \equiv 0,2\ (mod\ 3) \\ 2, & n \equiv 1\ (mod\ 3). \end{cases}$$

Theorem 1.5. [13] For $n \ge 2$, $sd\gamma_t(P_{3,n})$=1.

Theorem 1.6. [13] For $n \ge 4$,

$$sd\gamma_t(P_{4,n}) = \begin{cases} 1, & n \equiv 0,2\ (mod\ 5) \\ 2, & n \equiv 1\ (mod\ 5). \end{cases}$$

Theorem 1.7. [13] For $n, m \ge 3, sd\gamma_t(P_{m,n}) \le 3$.

Theorem 1.8. [13] For $n \ge 2, \gamma_{tr}(P_2 \times P_n) = 2\left\lceil\frac{n+2}{3}\right\rceil$ and $\gamma_t(P_2 \times P_n) = \gamma_t(P_2 \times P_{n-3}) + 2$.

Theorem 1.9. [13] For any graph $\gamma_t(P_4 \times P_n) = \gamma_t(P_4 \times P_{n-5}) + 6$.

2 Main Results

In this section we determine the value of the total restrained domination subdivision number of $P_n \times P_n$ for $n \geq 4$.

Theorem 2.1. Let G be the Cartesian product graph of $P_n \times P_n$ of order $n \geq 4$, Then $\mathrm{sd}\gamma_{\mathrm{tr}}(G) = 1$.

Proof: Let G be the Cartesian product graph of $P_n \times P_n$ of order $n \geq 4$.

Consider $V(G)=\{(u_{1,}v_1), (u_{1,}v_2), (u_{1,}v_3),\ldots, (u_{1,}v_n)$

$(u_{2,}v_1), (u_{2,}v_2), (u_{2,}v_3),\ldots, (u_{2,}v_n)$

$(u_{3,}v_1), (u_{3,}v_2), (u_{3,}v_3),\ldots, (u_{3,}v_n)$

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$(u_{n,}v_1), (u_{n,}v_2), (u_{n,}v_3),\ldots, (u_{n,}v_n)\}$.

Let D_n be a total restrained dominating set of $(P_n \times P_n)$ and $\mathfrak{D}_n$ be a total restrained dominating set of the first row $\mathcal{R}_1$, first column $\mathcal{C}_1$, last row $\mathcal{R}_n$ and last column $\mathcal{C}_n$ of $(P_n \times P_n)$. Let $(P_n \times P_n)'$ be the graph that is obtained from $(P_n \times P_n)$ by subdividing the edge $(u_i, v_j)(u_{i+1}, v_j)$, $i, j = 1,2,3,4 \ldots \ldots \ldots, n$ with the vertex x. Let D' be the total restrained dominating set of $(P_n \times P_n)'$. We show that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_n \times P_n)) = 1$.We prove the theorem by considering the following cases.

Case (i): For $n \equiv 0\ (mod\ 4)$.

Put $n = 4$. Let D be a total restrained dominating set of $(P_4 \times P_4)$. We show that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_4 \times P_4)) = 1$. We have $\gamma_{tr}(P_4 \times P_4) = 6$ [see Figure 3].

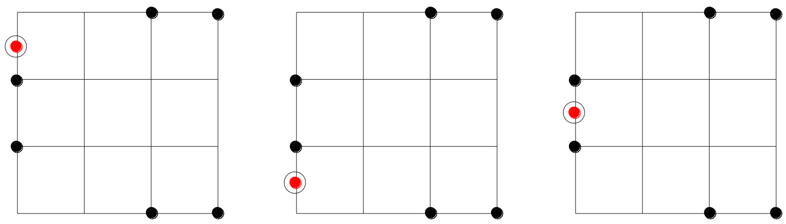

Figure 1: $(P_4 \times P_4)'$.

If $(u_i, v_j) \notin D$ and $(u_{i+1}, v_j) \in D$ or $(u_i, v_j) \in D$ and $(u_{i+1}, v_j) \notin D$ or $(u_i, v_j) \in D$ and $(u_{i+1}, v_j) \in D$, then the last two columns of $(P_4 \times P_4)'$ is a block B. We need four vertices for totally restrained domination of block B. To totally restrained dominate the first column of $(P_4 \times P_4)'$ we need three vertices and then $x \in D'$[see Figure 1]. The second column of $(P_4 \times P_4)'$ can be totally restrained dominated from the adjacent columns of block B and the first column of $(P_4 \times P_4)'$. We get $\gamma_{tr}(P_4 \times P_4)' = 7 > \gamma_{tr}(P_4 \times P_4)$. Therefore, the domination number is increased by subdividing one edge. Hence we proved that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_4 \times P_4)) = 1$.

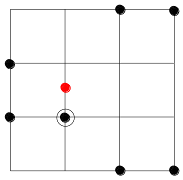

Figure 2: $(P_4 \times P_4)'$

If $(u_i, v_j) \notin D$ and $(u_{i+1}, v_j) \notin D$.The last two columns of $(P_4 \times P_4)'$ is a block B. We need four vertices for totally restrained domination of block B. To totally restrained dominate the first column of $(P_4 \times P_4)'$ we need two vertices and for the second column we need one vertex [see Figure 2]. We get $\gamma_{tr}(P_4 \times P_4)' = 7 = \gamma_{tr}(P_4 \times P_4) + 1 > \gamma_{tr}(P_4 \times P_4)$. Therefore, the domination number is increased by subdividing one edge and we get $\mathrm{sd}\gamma_{\mathrm{tr}}((P_4 \times P_4)) = 1$.

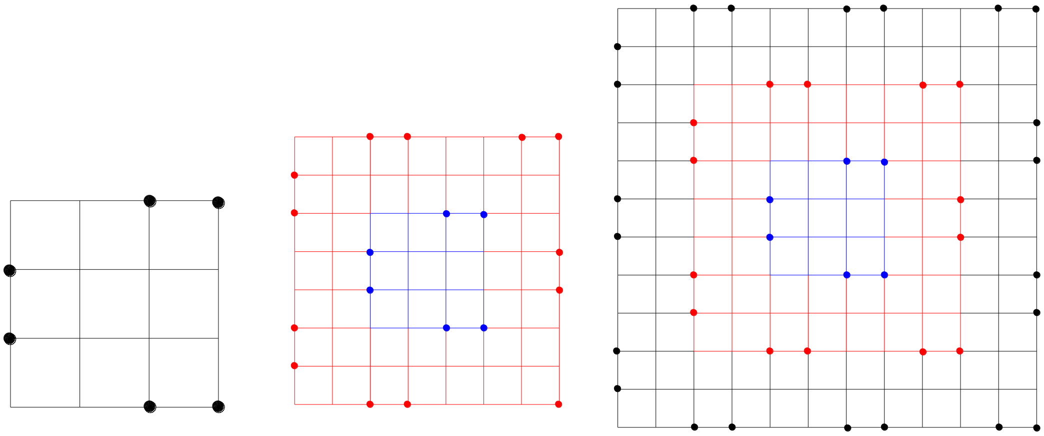

Figure 3: *Total restrained dominating set of* $P_4 \times P_4$, $P_8 \times P_8$ *and* $P_{12} \times P_{12}$.

Now we construct $P_n \times P_n$ from $P_{n-4} \times P_{n-4}$, $n \geq 8$ by adding two columns (two rows) at the first and two columns (two rows) at the last of $P_{n-4} \times P_{n-4}$, respectively [see Figure 3]. The union of the first row, the first column, the last row and the last column of $P_n \times P_n$ is a cycle of length $4n - 4$. Consider $D_n = D_{n-4} \cup \mathfrak{D}_n$ where D_{n-4} is the totally restrained dominating set of $P_{n-4} \times P_{n-4}$. By Theorem 1.1 and Theorem 1.3, we have $\gamma_{tr}(P_n \times P_n) = \frac{n^2+2n}{4}$ if $n \equiv 0(mod\ 4)$.

In this case $n \geq 4$ and for $\mathfrak{D}_n = \mathcal{R}_1 \cup \mathcal{C}_1 \cup \mathcal{R}_n \cup \mathcal{C}_n$ where

$\mathcal{R}_1 = \{ (u_1, v_3), (u_1, v_4), (u_1, v_7), (u_1, v_8), \ldots, (u_1, v_{n-5}), (u_1, v_{n-4}), (u_1, v_{n-1}), (u_1, v_n)\}$

$\mathcal{R}_n = \{(u_n, v_3), (u_n, v_4), (u_n, v_7), (u_n, v_8), \ldots, (u_n, v_{n-5}), (u_n, v_{n-4}), (u_n, v_{n-1}), (u_n, v_n)\}$

$\mathcal{C}_1 = \{(u_2, v_1), (u_3, v_1), (u_6, v_1), (u_7, v_1), \ldots, (u_{n-6}, v_1), (u_{n-5}, v_1), (u_{n-2}, v_1), (u_{n-1}, v_1)\}$

$\mathcal{C}_n = \{(u_1, v_n), (u_4, v_n), (u_5, v_n), \ldots, (u_{n-8}, v_n), (u_{n-7}, v_n), (u_{n-4}, v_n), (u_{n-3}, v_n), (u_n, v_n)\}$.

Then we know that $\gamma_{tr}(P_n \times P_n) = \gamma_{tr}(P_{n-4} \times P_{n-4}) + \gamma_{tr}(\mathfrak{D}_n)$. Let $(P_n \times P_n)'$ be the graph obtained from $(P_n \times P_n)$ by subdividing an edge with one vertex. We can prove $\gamma_{tr}(P_n \times P_n)' = \gamma_{tr}(P_{n-4} \times P_{n-4}) + 1 + \gamma_{tr}(\mathfrak{D}_n) = \gamma_{tr}(P_n \times P_n) + 1$. [Since $\gamma_{tr}(P_4 \times P_4)' = \gamma_{tr}(P_4 \times P_4) + 1$]. Hence we get $sd\gamma_{tr}((P_n \times P_n)) = 1$ for $n \equiv 0(mod\ 4)$.

Case (ii): If $n \equiv 1(mod\ 4)$.

Put $n = 5$. Let D be the total restrained dominating set of $(P_5 \times P_5)$. We show that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_5 \times P_5)) = 1$.

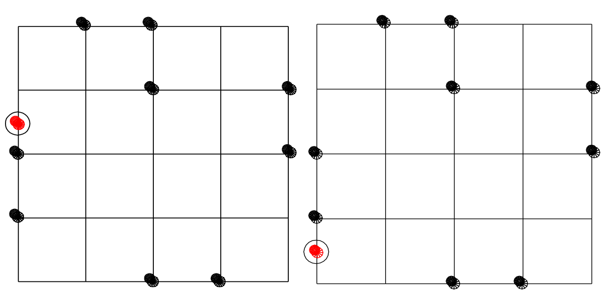

Figure 4: $(P_5 \times P_5)'$

If $(u_i, v_j) \notin D$ and $(u_{i+1}, v_j) \in D$ or $(u_i, v_j) \in D$ and $(u_{i+1}, v_j) \notin D$ or $(u_i, v_j) \in D$ and $(u_{i+1}, v_j) \in D$, then the last three columns of $(P_5 \times P_5)'$ is a block B. We need six vertices for totally restrained domination of block B. To totally restrained dominate the second column of $(P_5 \times P_5)'$ we need one vertex and for the first column of $(P_5 \times P_5)'$ we need three vertices. Then we get $x \in D'$ [see Figure 4]. We get $\gamma_{tr}(P_5 \times P_5)' = 10 > \gamma_{tr}(P_5 \times P_5)$ [since $\gamma_{tr}(P_5 \times P_5) = 9$]. Therefore, the domination number is increased by subdividing one edge. Hence we proved that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_5 \times P_5)) = 1$.

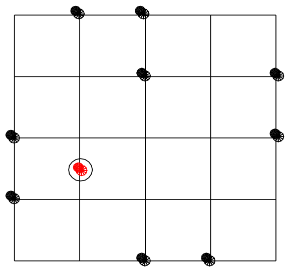

Figure 5: $(P_5 \times P_5)'$.

If $(u_i, v_j) \notin D$ and $(u_{i+1}, v_j) \notin D$.The last three columns of $(P_5 \times P_5)'$ is a block B. We need six vertices for totally restrained domination of B. To totally restrained dominate the first column of $(P_5 \times P_5)'$ we need two vertices and for the second column we need two vertices [see Figure 5]. We get $\gamma_{tr}(P_5 \times P_5)' = 7 = \gamma_{tr}(P_5 \times P_5) + 1 > \gamma_{tr}(P_5 \times P_5)$. Therefore the domination number is increased by subdividing one edge. Hence we proved that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_5 \times P_5)) = 1$.

Now we construct $P_n \times P_n$ from $P_{n-4} \times P_{n-4}$, $n \geq 9$ by adding two columns (two rows) at the first and two columns (two rows) at the last of $P_{n-4} \times P_{n-4}$, respectively [see Figure 6]. The union of the first row, the first column, the last row and the last column of $P_n \times P_n$ is a cycle of length $4n - 4$. Consider $D_n = D_{n-4} \cup \mathfrak{D}_n$ where D_{n-4} is the totally restrained dominating set of $P_{n-4} \times P_{n-4}$. By Theorem 1.1 and 1.3, we have $\gamma_{tr}(P_n \times P_n) = \frac{n^2+2n+1}{4}$ if $n \equiv 1(mod\ 4)$.

In this case $n \geq 4$ and for $\mathfrak{D}_n = \mathcal{R}_1 \cup \mathcal{C}_1 \cup \mathcal{R}_n \cup \mathcal{C}_n$ where

$\mathcal{R}_1 = \{ (u_1, v_3), (u_1, v_4), (u_1, v_7), (u_1, v_8), \dots, (u_1, v_{n-6}), (u_1, v_{n-5}), (u_1, v_{n-2}), (u_1, v_{n-1})\}$

$\mathcal{R}_n = \{ (u_n, v_2), (u_n, v_3), (u_n, v_6), (u_n, v_7), \dots, (u_n, v_{n-7}), (u_n, v_{n-6}), (u_n, v_{n-3}), (u_n, v_{n-2})\}$

$\mathcal{C}_1 = \{ (u_2, v_1), (u_3, v_1), (u_6, v_1), (u_7, v_1), \dots, (u_{n-7}, v_1), (u_{n-6}, v_1), (u_{n-3}, v_1), (u_{n-2}, v_1)\}$

$\mathcal{C}_n = \{(u_3, v_n), (u_4, v_n), (u_7, v_n), (u_8, v_n), \dots, (u_{n-6}, v_n), (u_{n-5}, v_n), (u_{n-2}, v_n), (u_{n-1}, v_n)\}$.

Then we know that $\gamma_{tr}(P_n \times P_n) = \gamma_{tr}(P_{n-4} \times P_{n-4}) + \gamma_{tr}(\mathfrak{D}_n)$. Let $(P_n \times P_n)'$ be the graph that is obtained from $(P_n \times P_n)$ by subdividing an edge with one vertex. Clearly we can prove $\gamma_{tr}(P_n \times P_n)' = \gamma_{tr}(P_{n-4} \times P_{n-4}) + 1 + \gamma_{tr}(\mathfrak{D}_n) = \gamma_{tr}(P_n \times P_n) + 1$ [$\because$ $\gamma_{tr}(P_5 \times P_5)' = \gamma_{tr}(P_5 \times P_5) + 1$]. Thus we proved that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_n \times P_n)) = 1$ for $n \equiv 1(mod\ 4)$.

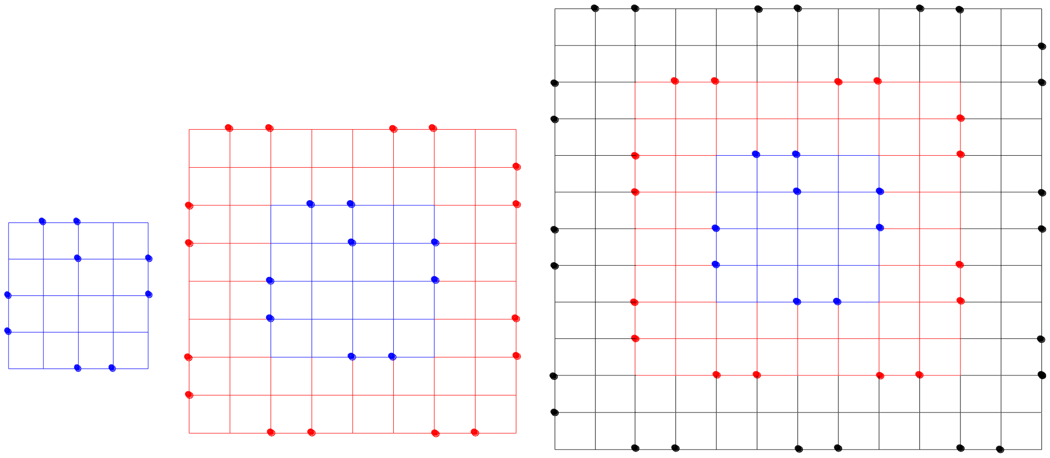

Figure 6: *Total restrained dominating set of $P_5 \times P_5$, $P_9 \times P_9$ and $P_{13} \times P_{13}$.*

Case (iii): If $n \equiv 2(mod\ 4)$

Put $n = 6$. Let D be the total restrained dominating set of $(P_6 \times P_6)$. We show that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_6 \times P_6)) = 1$. We have $\gamma_{tr}(P_6 \times P_6) = 12$.

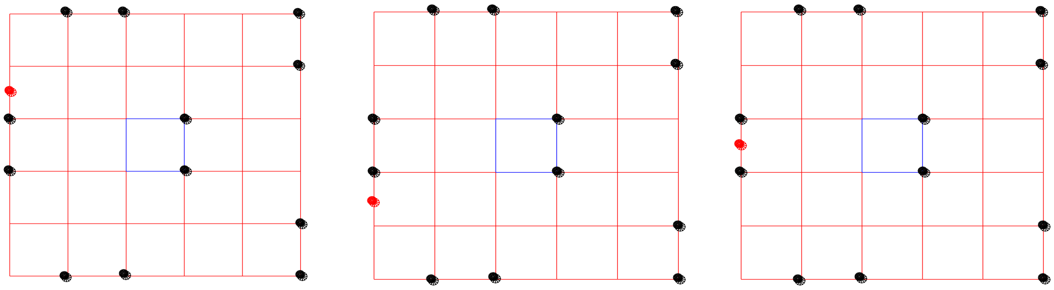

Figure 7: $(P_6 \times P_6)'$

If $(u_i, v_j) \notin D$ and $(u_{i+1}, v_j) \in D$ or $(u_i, v_j) \in D$ and $(u_{i+1}, v_j) \notin D$ or $(u_i, v_j) \in D$ and $(u_{i+1}, v_j) \in D$ then the last four columns of $(P_6 \times P_6)'$ is a block B. We need eight vertices for totally restrained domination of the block B. To totally restrained dominate the second column of $(P_6 \times P_6)$ we need two vertices and for the first column of $P_6 \times P_6$ we need three vertices. Then $x \in D'$[see Figure 7]. We get $\gamma_{tr}(P_6 \times P_6)' = 13 > \gamma_{tr}(P_6 \times P_6)$ [since $\gamma_{tr}(P_6 \times P_6) = 12$]. Therefore the domination number is increased by subdividing one edge. Hence we proved that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_6 \times P_6)) = 1$.

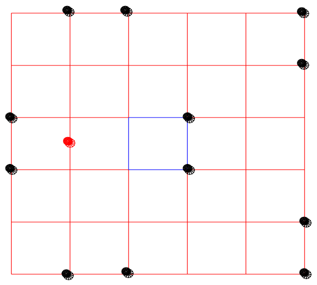

Figure 8: $(\mathrm{P}_6 \times \mathrm{P}_6)'$.

If $(u_i, v_j) \notin D$ and $(u_{i+1}, v_j) \notin D$. The last four columns of $(P_6 \times P_6)'$ is a block B. We need eight vertices for totally restrained domination of block B. To totally restrained dominate the first column of $P_6 \times P_6$ we need two vertices and for the second column of $(P_6 \times P_6)$ we need three vertices [see Figure 8]. We get $\gamma_{tr}(P_6 \times P_6)' = 13 = \gamma_{tr}(P_6 \times P_6) + 1 > \gamma_{tr}(P_6 \times P_6)$. Therefore, the domination number is increased by subdividing one edge. Hence we proved that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_6 \times P_6)) = 1$.

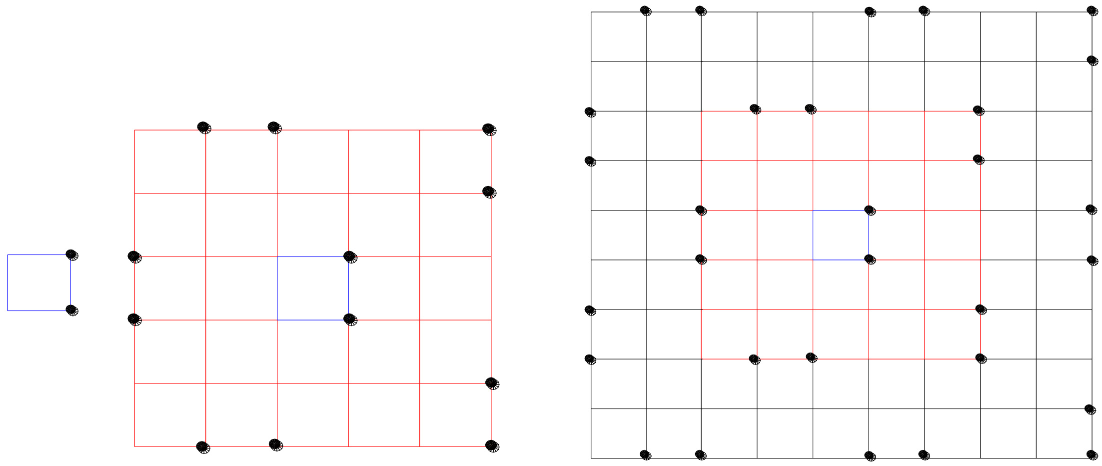

Figure 9: *Total restrained dominating set of* $\mathrm{P}_2 \times \mathrm{P}_2, \mathrm{P}_6 \times \mathrm{P}_6, \mathrm{P}_{10} \times \mathrm{P}_{10}$.

Now we construct $P_n \times P_n$ from $P_{n-4} \times P_{n-4}$, $n \geq 10$ by adding two columns (two rows) at the first and two columns (two rows) at the last of $P_{n-4} \times P_{n-4}$ respectively [see Figure 9]. The union of the first row, the first column, the last row and the last column of $P_n \times P_n$ is a cycle of length $4n - 4$. Consider $D_n = D_{n-4} \cup \mathfrak{D}_n$ where D_{n-4} is the totally restrained dominating set of $P_{n-4} \times P_{n-4}$.

By Theorem 1.1 and 1.3, we have $\gamma_{tr}(P_n \times P_n) = \frac{n^2+2n}{4}$ if $n \equiv 2(mod\ 4)$.

In this case $n \geq 4$ and for $\mathfrak{D}_n = \mathcal{R}_1 \cup \mathcal{C}_1 \cup \mathcal{R}_n \cup \mathcal{C}_n$ where

$\mathcal{R}_1 = \{ (u_1, v_2), (u_1, v_3), (u_1, v_6), (u_1, v_7), \ldots, (u_1, v_{n-8}), (u_1, v_{n-7}), (u_1, v_{n-4}), (u_1, v_{n-3}), (u_1, v_n)\}$
$\mathcal{R}_n = \{(u_n, v_2), (u_n, v_3), (u_n, v_6), (u_n, v_7), \ldots, (u_n, v_{n-8}), (u_n, v_{n-7}), (u_n, v_{n-4}), (u_n, v_{n-3}), (u_n, v_n)\}$
$\mathcal{C}_1 = \{ (u_3, v_1), (u_4, v_1), (u_7, v_1), (u_8, v_1), \ldots, (u_{n-7}, v_1), (u_{n-6}, v_1), (u_{n-3}, v_1), (u_{n-2}, v_1)\}$
$\mathcal{C}_n = \{ (u_1, v_n), (u_2, v_n), (u_5, v_n), (u_6, v_n), \ldots, (u_{n-5}, v_n), (u_{n-4}, v_n), (u_{n-1}, v_n), (u_n, v_n)\}$.

Then we know that $\gamma_{tr}(P_n \times P_n) = \gamma_{tr}(P_{n-4} \times P_{n-4}) + \gamma_{tr}(\mathfrak{D}_n)$. Let $(P_n \times P_n)'$ be the graph that is obtained from $(P_n \times P_n)$ by subdividing an edge with one vertex. We can prove $\gamma_{tr}(P_n \times P_n)' = \gamma_{tr}(P_{n-4} \times P_{n-4}) + 1 + \gamma_{tr}(\mathfrak{D}_n) = \gamma_{tr}(P_n \times P_n) + 1$. [since $\gamma_{tr}(P_6 \times P_6)' = \gamma_{tr}(P_6 \times P_6) + 1$]. Then we get $sd\gamma_{tr}((P_n \times P_n)) = 1$ for $n \equiv 2(mod\ 4)$.

Case (iv): If $n \equiv 3(mod\ 4)$.
Put $n = 7$. Let D be the total restrained dominating set of $(P_7 \times P_7)$. We show that $sd\gamma_{tr}((P_7 \times P_7)) = 1$.

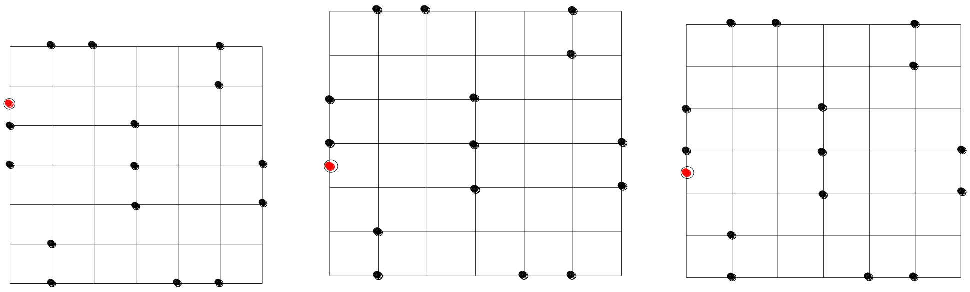

Figure 10: $(P_7 \times P_7)'$.

If $(u_i, v_j) \notin D$ and $(u_{i+1}, v_j) \in D$ or $(u_i, v_j) \in D$ and $(u_{i+1}, v_j) \notin D$ or $(u_i, v_j) \in D$ and $(u_{i+1}, v_j) \in D$ then the last five columns of $(P_7 \times P_7)'$ is a block B. we need ten vertices for totally restrained domination of block B. To totally restrained dominate the first and second column of $(P_7 \times P_7)$ we need three vertices. Then we get $x \in D'$[see Figure 10]. We get $\gamma_{tr}(P_7 \times P_7)' = 16 > \gamma_{tr}(P_7 \times P_7)$ [since $\gamma_{tr}(P_7 \times P_7) = 15$]. Therefore the domination number is increased by subdividing one edge and we get $sd\gamma_{tr}((P_7 \times P_7)) = 1$.

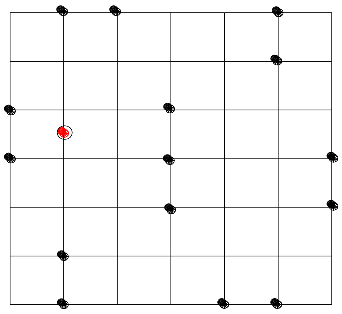

Figure 11: $(P_7 \times P_7)'$.

If $(u_i, v_j) \notin D$ and $(u_{i+1}, v_j) \notin D$ then the last five columns of $(P_7 \times P_7)'$ is a block B. We need ten vertices for totally restrained domination of B. To totally restrained dominate the first column of $P_7 \times P_7$ we need two vertices and for the second column of $(P_7 \times P_7)$ we need four vertices [see

Figure 11]. We get $\gamma_{tr}(P_7 \times P_7)' = 16 = \gamma_{tr}(P_7 \times P_7) > \gamma_{tr}(P_7 \times P_7)$. Therefore the domination number is increased by subdividing one edge. Hence we proved that $\mathrm{sd}\gamma_{\mathrm{tr}}((P_7 \times P_7)) = 1$.

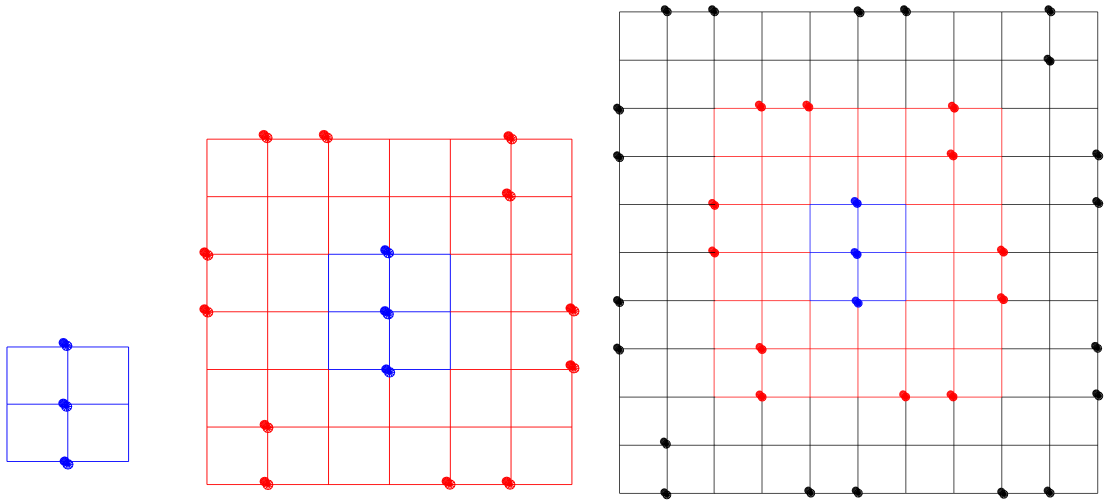

Figure 12: *Total restrained dominating set of* $P_3 \times P_3$, $P_7 \times P_7$ *and* $P_{11} \times P_{11}$.

Now we construct $P_n \times P_n$ from $P_{n-4} \times P_{n-4}$, $n \geq 11$ by adding two columns (two rows) at the first and two columns (two rows) at the last of $P_{n-4} \times P_{n-4}$ respectively [see Figure 12]. The union of the first row, the first column, the last row and the last column of $P_n \times P_n$ is a cycle of length $4n - 4$. Consider $D_n = D_{n-4} \cup \mathfrak{D}_n$ where D_{n-4} is the totally restrained dominating set of $P_{n-4} \times P_{n-4}$.

By Theorem 1.1 and 1.3, we have $\gamma_{tr}(P_n \times P_n) = \frac{n^2+2n-3}{4}$ if $n \equiv 3(mod\ 4)$.

In this case $n \geq 4$ and for $\mathfrak{D}_n = \mathcal{R}_1 \cup \mathcal{C}_1 \cup \mathcal{R}_n \cup \mathcal{C}_n$ where

$\mathcal{R}_1 = \{ (u_1, v_2), (u_1, v_5), (u_1, v_6), \ldots, (u_1, v_{n-5}), (u_1, v_{n-2}), (u_1, v_{n-1})\}$

$\mathcal{R}_n = \{ (u_n, v_2), (u_n, v_3), (u_n, v_6), \ldots., (u_n, v_{n-5}), (u_n, v_{n-4}), (u_n, v_{n-1})\}$

$\mathcal{C}_1 = \{ (u_4, v_1), (u_5, v_1), (u_8, v_1), (u_9, v_1), \ldots, (u_{n-7}, v_1), (u_{n-6}, v_1), (u_{n-3}, v_1), (u_{n-2}, v_1)\}$

$\mathcal{C}_n = \{ (u_3, v_n), (u_4, v_n), (u_7, v_n), (u_8, v_n), \ldots, (u_{n-8}, v_n), (u_{n-7}, v_n), (u_{n-4}, v_n), (u_{n-3}, v_n)\}$.

Then we know that $\gamma_{tr}(P_n \times P_n) = \gamma_{tr}(P_{n-4} \times P_{n-4}) + \gamma_{tr}(\mathfrak{D}_n)$. Let $(P_n \times P_n)'$ be the graph that is obtained from $(P_n \times P_n)$ by subdividing an edge with one vertex. We can prove $\gamma_{tr}(P_n \times P_n)' = \gamma_{tr}(P_{n-4} \times P_{n-4}) + 1 + \gamma_{tr}(\mathfrak{D}_n) = \gamma_{tr}(P_n \times P_n) + 1$[Since $\gamma_{tr}(P_7 \times P_7)' = \gamma_{tr}(P_7 \times P_7) + 1$].

Hence we get $\mathrm{sd}\gamma_{\mathrm{tr}}((P_n \times P_n)) = 1$ for $n \equiv 3(mod\ 4)$.

From all the above four cases we get $\mathrm{sd}\gamma_{\mathrm{tr}}((P_n \times P_n)) = 1$. ■

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