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Strong efficient domination and strong independent saturation number of graphs

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Abstract

A subset S of $V(G)$ of a graph G is called a strong (weak) efficient dominating set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ ($|N_w[v] \cap S| = 1$) where $N_s(v) = \{ u \in V(G) : uv \in E(G), \deg(u) \geq \deg(v) \}$ and $N_w(v) = \{ u \in V(G) : uv \in E(G), \deg(v) \geq \deg(u)\}$, $Ns[v] = Ns(v) \cup \{v\}$, $Nw[v] = Nw(v) \cup \{v\}$. The minimum cardinality of a strong (weak) efficient dominating set is called strong (weak) efficient domination number of G and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph G is strong efficient if there exists a strong efficient dominating set.

Keywords: Strong efficient dominating set, strong independent saturation number.

AMS Subject Classification (2010): 05C6.

1 Introduction

Let $G=(V, E)$ be a simple graph. A subset $S \subseteq V(G)$ is called a dominating set if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S [3]. The degree of a vertex v in a graph is the number of edges of G incident with v and is denoted by deg (v). The maximum degree of the vertices of G is denoted by $\Delta(G)$. A subset $D \subseteq V(G)$ is said to be a strong dominating set[4] if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $\deg(u) \geq \deg(v)$. A subset S of $V(G)$ is called strong (weak) efficient dominating set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$($|N_w[v] \cap S| = 1$). The minimum cardinality of a strong (weak) efficient dominating set of G is called strong (weak) efficient domination number and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph G is called strong efficient if and only if there exists a strong efficient dominating set of G. Not all graphs are strong efficient. In this paper we define strong independent saturation number of a graph and find some results on efficient domination in some graphs.

For all graph theoretic terminologies and notations, we follow Harary [2]. The following definitions are necessary for the present study.

Definition 1.1. [3] A subset S of $V(G)$ of a graph G is called a dominating set if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Definition 1.2. [4] A subset S of $V(G)$ is called a strong dominating set of G if for every $v \in V - S$ there exists $u \in S$ such that u and v are adjacent and $\deg u \geq \deg v$.

Definition 1.3. [1] A subset S of $V(G)$ is called an efficient dominating set of G if for every $v \in V(G)$, $|N[v] \cap S| = 1$.

2 Main Results

Definition 2.1. Let $G = (V, E)$ be a simple graph. A subset S of $V(G)$ is called a strong (weak) efficient dominating set if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ ($|N_w[v] \cap S| = 1$).

Remark 2.2. $N_s(v) = \{u \in V(G) : uv \in E(G), \deg(u) \geq \deg(v)\}$, $N_w(v) = \{u \in V(G) : uv \in E(G), \deg(v) \geq \deg(u)\}$, $N_s[v] = N_s(v) \cup \{v\}$ and $N_w[v] = N_w(v) \cup \{v\}$. The minimum cardinality of a strong (weak) efficient dominating set is called strong (weak) efficient domination number of G and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph G is strong efficient if there exists a strong efficient dominating set.

Example 2.3. $\{v_2, v_5, v_8\}$ is the strong efficient dominating set of P_9 and $\gamma_{se}(P_9) = 3$.

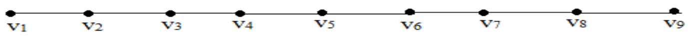

Figure 1: *Path graph* P_9.

Theorem 2.4. $\gamma_{se}(G) = 1$ if and only if G has a full degree vertex.

Theorem 2.5. $\gamma_{se}(G) = 2$ if and only if G is obtained from a graph H of order $n - 2$ by attaching 2 independent vertices u, v such that if X, Y are the sets of vertices in H made adjacent with u, v respectively and $deg_G\, u \geq deg_H\, w$ for all $w \in X$ and $deg_G\, v \geq deg_H\, w$ for all $w \in Y$, then

(i) $X \cup Y = V(H)$.

(ii) If $w \in X \cap Y$, then $deg_H\, w > |X|$ or $|Y|$.

(iii). If $deg_H\, w \leq |X|, |Y|$, then $w \notin X \cap Y$.

Proof: Suppose $\gamma_{se}(G) = 2$.

Let D be a γ_{se} - set of G. Let $D = \{u, v\}$. Clearly u and v are not adjacent. Let $X = N_w(u)$, $Y = N_w(v)$. Let $H = \langle V\text{-}D \rangle$. Then, $deg_G\, u \geq deg_H\, w$ for all $w \in X$ and $deg_G\, v \geq deg_H w$ for all $w \in Y$ and $X \cup Y = V(H)$. Since D is a strong efficient dominating set of G, Conditions (ii) and (iii) are satisfied.

Conversely, let $w \in V(H)$. By condition (i), $\{u,v\}$ is an independent strong dominating set of G. By conditions (ii) and (iii), $\{u,v\}$ is a strong efficient dominating set of G. Therefore, $\gamma_{se}(G) \leq 2$. Since G has no full degree vertex, $\gamma_{se}(G) \geq 2$. Hence, $\gamma_{se}(G) = 2$. ■

Theorem 2.6. $\gamma_{se}(G)$ =3 if and only if G is obtained from a graph H of order n-3 by attaching 3 independent vertices u, v and w such that if X, Y, Z are the sets of vertices in H made adjacent with u, v, w respectively and $deg_G u \geq deg_H x$ for all $x \in X$, $deg_G v \geq deg_H x$ for all $x \in Y$ and $deg_G w \geq deg_H x$ for all $x \in Z$ then

(i) $X \cup Y \cup Z = V(H)$

(ii) If $x \in X \cap Y$, then $deg_H x > |X|$ or $|Y|$. If $x \in Y \cap Z$ then $deg_H x > |Y|$ or $|Z|$. If $x \in X \cap Z$, then $deg_H x > |X|$ or $|Z|$. *If* $x \in X \cap Y \cap Z$, then $deg_H x > |X|, |Y|$ or $|Y|, |Z|$ or $|X|, |Z|$.

(iii) If $deg_H x < |X|, |Y|, |Z|$ then $x \notin X \cap Y \cap Z$.

Proof: Suppose $\gamma_{se}(G) = 3$. Let $D = \{u,v,w\}$ be a γ_{se} – set of G. Clearly $\{u,v,w\}$ is independent. Let $X = N_w(u)$, $Y = N_w(v)$ and $Z = N_w(w)$. Let $H = \langle V - D \rangle$. Then $deg_G u \geq deg_H x$ for all $x \in X$, $deg_G v \geq deg_H x$ for all $x \in Y$ and $deg_G w \geq deg_H x$ for all $x \in Z$ and $X \cup Y \cup Z = V(H)$. Since D is a strong efficient dominating set of G, conditions (ii), and (iii) are satisfied.

Conversely, suppose G satisfies the hypothesis. Let $x \in V(G)$. By condition (i), $\{u,v,w\}$ is a strong independent dominating set of G. By condition (ii), and (iii) $\{u,v,w\}$ is a strong efficient dominating set of G. Therefore, $\gamma_{se}(G) \leq 3$. Since G has no full degree vertex and every strong efficient dominating set is minimal, $\gamma_{se}(G) \neq 1, 2$. Therefore, $\gamma_{se}(G) = 3$. ■

Definition 2.7. Let $I^s(v) = max\ \{|S| : S$ is a minimal strong independent dominating set of G containing $v\}$. $I^s(G) = min\ \{ I^s(v) : v \in V(G) \}$. $I^s(G)$ is called strong independent saturation number of G.

Example 2.8. For the graph G given below, $\beta_s(G) = \Gamma_{se}(G) = \Gamma_s(G)$.

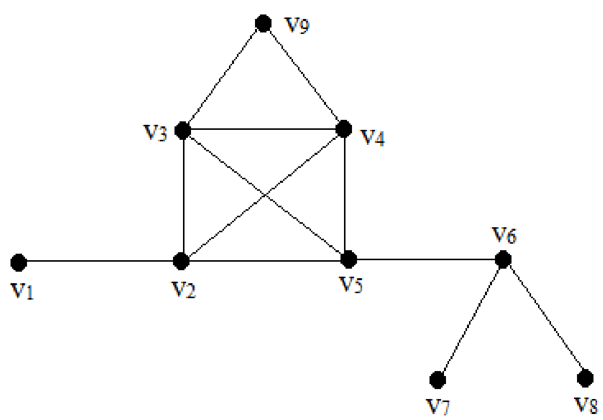

Figure 2: *A graph G with* $\beta_s(G) = \Gamma_{se}(G) = \Gamma_s(G)$.

$\{v_1, v_5, v_7, v_8, v_9\}$, $\{v_1, v_3, v_6\}$, $\{v_1, v_4, v_6\}$ are minimal strong independent dominating sets of G containing v_1. $\{v_1, v_5, v_7, v_8, v_9\}$ is of maximum cardinality. So, $I^s(v_1) = 5$. $\{v_2, v_6, v_9\}$ is the minimal strong independent dominating set of G containing v_2. Therefore, $I^s(v_2) = 3$. $\{v_1, v_3, v_6\}$ is the minimal strong independent dominating set of G containing v_3. Hence, $I^s(v_3) = 3$. $\{v_1, v_4, v_6\}$ is the minimal strong independent dominating set of G containing v_4 and consequently we have $I^s(v_4) = 3$. $\{v_1, v_5, v_7, v_8, v_9\}$ is the minimal strong independent dominating set of G containing v_5.
$I^s(v_5) = 5$. $\{v_1, v_3, v_6\}$, $\{v_1, v_4, v_6\}$, $\{v_2, v_6, v_9\}$ are the minimal strong independent dominating sets of G containing v_6. Hence, $I^s(v_6) = 3$.

$\{v_1, v_5, v_7, v_8, v_9\}$ is the minimal strong independent dominating set of G containing v_7.
Therefore, $I^s(v_7) = 5$. $\{v_1, v_5, v_7, v_8, v_9\}$ is the minimal strong independent dominating set of G containing v_8 and hence $I^s(v_8) = 5$. $\{v_2, v_6, v_9\}$, $\{v_1, v_5, v_7, v_8, v_9\}$ are the minimal strong independent dominating set of G containing v_9. $\{v_1, v_5, v_7, v_8, v_9\}$ is of maximum cardinality. Hence, $I^s(v_9) = 5$. Now, $I^s(G) = \min\{3, 5\} = 3$. We have, $\{v_1, v_3, v_6\}$, $\{v_1, v_4, v_6\}$, $\{v_2, v_6, v_9\}$ are the minimal strong dominating sets of G. Therefore, $\gamma_s(G) = 3$.

$\{v_1, v_3, v_6\}$, $\{v_1, v_4, v_6\}$, $\{v_2, v_6, v_9\}$ are the minimal strong independent dominating sets of G. Therefore, $i_s(G) = 3$. As the above three sets are the minimal strong efficient dominating sets of G we have, $\gamma_{se}(G) = 3$. Therefore, $\gamma_s(G) = i_s(G) = \gamma_{se}(G) = I^s(G)$.
$S = \{v_1, v_5, v_7, v_8, v_9\}$ is the independent strong dominating set of maximum cardinality and $\beta_s(G) = 5$. Since S is the strong dominating set of maximum cardinality, $\Gamma_s(G) = 5$. S is also a minimal strong efficient dominating set of maximum cardinality which implies that $\Gamma_{se}(G) = 5$. Hence, $\beta_s(G) = \Gamma_{se}(G) = \Gamma_s(G)$.

Remark 2.9. There is no relationship between $I^s(G)$ and $\gamma_{se}(G)$. There are graphs for which $I^s(G) = \gamma_{se}(G)$, $I^s(G) < \gamma_{se}(G)$ and $I^s(G) > \gamma_{se}(G)$.

In Example 2.8 we have a graph G with $I^s(G) = \gamma_{se}(G)$. The existence of graphs with $I^s(G) < \gamma_{se}(G)$ and $I^s(G) > \gamma_{se}(G)$ are given in Illustration 2.10 and 2.11.

Illustration 2.10. For the following graph G, $I^s(G) < \gamma_{se}(G)$.

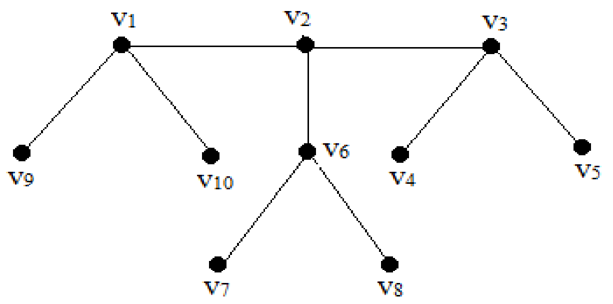

Figure 3: *A graph G with* $I^s(G) < \gamma_{se}(G)$.

$S_1 = \{v_1, v_3, v_6\}$, $S_2 = \{v_2, v_4, v_5, v_7, v_8, v_9, v_{10}\}$ are the only minimal strong independent dominating sets of G. $|S_1| = 3$, $|S_2| = 7$. $I^s(v_1) = I^s(v_3) = I^s(v_6) = 3$.

$I^s(v_2) = I^s(v_4) = I^s(v_5) = I^s(v_7) = I^s(v_8) = I^s(v_9) = I^s(v_{10}) = 7$.So, $I^s(G) = \min\{3, 7\} = 3$.
$N_s[v_2] \cap S_1 = \{v_1, v_2, v_3, v_6\} \cap S_1 = \{v_1, v_3, v_6\}$ and $|N_s[v_2] \cap S_1| = 3 > 1$.

This implies that S_1 is not a strong efficient dominating set of G. In S_2, v_2 strongly dominates v_1, v_3, v_6. and v_4, v_5, v_7, v_8, v_9, v_{10} are strong isolates. Therefore, S_2 is a strong efficient dominating set of G and $\gamma_{se}(G) = 7$.

Hence, for the given graph, $I^s(G) < \gamma_{se}(G)$.

Illustration 2.11. For the path graph P_6, $I^s(P_6) > \gamma_{se}(P_6)$.

Figure 4: *Path graph P_6.*

Let $S_1 = \{v_2, v_5\}$, $S_2 = \{v_1, v_3, v_5\}$, $S_3 = \{v_2, v_4, v_6\}$. S_1, S_2 and S_3 are the minimal strong independent dominating sets of P_6. $N_s[v_4] \cap S_2 = \{v_3. v_5\}$ and $N_s[v_3] \cap S_3 = \{v_2. v_4\}$.

Hence, $|N_s[v_4] \cap S_2| = 2 > 1$ and $|N_s[v_3] \cap S_3| = 2 > 1$ and S_2 and S_3 are not strong efficient dominating sets of P_6.

v_2 strongly dominates v_1 and v_3 andv_5 strongly dominates v_4 and v_6. Therefore, $|N_s[v_i] \cap S_1| = 1$ for all $v_i \in S_1$ and S_1 is the strong efficient dominating set of P_6. Therefore, $\gamma_{se}(P_6) = 2$.
Now, $I^s(v_1) = I^s(v_3) = I^s(v_5) = I^s(v_2) = I^s(v_4) = I^s(v_6) = 3$ implies that $I^s(P_6) = 3$.
Therefore, $I^s(P_6) > \gamma_{se}(P_6)$.

Result 2.12. $I^s(P_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & n \geq 2, n \neq 3 \\ 0, & n = 3. \end{cases}$

Proof: When n=2, the result is obvious. When n=3, $I^s(P_3) = 0$.

Suppose $n \geq 4$. Let $v_1, v_2, .., v_n$ be the vertices of the path P_n.

Case (i): n is odd.
Let $S_1 = \{v_1, v_3, .., v_n\}$ and $S_2 = \{v_2, v_4, .., v_{n-1}\}$.
Therefore, S_1 and S_2 are the minimal independent strong dominating sets containing $v_1, v_3, .., v_n$ and $v_2, v_4, .., v_{n-1}$ respectively.

$$|S_1| = \frac{n-1}{2} + 1 = \frac{n+1}{2}.$$

$$|S_2| = \frac{n-1-2}{2} + 1 = \frac{n-1}{2}.$$

$$I^s(P_n) = \min\left\{\frac{n+1}{2}, \frac{n-1}{2}\right\} = \frac{n-1}{2}.$$

Therefore, $I^s(P_n) = \dfrac{n-1}{2}$ when n is odd.

Case (ii): n is even.

Let $S_3 = \{ v_1, v_3,.., v_{n-1} \}$ and $S_4 = \{ v_2, v_4,.., v_n \}$.

Therefore, S_3 and S_4 are the minimal independent strong dominating sets containing $v_1, v_3,.., v_{n-1}$ and $v_2, v_4,.., v_n$ respectively.

$$|S_3| = \frac{n-1-1}{2} + 1 = \frac{n}{2}.$$

$$|S_4| = \frac{n-2}{2} + 1 = \frac{n}{2}.$$

$$I^s(P_n) = \min\left\{\frac{n}{2}, \frac{n}{2}\right\} = \frac{n}{2}.$$

Therefore, $I^s(P_n) = \dfrac{n}{2}$ when n is even. That is, $I^s(P_n) = \left\lfloor \dfrac{n}{2} \right\rfloor$, $n \geq 4$. ■

Result 2.13. $I^s(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 3$.

Proof: Case (i): n is odd.

Let $S_1 = \{ v_1, v_3,.., v_{n-2} \}$ and $S_2 = \{ v_2, v_4,.., v_{n-1} \}$.

$\therefore$ S_1 and S_2 are the minimal independent strong dominating sets containing $v_1, v_3,.., v_{n-2}$ and $v_2, v_4,.., v_{n-1}$ respectively.

$$|S_1| = \frac{n-2-1}{2} + 1 = \frac{n-1}{2}.$$

$$|S_2| = \frac{n-1-2}{2} + 1 = \frac{n-1}{2}$$

Let $S_3 = \{v_3, v_5, v_7. ..., v_n\}$

$$|S_3| = \frac{n-3}{2} + 1 = \frac{n-1}{2}.$$

$$I^s(C_n) = \min\left\{\frac{n+1}{2}, \frac{n-1}{2}, \frac{n-1}{2}\right\} = \frac{n-1}{2}.$$

$I^s(C_n) = \dfrac{n-1}{2}$, when n is odd.

Case (ii): n is even.

Let $S_3 = \{ v_1, v_3,.., v_{n-1} \}$ and $S_4 = \{ v_2, v_4,.., v_n \}$.

Therefore, S_3 and S_4 are the minimal independent strong dominating sets containing $v_1, v_3,.., v_{n-1}$ and $v_2, v_4,.., v_n$ respectively.

$$|S_3| = \frac{n-1-1}{2} + 1 = \frac{n}{2}.$$

$$|S_4| = \frac{n-2}{2} + 1 = \frac{n}{2}.$$

$$I^s(C_n) = \min\left\{\frac{n}{2}, \frac{n}{2}\right\} = \frac{n}{2}.$$

Therefore, $I^s(C_n) = \frac{n}{2}$ when n is even.

$I^s(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 3$. ■

Result 2.14. $I^s(K_n) = 1 \; \forall \, n \in N$.

Result 2.15. $I^s(K_{m,n}) = \begin{cases} n & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$

Proof: Let G be a complete bipartite graph $K_{m,n}$.
Let (V_1, V_2) be the partition of G with $|V_1| = m$ and $|V_2| = n$.
Let $v_1, v_2, ..., v_m \in V_1(G)$ and $u_1, u_2, ..., u_n \in V_2(G)$.

Case (i): $m = n$.

$V_1(G)$ and $V_2(G)$ are the minimal independent strong dominating sets containing $v_1, v_2, ..., v_m$ and $u_1, u_2, ..., u_n$ respectively.
$I^s(v_1) = I^s(v_2) = \ldots = I^s(v_m) = n$.
$I^s(u_1) = I^s(u_2) = \ldots = I^s(u_n) = n$.
Therefore, $I^s(K_{m,n}) = n$, when $m = n$.

Case (ii): $m < n$.

Since $\deg u_i < \deg v_j$, for all i = 1 to m, j = 1 to n, there is no minimal independent strong dominating set containing $u_1, u_2, ..., u_n$.
$I^s(u_1) = I^s(u_2) = \ldots = I^s(u_n) = 0$.
$\{v_1, v_2, ..., v_m\}$ is the minimal independent strong dominating set containing $v_1, v_2, ..., v_m$.
$I^s(v_1) = I^s(v_2) = \ldots = I^s(v_m) = m$.
Therefore, $I^s(K_{m,n}) = \min\{0, m\} = 0$.

Case (iii): $m > n$.

The proof is similar to case (ii). ■

Result 2.16. $I^s(D_{r,s}) = \begin{cases} r+1 & \text{if } r = s \\ 0 & \text{otherwise} \end{cases}$

Proof: Let $G = D_{r,s}$.

Let u and v be the central vertices of G. Let $u_1, u_2, ..., u_r$ be the vertices adjacent to u and $v_1, v_2, ..., v_s$ be the vertices adjacent to v.

Case (i): $r = s$.

$\{ u, v_1, v_2, ..., v_s \}$ and $\{ v, u_1, u_2, ..., u_r \}$ are the minimal independent strong dominating sets containing $u, v_1, v_2, ..., v_s$ and $v, u_1, u_2, ..., u_r$ respectively.

$I^s(u) = I^s(v_1) = I^s(v_2) = \ldots = I^s(v_s) = r + 1$.

$I^s(v) = I^s(u_1) = I^s(u_2) = \ldots = I^s(u_r) = r + 1$.

Therefore, $I^s(D_{r,s}) = r + 1$.

Case (ii): $r < s$.

$\deg v = \Delta(G) = r + 1$.

v strongly dominates $u, v_1, v_2, ..., v_s$.

$\{ v, u_1, u_2, ..., u_r \}$ is the only the minimal independent strong dominating set of G containing $v, u_1, u_2, ..., u_r$.

$I^s(v) = I^s(u_1) = I^s(u_2) = \ldots = I^s(u_r) = r + 1$.

Since $\deg u < \deg v$, there is no minimal independent strong dominating set containing $u, v_1, v_2, ..., v_s$.

Therefore, $I^s(u) = I^s(v_1) = I^s(v_2) = \ldots = I^s(v_s) = 0$.

$I^s(D_{r,s}) = \min\{r + 1, 0\} = 0$.

Case (iii): $r > s$.

The proof is similar to case (ii). ■

References

[1] D. W. Bange, A. E. Barkauskas and P. J. Slater, *Efficient dominating sets in graphs*, Application of Discrete Mathematics, SIAM, Philadephia, (1988), 189-199.

[2] F. Harary, *Graph Theory*, Addison – Wesley, 1969.

[3] T.W. Haynes, Stephen T. Hedetniemi and Peter J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, INC, New York, 1998.

[4] E. Sampathkumar and L. Pushpa Latha. *Strong weak domination and domination balance in a graph*, Discrete Math., 161(1996), 235-242.