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Vertex capacity in a graph structure

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Abstract

In a graph structure $G = (V, E_1, E_2 \dots E_k)$, the capacity of a vertex $c(v)$ is the number of different E_i edges incident at v. In this paper, we have made an attempt to answer the problem of examining the graph structures for which sum of vertex capacities is equal to sum of number of relations and the number of edges in the structure for some families of graphs viewed as graph structures.

Keywords: Graph structure, E_i – connected, E_i –edges, E_i-path, capacity of a vertex, k-edge colorable, k-edge chromatic, edge chromatic partition, matching.

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1 Introduction

A graph structure $G = (V, E_1, E_2, \dots, E_k)$ consists of a non-empty set V together with relations $E_1, E_2, \dots, E_k$ on V which are mutually disjoint such that each E_i, $1 \le i \le k$ is symmetric and irreflexive. If $(u,v) \in E_i$ for some i, $1 \le i \le k$, we call it $\underline{E_i}$-edge and is denoted by uv. E. Sampath kumar [2] introduced this concept in his lecture notes "Generalized Graph Structures".

A parameter, the capacity of a vertex similar to the degree of a vertex, was introduced and the problem of examining the graph structures in which sum of the vertex capacities is equal to sum of number of relations and total number of edges in the structure was posed in [2]. For any graph, there may be many structure representations. We consider only the standard graphs such as cycles and paths and see under which the above problem has a positive solution. The structure representation naturally arising out of chromatic partitioning is also dealt with.

For the definitions and notations we follow [1] and [2].

The following definitions are related which will be used in the subsequent theorems

1. The capacity of a vertex c(v) in a graph structure G = $(V, E_1, E_2, \dots, E_k)$ is the number of different E_i-edges incident at v. Clearly, $c(v) \le \deg(v)$.
2. In a graph structure G, let $\delta_c(G)$ denote the minimum capacity of a vertex and $\Delta_c(G)$ denote the maximum capacity of a vertex in G. Clearly, if $G = (V, E_1, E_2, \dots, E_k)$ is a graph structure, then for any vertex v, $\delta_c(G) \le c(v) \le \Delta_c(G) \le k$.

3. Let S be a set of vertices in G. Then the capacity of S is the sum of the capacities of vertices in S.
4. A graph structure $G = (V, E_1, E_2, \ldots, E_k)$ is said to be connected if the underlying graph of G is connected.
5. An E_i-path between two vertices u and v consists of only E_i-edges. An E_i-cycle consists of only E_i-edges.
6. A set S of vertices in a graph structure $G = (V, E_1, E_2, \ldots, E_k)$ is E_i-connected for some i if any two vertices in S are connected by an E_i-path. A set of vertices may be E_i-connected for more than one i.
7. A graph structure is totally edge disconnected if no two E_i-edges are adjacent. Then the graph structure has E_i-components.

2 Main Results

Theorem 2.1. Let $G = (V, E_1, E_2, \ldots, E_k)$ be a structure representation of a cycle of length n with K relations $E_1, E_2, \ldots, E_k$ and each E_i is a set of edges, $1 \le i \le K$, $K \ge 2$. Let $G_i = G[E_i]$ be a sub graph of G induced by the edges in E_i. If the vertices in G_i are E_i –connected, then the sum of the vertex capacities is equal to the sum of the number of relations and the total number of edges in G. That is, $\sum_{v \in G} c(v) = |E| + K$, where $|E|$ denotes the number of edges in G.

Proof: Let $G = (V, E_1, E_2, \ldots, E_k)$ be a structure representation of a cycle of length n with K relations $E_1, E_2, \ldots, E_k$ and each E_i is a set of edges, $1 \le i \le K$, $K \ge 2$. Let G_i be a sub graph of G induced by the edges in E_i. Let the vertices in G_i be E_i-connected $\forall\, i$.

Since G is a cycle of length n, $|V| = n$ and $|E| = n$. Since G_i is a subgraph of G induced by the edges in E_i, $\forall\ 1 \le i \le K$, we have K such induced E_i-connected subgraphs.
That is, each G_i is an E_i–path. Let the number of edges in G_i be e_i, for all i. Hence each G_i has e_i+1 vertices and $e_1 + e_2 + \ldots + e_k = n$. Also each terminal vertex of G_i is counted in one more G_j. Since each G_i is an E_i –path, G_i has only E_i-edges and so the capacity of each vertex in G_i is 1. That is, $\sum_{v \in G_i} c(v) = e_i+1$, $1 \le i \le K$.

$$\text{Hence, } \sum_{v \in G} c(v) = \sum_{v \in G_i} c(v) = \sum_{v \in G_1} c(v) + \sum_{v \in G_2} c(v) + \ldots\ldots\ldots + \sum_{v \in G_k} c(v)$$
$$= (e_1+1) + (e_2+1) + \ldots\ldots\ldots + (e_k+1)$$
$$= (e_1 + e_2 + \ldots\ldots\ldots + e_k) + (1 + 1 + \ldots\ldots + 1)(k \text{ times})$$
$$= n + K$$
$$= |E| + K. \quad \blacksquare$$

Corollary 2.2. In the above theorem, if G_i is not E_i-connected for some i, $1 \le i \le K$, then $\sum_{v \in G} c(v) \ne |E| + K$.

Proof: Let $G = (V, E_1, E_2, ..., E_k)$ be a structure representation of a cycle of length n and G_r be not E_r –connected. Then G_r has at least two components. Assume that G_r has two components namely, $G_r^{(1)}$ and $G_r^{(2)}$. Let the number of edges in G_i be e_i for all i. Let $G_r^{(1)}$ has s edges. Then $G_r^{(2)}$ has $e_r - s$ edges. Since each G_i is an E_i –path, G_i has e_i+1 vertices. Then $G_r^{(1)}$ has s+1 vertices and $G_r^{(2)}$ has e_r–s+1 vertices. Further each terminal vertex of G_i is counted in one more G_j.

As in Theorem 2.1, the capacity of each vertex in G_i is 1 including $G_r^{(1)}$ and $G_r^{(2)}$.

That is, $\sum_{v\in G_i} c(v) = e_i + 1,\ 1 \le i \le K,\ i \ne r$ and $\sum_{v\in G_r^{(1)}} c(v) = s + 1;\ \sum_{v\in G_r^{(2)}} c(v) = e_r - s + 1$.

$$\begin{aligned}
\text{Hence, } \sum_{v\in G} c(v) &= \sum_{v\in G_1} c(v) + \sum_{v\in G_2} c(v) + ... + \sum_{v\in G_r} c(v) + ... + \sum_{v\in G_K} c(v) \\
&= \sum_{v\in G_1} c(v) + \sum_{v\in G_2} c(v) + ... + \sum_{v\in G_r^{(1)}} c(v) + \sum_{v\in G_r^{(2)}} c(v) .. + \sum_{v\in G_K} c(v) \\
&= (e_1+1) + (e_2+1) + \ldots + (e_{r-1}+1) + (s+1) + (e_r - s+1) + (e_{r+1}+1) + \ldots\ldots + (e_k+1) \\
&= (e_1 + e_2 + \ldots + e_r\text{-}1 + e_r + e_r{+}1 + \ldots\ldots + e_k) + (1 + 1 + \ldots + s + 1 - s + 1 + \ldots + 1) \\
&= n + (1+1+\ldots+1)\ (k+1 \text{ times}) \\
&= n + (K+1) = |E| + (K+1) \\
&> |E| + \text{K}.
\end{aligned}$$

That is, $\sum_{v\in G} c(v) \ne |E| + K$. ■

Corollary 2.3. In the above theorem, if there is only one relation say E_1, then $\sum_{v\in G} c(v) \ne |E| + K$.

Proof: Since G is a structure representation of a cycle of length n, it has n vertices. Let G has only one relation E_1. Then $|V| = n$; $|E| = n$ and $K = 1$.

Since G has only one relation, capacity of each vertex in G is 1.

$$\begin{aligned}
\text{That is, } \sum_{v\in G} c(v) &= (1+1+\ldots+1)(n \text{ times}) \\
&= n = |E| \ne |E| + K.
\end{aligned}$$
■

Theorem 2.4. Let $G = (\text{V}, E_1, E_2, \ldots, E_k)$ be a structure representation of a path P of length n with k-relations $E_1, E_2, \ldots, E_k$ and each E_i is a set of edges. Let $G_i = G[E_i]$ be a subgraph of G induced by the edges in E_i. If the vertices in G_i are E_i –connected, then the sum of the vertex capacities is equal to sum of the number of relations and the total number of edges.

Proof: Let $G = (\text{V}, E_1, E_2, \ldots, E_k)$ be a structure representation of a path P of length n with k-relations $E_1, E_2, \ldots, E_k$ and each E_i is a set of edges, $1 \le i \le K$. Let the vertices in G_i are E_i–connected for all i.

Since G is a path of length n, $|V| = n + 1$; and $|E| = n$. Each G_i is an E_i –path. So each G_i has number of vertices one more than the number of edges and the terminal vertices of G_i are counted twice except the starting and ending vertex of the path P.

Let the number of edges in G_i be e_i. Then the number of vertices in G_i is e_i+1 and the capacity of each vertex in G_i is1 $e_1 + e_2 + \ldots + e_k = n$.

$$\sum_{v\in G} c(v) = \sum_{v\in G_i} c(v) = (e_1+1) + (e_2+1) + \ldots\ldots\ldots + (e_k+1)$$

$$= (e_1 + e_2 + \ldots + e_k) + (1 + 1 + \ldots\ldots + 1)(k \text{ times})$$

$$= n + K = |E| + K. \qquad \blacksquare$$

Corollary 2.5. In the above theorem, if at least one of the E_i is not E_i-connected, $1 \le i \le K$, then $\sum_{v\in G} c(v) \ne |E| + K$.

Proof: The proof is same as Corollary 2.2. The only difference is, in corollary 2.2, G is a structure representation of a cycle. Here, G is a structure representation of a path. ■

Theorem 2.6. Let G be a graph with n edges whose edge chromatic number is K. Let G' = (V, E_1, E_2,…, E_k) be the structure representation given by the chromatic partition of G with k-relations E_1, E_2,…, E_k and each E_i is an edge color class of G, $1 \le i \le K$. Then the sum of the vertex capacities is not equal to the sum of number of relations and the total number of edges. That is, $\sum_{v\in G'} c(v) \ne |E| + K$.

Proof: Let G be a graph with n edges whose edge chromatic number is K. Let G' = (V, E_1, E_2,…, E_k) be the structure representation given by the chromatic partition of G with k-relations E_1, E_2,…, E_k and each E_i is an edge color class of G, $1 \le i \le K$.

Let $G_i = G'[E_i]$ be a sub graph of G' induced by the edges in E_i. Since in an edge coloring, no two adjacent edges receive the same color, no two edges in G_i are adjacent. Hence each G_i is a matching.

Let the number of edges in G_i be e_i. Then $e_1 + e_2 + \ldots + e_k = n$. Since no two edges are adjacent in G_i, no two edges have the same starting vertex and ending vertex. Therefore, the number of vertices in G_i is $2e_i$ and each has capacity1, that is, $\sum_{v\in G_i} c(v) = 2e_i$.

Therefore, $\sum_{v\in G'} c(v) = \sum_{v\in G_i} c(v), 1 \le i \le K$

$$= 2e_1 + 2e_2 + \ldots\ldots + 2e_k$$

$$= 2(e_1 + e_2 + \ldots + e_k) = 2n = 2\,|E| + K$$

$$\ne |E| + K. \qquad \blacksquare$$

We concluded that this work can be extended to determine minimum δ, for which $\sum_{v\in G} c(v) = |E| + k$ and compare it with the chromatic number of the given graph.

References

[1] F. Harary, *Graph Theory*, Addition – Wesley Publishing Co., 1969.

[2] E. Sampathkumar, *Generalized Graph Structures*, Lecture notes.