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L- edge colouring of graphs

R. B. Gnana Jothi, A. Uma Devi
V.V. Vanniaperumal College for Women,
Virudhunagar- 626 001, Tamilnadu, INDIA.

Abstract

A graph $G = (V, E)$ is said to be topogenic if it admits a topogenic set indexer, which is a set indexer $f: V \to 2^X$ such that $f(V) \bigcup f^{\oplus}(E)$ is a topology on X. A list colouring of a graph $G = (V,E)$ with a colour list $C = \{c(v_1), c(v_2), \ldots c(v_n)\}$ for $V = \{v_1, v_2, \ldots v_n\}$ is a proper colouring of V by element of $\bigcup_j c(v_j)$ so that the adjacent vertices u,v are coloured differently and the colour for $v \in V$ is in $C(V)$. L- edge colouring of a graph G is an assignment $L:E(G) \to 2^X - \varphi$ such that no two adjacent edges receive the same label where X is a ground set. A graph G is to be L- edge colourable if there exists an L- edge colouring of G. A comparative study of topogenic graphs and L- edge colourable graphs has been made in this paper.

Keywords: L-edge colouring, null set magic, exactly null set magic ,topogenic.

AMS Subject Classification (2010): 05C10.

1 Introduction

Graph colorings are useful to solve various problems ranging from scheduling to the channel assignment problem. List colorings, in particular, arise naturally from applications with restrictions on which values can be assigned to certain objects. Topogenic graph was introduced by B.D. Acharya, K.A. Germina and Jisha Elizabath Joy in [1]. The concept of null set magic labeling was introduced in [2]. The idea of list edge colouring was introduced in [3]. This motivated us to define a new type of edge colouring '*L*- edge colouring'.

L- edge colouring of a graph G is an assignment $L:E(G) \to 2^X - \varphi$ such that no two adjacent edges receive the same label, where X is a ground set. A graph G is said to be L- edge colourable, if there exists an L- edge colouring of G.

Definition 1.1. A graph G is said to be null set-magic if its edges can be assigned distinct subsets of a nonempty set X with $2^{|X|-1} \leq |E(G)| \leq 2^{|X|} - 1$ such that for every vertex u of G intersection of the subsets assigned to the edges incident at u is φ, such a set assignment to the edges of G is called a null set magic labeling of G.

Definition 1.2. A null set magic graph G is said to be exactly null set magic if $|P(X)|$ -1 = $|E(G)|$ where $P(X)$ denotes the power set of X.

Definition 1.3. A graph $G = (V,E)$ is said to be topogenic with respect to a non empty ground set X if it admits a topogenic set indexer, which is a set indexer $f: V \to 2^X$ such that $f(V) \cup f^{\oplus}(E)$ is a topology on X.

Definition 1.4. L- edge colouring of a graph G is an assignment $L: E(G) \to 2^X - \varphi$ such that no two adjacent edges receive the same label where X is a ground set. A graph G is to be L- edge colourable if there exists an L- edge colouring of G.

2 Main Results

Theorem 2.1. Any topogenic graph is L-edge colourable.

Proof: Let $G = (V,E)$ be a topogenic graph with respect to a non empty ground set X. Therefore it admits a topogenic set indexer. Hence there exists an injective set valued function $f: V \to 2^X$ such that the induced edge function $f^{\oplus}: E \to 2^X - \{\phi\}$ is also injective. Therefore the edges of G receive distinct labels. Hence G is L- edge colourable. ■

Theorem 2.2. Exactly null set magic graphs are L- edge colourable.

Proof: Let $G = (V,E)$ be an exactly null set magic graph. Therefore its edges can be assigned distinct subsets of a nonempty set X, with $2^{|X|-1} \le |E(G)| \le 2^{|X|} - 1$ such that for every vertex u of G intersection of the subsets assigned to the edges incident at u is φ. Since the edges of G receive distinct labels, G is L- edge colourable. ■

Theorem 2.3. $P_m + K_1$ is exactly null set magic for $m = 2^n$ with $n \ge 3$ and hence L-edge colourable.

Proof: First we prove that $P_m + K_1$ is exactly null set magic.

$$\gamma(P_m + K_1) = 2^n + 1;\ \varepsilon(P_m + K_1) = 2^{n+1} - 1.$$

Let $V(P_m + K_1) = \{u_1, v_1, v_2, \ldots, v_{2^n}\}$ and

$$E(P_m + K_1) = \{uv_i;\ i = 1, 2, \ldots, 2^n\} \cup \{u_i v_{i+1};\ i = 1, 2, \ldots, 2^n - 1\}.$$

Label the edges of $P_m + K_1$ with non-empty subsets of $X = \{1, 2, \ldots, n+1\}$ as follows.

Case 1: n is odd

$$l(uv_i) = \{2i-1\},\ i = 1, 2, \ldots \frac{n+1}{2}$$

$$l(uv_i) = \left\{1,\ i - \left(\frac{n-1}{2}\right)\right\},\ i = \frac{n+3}{2}, \frac{n+5}{2}, \ldots \frac{3n+1}{2}$$

$$l(uv_i) = \left\{2,\ i - \left(\frac{3n-3}{2}\right)\right\},\ i = \frac{3n+3}{2}, \frac{3n+5}{2}, \ldots \frac{5n-1}{2}$$

$$l\,(uv_i) = \left\{3,\ i-\left(\frac{5n-7}{2}\right)\right\},\ \mathrm{i} = \frac{5n+1}{2},\ \frac{5n+3}{2},\ldots\ \frac{7n-5}{2}$$

$$\vdots$$

$$l\,(uv_i) = \left\{n,\ i-\left(\frac{n^2-1}{2}\right)\right\},\ i = \frac{n^2+2n+1}{2}$$

$$l\,(uv_i) = \left\{1, 2,\ i-\left(\frac{n^2+2n-3}{2}\right)\right\},\ i = \frac{n^2+2n+3}{2},\ \frac{n^2+2n+5}{2},\ \ldots..\ \frac{n^2+4n-1}{2}$$

$$l\,(\mathrm{uv_i}) = \left\{1, 3,\ i-\left(\frac{n^2+4n-7}{2}\right)\right\},\ i = \frac{n^2+4n+1}{2},\ \frac{n^2+4n+3}{2},\ \ldots..\ \frac{n^2+6n-5}{2}$$

$$\vdots$$

$$l\,(uv_i) = \left\{1, n,\ i-\left(\frac{2n^2-n-1}{2}\right)\right\},\ i = \frac{2n^2+n+1}{2}$$

Label the edges upto $\left\lceil \frac{2^{n+1}-1}{2} \right\rceil - \left(\frac{n+3}{2}\right)$ number of edges where $\lceil\ \ \rceil$ denotes the ceiling function.

$$l\left[uv_{2^n-\left(\frac{n+1}{2}\right)}\right] = \{1, 2, 3, \ldots, n+1\}$$

$$l\left[uv_{2^n-\left(\frac{n-1}{2}\right)}\right] = \{1, 2, 3, \ldots, n\}$$

$$l\left[uv_{2^n+1-i}\right] = \{2i\},\ i = 1,2,\ \ldots\ \frac{n-1}{2}$$

$$l\left[v_i\, v_{i+1}\right] = \left[l\left[u\, v_i\right]\right]^C,\ i = 1,2,\ \ldots\ \frac{2n^2+n+1}{2}$$

$$l\left[v_i\, v_{i+1}\right] = \{n+1\},\ i = \frac{2n^2+n+3}{2}$$

$$l\left[v_i\, v_{i+1}\right] = \left[l\left[u\, v_{i+1}\right]\right]^C,\ i = \frac{2n^2+n+7}{2},\ \frac{2n^2+n+5}{2},\ \ldots\ 2^n-1$$

A_1 = Set of all singleton sets assigned

$$= \left\{l(uv_i) : i = 1, 2, \ldots, \frac{n+1}{2}\right\} \cup \left\{l\left(uv_{2^n+1-i}\right) : i = 1, 2, \ldots, \frac{n-1}{2}\right\} \cup \left\{l\left(v_i v_{i+1}\right) : i = \frac{2n^2+n+3}{2}\right\}$$

$$= \{\{1\},\ \{3\},\ \ldots\{n\}\} \cup \{\{2\},\ \{4\},\ \ldots\{n-1\}\} \cup \{n+1\}$$

$$= \{\{1\},\ \{2\}, \{3\},\ \ldots\{n-1\}, \{n\}, \{n+1\}\}$$

A_2 = Set of all two element sets assigned

$$=\left\{l(uv_i):i=\frac{n+3}{2},\frac{n+5}{2},\,...\,\frac{3n+1}{2}\right\}\cup\left\{l(uv_i):i=\frac{3n+3}{2},\frac{3n+5}{2},\,...\,\frac{5n-1}{2}\right\}$$

$$\cup\left\{l(uv_i):i=\frac{5n+1}{2},\frac{5n+3}{2},\,...\,\frac{7n-5}{2}\right\}\cup\left\{l(uv_i):i=\frac{n^2+2n+1}{2}\right\}$$

$$=\left\{\{1,2\},\{1,3\},...\{1,n+1\}\right\}\cup\left\{\{2,3\},\{2,4\},...\{2,n+1\}\right\}$$

$$\cup\left\{\{3,4\},\{3,5\},...\{3,n+1\}\right\}\cup...\cup\left\{\{n,n+1\}\right\}$$

A_3 = set of all three element sets assigned

⋮

A_n = set of all n element sets assigned

It is noted that $\{A_i\},\ i=1,2,3,...n+1$ are distinct and $\bigcup_{i=1}^{i=n+1} A_i = P(X)\setminus\phi$.

Case 2: *n* is even

$$l(uv_i)=\{2i-1\},\ i=1,2,\ldots\ \frac{n}{2}$$

$$l(uv_i)=\left\{1,\ i-\left(\frac{n-2}{2}\right)\right\},\ i=\frac{n+2}{2},\frac{n+4}{2},\ ...\frac{3n}{2}$$

$$l(uv_i)=\left\{2,\ i-\left(\frac{3n-4}{2}\right)\right\},\ i=\frac{3n+2}{2},\frac{3n+4}{2},...\frac{5n-2}{2}$$

$$l(uv_i)=\left\{3,\ i-\left(\frac{5n-8}{2}\right)\right\},\ i=\frac{5n}{2},\frac{5n+2}{2},...\frac{7n-6}{2}$$

⋮

$$l(uv_i)=\left\{n,\ i-\left(\frac{n^2-2}{2}\right)\right\},\ i=\frac{n^2+2n}{2}$$

$$l(uv_i)=\left\{1,2,\ i-\left(\frac{n^2+2n-4}{2}\right)\right\},\ i=\frac{n^2+2n+2}{2},\frac{n^2+2n+4}{2},\ .....\ \frac{n^2+4n-2}{2}$$

$$l(uv_i)=\left\{1,3,\ i-\left(\frac{n^2+4n-8}{2}\right)\right\},\ i=\frac{n^2+4n}{2},\frac{n^2+4n+2}{2},\ .....\ \frac{n^2+6n-6}{2}$$

⋮

$$l(uv_i)=\left\{1,n,\ i-\left(\frac{2n^2-n-2}{2}\right)\right\},\ i=\frac{2n^2+n}{2}$$

Label the edges upto $\left\lceil\frac{2^{n+1}-1}{2}\right\rceil-\left(\frac{n+4}{2}\right)$ number of edges where $\lceil\quad\rceil$ denotes the ceiling function.

$$l\left[uv_{2^n-\left(\frac{n+2}{2}\right)}\right] = \{1,2,3,\ldots,n+1\}$$

$$l\left[uv_{2^n-\left(\frac{n}{2}\right)}\right] = \{1,2,3,\ldots,n\}$$

$$l\left[uv_{2^n+1-i}\right] = \{2i\},\ i = 1,2,\ \ldots\ \frac{n}{2}$$

$$l[v_i\, v_{i+1}] = [l[u\, v_i]]^C,\ i = 1,2,\ \ldots\ \frac{2^{n+1}-2n+2}{2}$$

$$l[v_i\, v_{i+1}] = [l[u\, v_{i+1}]]^C,\ i = \frac{2^{n+1}-2n+4}{2}, \frac{2^{n+1}-2n+6}{2},\ \ldots\ 2^n-1.$$

A_1 = Set of all singleton sets assigned

$$= \left\{l(uv_i): i=1,2,\ldots,\frac{n}{2}\right\} \cup \left\{l\left(uv_{2^n+1-i}\right): i=1,2,\ldots,\frac{n}{2}\right\}$$

$$= \{\{1\},\{3\},\ldots\{n-1\}\} \cup \{\{2\},\{4\},\ldots\{n\}\}$$

$$= \{\{1\},\{2\},\{3\},\ldots\{n-1\},\{n\},\{n+1\}\}$$

A_2 = Set of all two element sets assigned

$$= \left\{l(uv_i): i = \frac{n+2}{2}, \frac{n+4}{2},\ \ldots\ \frac{3n}{2}\right\} \cup \left\{l(uv_i): i = \frac{3n+2}{2}, \frac{3n+4}{2},\ \ldots\ \frac{5n-2}{2}\right\}$$

$$\cup \left\{l(uv_i): i = \frac{5n}{2}, \frac{5n+2}{2},\ \ldots\ \frac{7n-6}{2}\right\} \cup \left\{l(uv_i): i = \frac{n^2+2n}{2}\right\}$$

$$= \{\{1,2\},\{1,3\},\ldots\{1,n+1\}\} \cup \{\{2,3\},\{2,4\},\ldots\{2,n+1\}\}$$

$$\cup \{\{3,4\},\{3,5\},\ldots\{3,n+1\}\} \cup \ldots \cup \{\{n,n+1\}\}$$

A_3 = set of all three element sets assigned

$\vdots$

A_n = set of all n element sets assigned

It is noted that $\{A_i\},\ i=1,2,3,\ldots n+1$ are distinct and $\bigcup_{i=1}^{n+1} A_i = P(X)\setminus\phi$. Therefore P_m+K_1 is exactly null set magic. Hence, it is *L*- edge colourable. ■

Illustration 2.4. $P_{2^5} + K_1$ is exactly null set magic and hence *L*-edge colourable.

$$\gamma\left(P_{2^5} + K_1\right) = 2^5+1 = 33.$$

$$\varepsilon\left(P_{2^5} + K_1\right) = 2^6 - 1 = 63.$$

Let $V\left(P_{2^5} + K_1\right) = \{u, v_1, v_2, \ldots, v_{32}\}$ and

$$E\left(P_{2^5} + K_1\right) = \{u v_i : i = 1, 2, \ldots, 32\} \cup \{v_i v_{i+1} : i = 1, 2, \ldots 31\}.$$

Label the edges of $P_{2^5} + K_1$ with nonempty subsets of X = $\{1, 2, 3, 4, 5, 6\}$ as follows:

$$l(uv_i) = \begin{cases} \{2i-1\}, \ i = 1, 2, 3 \\ \left\{1, \ i - \left(\frac{n-1}{2}\right)\right\}, \ i = 4, 5, 6, 7, 8 \\ \left\{2, \ i - \left(\frac{3n-3}{2}\right)\right\}, \ i = 9, 10, 11, 12 \\ \left\{3, \ i - \left(\frac{5n-7}{2}\right)\right\}, \ i = 13, 14, 15 \\ \left\{4, \ i - \left(\frac{7n-13}{2}\right)\right\}, i = 16, 17 \\ \left\{5, \ i - \left(\frac{n^2-1}{2}\right)\right\}, \ i = 18 \\ \left\{1, 2, \ i - \left(\frac{n^2+2n-3}{2}\right)\right\}, \ i = 19, 20, 21, 22 \\ \left\{1, 3, \ i - \left(\frac{n^2+4n-7}{2}\right)\right\}, \ i = 23, 24, 25 \\ \left\{1, 4, \ i - \left(\frac{n^2+2n+7}{2}\right)\right\}, \ i = 26, 27 \\ \left\{1, 5, \ i - \left(\frac{2n^2-n-1}{2}\right)\right\}, \ i = 28 \end{cases}$$

$$l\left[uv_{2^n - \left(\frac{n+1}{2}\right)}\right] = \{1, 2, \ldots, n+1\}$$

$$l(uv_{29}) = \{1, 2, 3, 4, 5, 6\}$$

$$l\left[uv_{2^n - \left(\frac{n-1}{2}\right)}\right] = \{1, 2, 3, \ldots, n\}$$

$l\,(uv_{30}) = \{1, 2, 3, 4, 5\}$

$l\left[uv_{2^n+1-i}\right] = \{2i\}\,,\ i = 1,2,\ \ldots\ \dfrac{n-1}{2}$

$l\,(uv_{32}) = \{2\}\,,\ l\,(uv_{31}) = \{4\}$

$l\left[v_i\,v_{i+1}\right] = \left[l\left[u\,v_i\right]\right]^C,\ i = 1,2,\ \ldots\ \dfrac{2n^2+n+1}{2}$

$l\left[v_i\,v_{i+1}\right] = \left[l\left[u\,v_i\right]\right]^C,\ i = 1,2,\ \ldots 28$

$l\left[v_i\,v_{i+1}\right] = \{n+1\}\,,\ i = \dfrac{2n^2+n+3}{2}$

$l\left[v_{29}\,v_{30}\right] = \{6\}$

$l\left[v_i\,v_{i+1}\right] = \left[l\left[u\,v_{i+1}\right]\right]^C,\ i = \dfrac{2n^2+n+7}{2},\ \dfrac{2n^2+n+5}{2},\ \ldots\ 2^n-1$

$l\left[v_i\,v_{i+1}\right] = \left[l\left[u\,v_{i+1}\right]\right]^C,\ i = 31,\ 30$

$l\left[v_{30}\,v_{31}\right] = \{1, 2, 3, 5, 6\}$

$l\left[v_{31}\,v_{32}\right] = \{1, 3, 4, 5, 6\}$

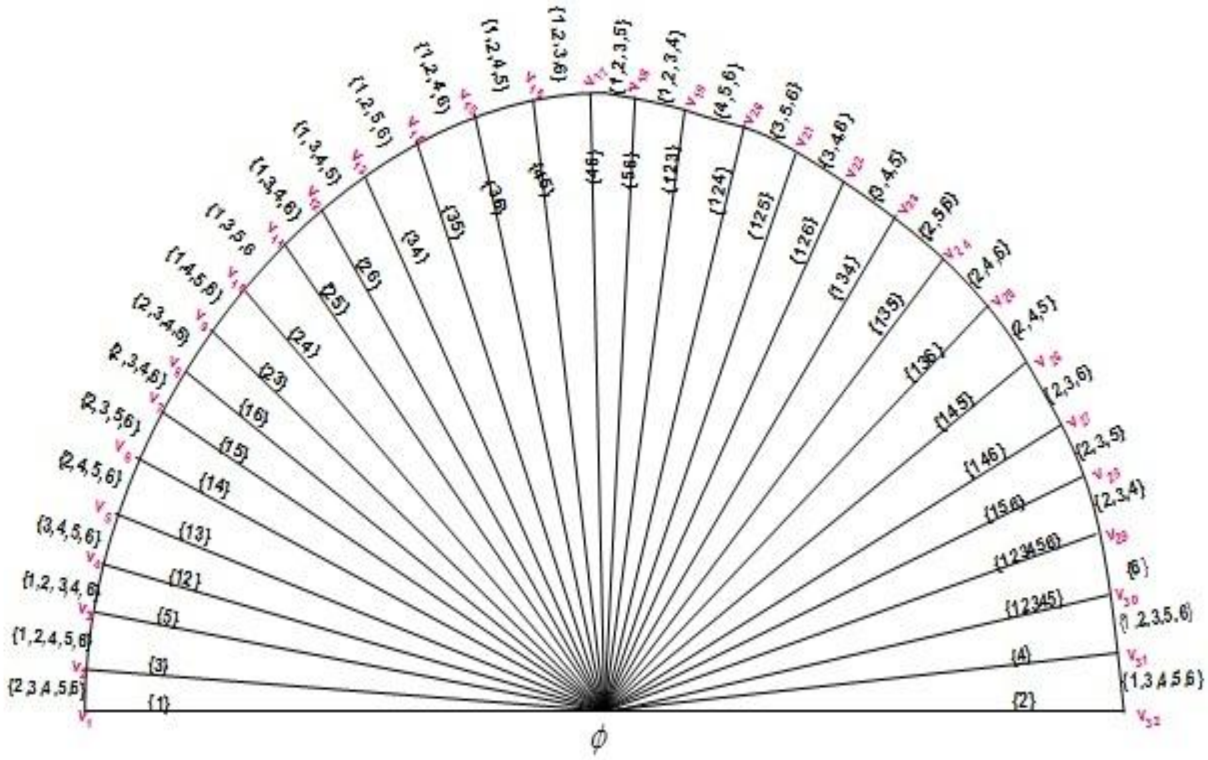

Figure 1: *L-edge colouring of* $P_{2^5} + K_1$.

A_1 = Set of all singleton sets assigned

$$=\{l(uv_i):i=1,2,3\}\cup\{l(uv_{2^n+1-i}):i=1,2\}\cup\left\{l(v_iv_{i+1}):i=\frac{2n^2+n+3}{2}\right\}$$

$$=\{\{1\},\{3\},\{5\}\}\cup\{\{2\},\{4\}\}\cup\{6\}$$

$$=\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$$

A_2 = Set of all two element sets assigned

$$=\{l(uv_i):i=4,5,6,7,8\}\cup\{l(uv_i):i=9,10,11,12\}$$
$$\cup\{l(uv_i):i=13,14,15\}\cup\{l(uv_i):i=16,17\}\cup\{l(uv_i):i=18\}$$

$$=\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\}\}\cup\{\{2,3\},\{2,4\},\{2,5\},\{2,6\}\}$$
$$\cup\{\{3,4\},\{3,5\},\{3,6\}\}\cup\{\{4,5\},\{4,6\}\}\cup\{5,6\}$$

$$=\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},$$
$$\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$$

A_3 = set of all three element sets assigned

$$=\{l(uv_i):i=19,20,\ldots 28\}\cup\{l(v_i,v_{i+1}):i=19,20,\ldots 28\}$$

$$=\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\}\}$$
$$\cup\{\{4,5,6\},\{3,5,6\},\{3,4,6\},\{3,4,5\},\{2,5,6\},\{2,4,6\},\{2,4,5\},\{2,3,6\},\{2,3,5\},\{2,3,4\}\}$$

A_4 = set of all four element sets assigned

$$=\{l(v_i,v_{i+1}):i=4,5,6,\ldots 19\}$$

$$=\{\{3,4,5,6\},\{2,4,5,6\},\{2,3,5,6\},\{2,3,4,6\},\{2,3,4,5\},\{1,4,5,6\},\{1,3,5,6\},$$
$$\{1,3,4,6\},\{1,3,4,5\},\{1,2,5,6\},\{1,2,4,6\},\{1,2,4,5\},\{1,2,3,6\},\{1,2,3,5\},\{1,2,3,4\}\}$$

A_5 = set of all five element sets assigned

$$=\{l(v_i,v_{i+1}):i=1,2,3\}\cup\{l(v_i,v_{i+1}):i=30,31\}\cup\{l(u,v_{30})\}$$

$$=\{\{2,3,4,5,6\},\{1,2,4,5,6\},\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,3,4,5,6\},\{1,2,3,4,5\}\}$$

$$A_6=\{l(uv_{29})\}=\{1,2,3,4,5,6\}$$

It is noted that $A_i,\ i=1,2,3,4,5,6$ are distinct and $\bigcup_{i=1}^{i=6}A_i=P(X)\setminus\phi$. From the above labeling it is clear that $P_{2^5}+K_1$ is exactly null set magic and hence L-edge colourable.

Observation 2.5. Some graphs are L- edge colourable but fail to be topogenic. K_4 is L-edge colourable since the edges of K_4 can be assigned non empty subsets of $X=\{1,2\}$ such that no two adjacent edges receive the same label.

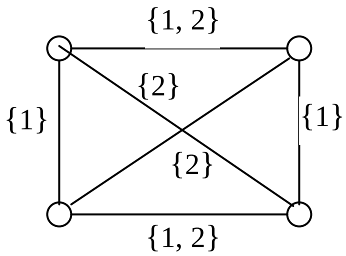

Figure 2: *L-edge colouring of* K_4.

It is proved that K_4 is not topogenic in [1].

3 Conclusion

Interrelation between null set magic labeling and *L*-edge colouring seems to be much interesting as well as application oriented. Using *L*-edge colouring, *L*-edge chromatic number may be defined. A trial towards determining the relation between the edge chromatic number and *L*-edge chromatic number may be much fruitful.

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