International Journal of Mathematics and Soft Computing
Vol.3, No.2 (2013), 17 - 20.

ISSN Print : 2249 – 3328
ISSN Online: 2319 – 5215

On disconnected domination number of a graph

S. Balamurugan
PG & Research Department of Mathematics,
Raja Doraisingam Government Arts College,
Sivagangai – 630 561, INDIA.
Email: balapoojaa2009@gmail.com

G. Prabakaran
Research Department of Mathematics,
Thiagarajar College, Madurai - 625009, INDIA.
E-mail: tcmath@gmail.com

Abstract

A dominating set D of graph $G = (V, E)$ is a disconnected dominating set, if the induced subgraph < D > is disconnected. The disconnected dominating number $\gamma_{dc}(G)$ of G is the minimum cardinality of a disconnected dominating set of G. In this paper, we relate this parameter to other parameters of graph G and obtain some bounds also.

Keywords: Domination, disconnected domination, independence number, disconnected domination number.

AMS Subject Classification (2010): 05C69.

1 Introduction

All the graphs considered in this paper are non-complete and of order $n \geq 2$. Let $G = (V, E)$ be a non complete graph of order at least 2. A subset D of vertices is a dominating set, if every vertex v in V-D is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. It is clear that every isolated vertex is in every dominating set. A comprehensive survy on the fundamentals of domination is given by Haynes et al [1]. Prof . E. Sampathkumar [3] introduced the concept of connected domination in graphs. A set $D \subseteq V$ is called a connected dominating set, if D is a dominating set and < D > is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. A set $D \subseteq V$ is said to be a disconnected dominating set if the induced subgraph < D > is disconnected. The disconnected dominating number $\gamma_{dc}(G)$ of G is the minimum cardinality of a disconnected dominating set of G. Two vertices are said to be independent, if they are not adjacent. A set D of vertices is said to be an independent set, if no two vertices of D are adjacent. The maximum cardinality of an independent set of G is called the independence number of G and is denoted by $\alpha_0(G)$.

Remark 1.1. In $K_{n,}$ no vertex subset is disconnected. Therefore, disconnected dominating set does not exist in complete graphs. Hence, we consider only non complete graphs.

Definition 1.2. A dominating set D of graph $G = (V, E)$ is a disconnected dominating set, if the induced subgraph $< D >$ is disconnected. The disconnected dominating number $\gamma_{dc}(G)$ of G is the minimum cardinality of a disconnected dominating set of G.

Proposition 1.3.

1. For any complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$,
   $\gamma_{dc}(K_{m,n}) = m$.
2. For any cycle C_n with $n \geq 4$,
   $\gamma_{dc}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.
3. For any path P_n with $n \geq 3$,
   $\gamma_{dc}(P_n) = \begin{cases} 2 & \text{if } n = 3 \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } n \geq 4 \end{cases}$
4. For any star $K_{1,n-1}$ with $n \geq 3$,
   $\gamma_{dc}(K_{1,n-1}) = n-1$.
5. For any double star $D_{r,s}$ with $2 \leq r \leq s$,
   $\gamma_{dc}(D_{r,s}) = r+1$.

2 Main Results

Theorem 2.1. For any graph G, $\gamma(G) \leq \gamma_{dc}(G) \leq \alpha_0(G)$, where $\alpha_0(G)$ is the independence number of G.

Proof: Clearly, $\gamma(G) \leq \gamma_{dc}(G)$.

Let D be a maximum independent set of vertices in G. Since G is not complete, D has at least two vertices. Since D is a maximum independent set, every vertex in $V - D$ is adjacent to some vertex in D. This implies that D is a dominating set. Since the vertices in D are independent, D is a disconnected dominating set. Therefore, $\gamma_{dc}(G) \leq \alpha_0(G)$. ■

Theorem 2.2. For any graph G, $\gamma_{dc}(G) = n$ if and only if $G = \overline{K_n}$.

Proof: If $G = \overline{K_n}$, then $\gamma_{dc}(G) = n$.

Let G be a graph with $\gamma_{dc}(G) = n$. If all the vertices are isolates, then $G = \overline{K_n}$. If G has non isolated vertices, then the set D of independent vertices is a disconnected dominating set and $|S| < n$, a contradiction. ■

Theorem 2.3. Given any integer $k \geq 0$, there exists a graph G such that $\gamma_{dc}(G) - \gamma(G) = k$.

Proof: Let G be the complete bipartite graph $K_{k+2,k+3}$. Then $\gamma(G) = 2$ and $\gamma_{dc}(G) = k+2$.

Therefore, $\gamma_{dc}(G)$ - $\gamma_{dc}(G)$ = $(k + 2) – 2 = k$. ■

Theorem 2.4. Given any integer $k \geq 0$, there exists a graph G such that $\alpha_0(G) - \gamma_{dc}(G) = k$.

Proof: Let G be the complete bipartite graph $K_{2, k+2}$. Then $\alpha_0(G) = k+2$ and $\gamma_{dc}(G) = 2$. Therefore, $\alpha_0(G) - \gamma_{dc}(G) = (k + 2) - 2 = k$. ■

Theorem 2.5. For any two integers a, b with $2 \leq a \leq b$, there exists a graph G such that $\gamma_{dc}(G) = a$ and $\alpha_0(G) = b$.

Proof: Let G be the complete bipartite graph $K_{a, b}$. Then $\gamma_{dc}(G) = \min\{a, b\} = a$ and $\alpha_0(G) = \max\{a, b\} = b$. Therefore, $\gamma_{dc}(G) = a$ and $\alpha_0(G) = b$. ■

Theorem 2.6. Let G be a graph. Then $\gamma_{dc}(G) = n - 1$ if and only if $G = K_{1, n-1}$ or $G = K_2 \cup (n - 2) K_1$.

Proof: If $G = K_{1,n-1,}$ then $\gamma_{dc}(G) = n$-1 or if $G = K_2 \cup (n - 2) K_{1,}$ then $\gamma_{dc}(G) = n$-1. Conversely, let G be a graph with $\gamma_{dc}(G) = n - 1$. ■

Let D be a γ_{dc}-set of G and $V – D = \{v\}$.

We claim that $< D >$ has no edge.

Suppose there exist two vertices x and y in D such that $xy \in E(G)$.

Case (i): v is adjacent to exactly one of x and y.

Let $xv \in E(G)$ and $vy \notin E(G)$. Then $D_1 = D – \{y\}$ is a disconnected dominating set of cardinality less than $\gamma_{dc}(G)$, which is a contradiction.

Case (ii): v is adjacent to neither x nor y.

In this case, $vx \notin E(G)$ and $vy \notin E(G)$. Then $D_1 = D – \{x\}$ or $D – \{v\}$ is a disconnected dominating set of cardinality less than $\gamma_{dc}(G)$, which is a contradiction.

Case (iii): v is adjacent to both x and y.

In this case, $vx, vy \in E(G)$. Then $D_1 = D$-$\{x\}$ or D-$\{y\}$ is a disconnected dominating of cardinality less than $\gamma_{dc}(G)$, which is a contradiction. Hence, $< D >$ has no edges.

Next, we show that $G = K_{1,n-1}$ or $G = K_2 \cup (n-2) K_{1.}$

Case (a): G is connected.

Let $D = \{u_1, u_2, \ldots, u_{n-1}\}$ be a γ_{dc}-set of G. Since D is a dominating set, v is adjacent to some u_i, $1 \leq i \leq n$-1. Suppose there exists $u_{j,}$ $1 \leq j \leq n$-1 such that u_j is not adjacent to v. Then there is no path between u_j and v, which contradicts the assumption that G is connected.

Therefore, each u_i is adjacent to v, $1 \leq i \leq n$-1. Hence $G = K_{1, n-1}$.

Case (b): G is disconnected.

Since $\gamma_{dc}(G) = n$-1 $< n$. $G \neq \overline{K_n}$.

Therefore, G has edges. Since $< D >$ is disconnected and $|V - D| = 1$, every edge has one end in D and other end in $V – D$.

Suppose there exist $x_{1,} x_{2,\ldots,} x_r$ in D such that $vx_i \in E(G)$, $1 \le i \le r$.
If $r = n$-1, then $G = K_{1,n\text{-}1,}$ a connected graph, a contradiction. Therefore, $r \le n$-2.
If $r \ge 2$, then $D_1 = D - \{x_{1,} x_{2,\ldots,} x_r\} \cup \{v\}$ is a disconnected dominating set of cardinality $n - r \le n$-2, which is a contradiction.
Therefore, $r = 1$ and hence $G = K_2 \cup (n\text{-}2)\, K_{1.}$ ■

References

[1] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Docker Inc.. New york, 1998.

[2] S.T. Hedetniemi and Renu Lasker, *Connected domination in Graphs*, Graph Theory and Combinatorics, Academic Press, London(1984), 209-217.

[3] E. Sambathkumar and H. B. Walikar, *The connected domination of a graph*, Math. Phys. Sci. 13(1979), 607-613.