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Total complementary acyclic domination in graphs

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Abstract

Let $G = (V, E)$ be a graph without isolates. A subset S of $V(G)$ is called a total dominating set of G if for every $v \in V$, there exists $u \in S$ such that u and v are adjacent. S is called a total complementary acyclic dominating set of G, if S is a total dominating set of G and $\langle V - S\rangle$ is acyclic. $V(G)$ is a total complementary acyclic dominating set of G (since G has no isolates). The minimum cardinality of a total complementary acyclic dominating set of G is called the total complementary acyclic domination number of G and is denoted by $\gamma^t_{c-a}(G)$. In this paper, characterization of graphs for which $\gamma^t_{c-a}(G)$ takes specific values are found.

Keywords: Domination, total domination, total complementary acyclic dominating set, total complementary acyclic domination number.
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1 Introduction

By a graph, we mean a finite, undirected, without loops and multiple edges. A set S of vertices of a graph $G = (V, E)$ is a dominating set of G if every vertex in $V - S$ is adjacent to some vertex of S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. Let G be a graph without isolated vertices. A set $S \subseteq V(G)$ is a total dominating set if every vertex in $V(G)$ is adjacent to a vertex in S. Every graph without isolated vertices has a total dominating set , since $S = V(G)$ is such a set. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of cardinality $\gamma_t(G)$, we call a $\gamma_t(G)$-set. Total domination in graphs was introduced by Cockayne, Dawes and Hedetniemi[1] and is well studied in graph theory. A detailed survey on this subject is available in Slater[3,4].

Definition 1.1. Let G be a graph without isolates. A subset S of $V(G)$ is said to be a total complementary acyclic dominating set(total c-a dominating set) of G if S is a total dominating set of G and $\langle V - S\rangle$ is acyclic.
For any graph G without isolates, $V(G)$ is a total c-a dominating set of G.

Definition 1.2. The minimum cardinality of a total c-a dominating set of G is called the total c-a domination number of G and is denoted by $\gamma^t_{c-a}(G)$.

Example 1.3. A total c-a dominating set of a graph G is given below.

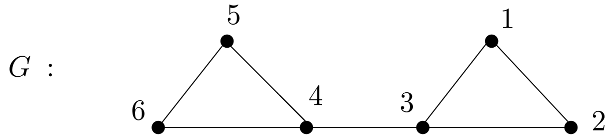

Figure 1: *Total c-a dominating set of G.*

$S = \{3, 4\}$ is a total dominating set and $\langle V - S\rangle$ is acyclic. Therefore, S is a total c-a dominating set.

Definition 1.4. A total c-a dominating set S of G is minimal if no proper subset of S is a total c-a dominating set of G.

Remark 1.5. Any superset of a total c-a dominating set of G is also a total c-a dominating set of G, since if S is a total c-a dominating set of G and $u \in V - S$, then $S \cup \{u\}$ is a total c-a dominating set of G. Therefore, total c-a domination is superhereditary.

Remark 1.6. A total c-a dominating set of G is minimal iff it is 1-minimal.

2 Main Results

In this section, some results on minimal total c-a dominating set and $\gamma^t_{c-a}(G)$ for standard graphs are discussed.

Theorem 2.1. A total c-a dominating set D of G is minimal if and only if for each vertex $u \in D$, one of the following conditions holds.

1. u has a private neighbour in $V - D$.
2. $\langle (V - D) \cup \{u\}\rangle$ contains a cycle.

Proof: Let D be a total c-a dominating set of G. Suppose D is minimal.
Let $u \in D$. Then $D - \{u\}$ is not a total c-a dominating set of G.
Therefore, $\langle (V - D) \cup \{u\}\rangle$ contains a cycle or u has a private neighbour in $V - D$ with respect to D.
Conversely, suppose for every u in D, one of the conditions holds.
If (1) holds, then $D - \{u\}$ is not a dominating set.
If (2) holds, then $D - \{u\}$ is not a complementary acyclic. Therefore, D is a minimal total c-a dominating set of G. ■

Lemma 2.2.

1. $\gamma^t_{c-a}(K_n) = n - 2,\ n \geq 4.$
2. $\gamma^t_{c-a}(K_{1,n}) = 2$
3. $\gamma^t_{c-a}(D_{r,s}) = 2$

4. $\gamma^t_{c-a}(W_n) = 2$

5. $\gamma^t_{c-a}(P_n) = \begin{cases} \frac{n}{2} & if\ n \equiv 0\ (mod\ 4) \\ \lfloor \frac{n}{2} \rfloor + 1 & otherwise \end{cases}$

Theorem 2.3. For any cycle C_n,

$$\gamma^t_{c-a}(C_n) = \begin{cases} \frac{n}{2} & if\ n \equiv 0\ (mod\ 4) \\ \frac{n}{2} + 1 & if\ n \equiv 2\ (mod\ 4) \\ \lceil \frac{n}{2} \rceil & if\ n\ is\ odd,\ n \geq 3 \end{cases}$$

Proof: **Case (i):** $n \equiv 0(mod\ 4)$.
Let $n = 4k$, $k \geq 1$.
Let $V(C_n) = \{v_1, v_2, ..., v_{4k}\}$ and $D = \{v_1, v_2, v_5, v_6, ..., v_{4k-3}, v_{4k-2}\}$. Then $|D| = 2k$. Clearly D is a total c-a dominating set of C_n. Therefore, $\gamma^t_{c-a}(C_n) \leq 2k$.

Let D_1 be a minimum total c-a dominating set of C_n. Let $v_i \in D_1$. Then either v_{i-1} or $v_{i+1} \in D_1$. If D_1 contains $k-1$ pairs of adjacent vertices, then D_1 can totally dominate at most 4k-4 vertices of C_n, which is a contradiction. Therefore, $|D_1| \geq 2k$.
Hence, $\gamma^t_{c-a}(C_n) = 2k = \frac{n}{2}$.
Case (ii): $n \equiv 2(mod\ 4)$.
Let $n = 4k + 2$, $k \geq 1$.
Let $V(C_n) = \{v_1, v_2, ..., v_{4k+2}\}$ and $D = \{v_1, v_2, v_5, v_6, ..., v_{4k+1}, v_{4k+2}\}$. Then $|D| = 2k + 2$. Clearly D is a total c-a dominating set of C_n. Therefore, $\gamma^t_{c-a}(C_n) \leq 2k + 2$.

Let D_1 be a minimum total c-a dominating set of C_n. Let $v_i \in D_1$. Then either v_{i-1} or $v_{i+1} \in D_1$. If D_1 contains k pairs of adjacent vertices, then D_1 can totally dominate at most 4k vertices of C_n, which is a contradiction. Therefore, $|D_1| \geq 2k + 2$.
Hence, $\gamma^t_{c-a}(C_n) = 2k + 2 = \frac{n}{2} + 1$.
Case (iii): $n \equiv 1(mod\ 4)$.
Let $n = 4k + 1$, $k \geq 1$.
Let $V(C_n) = \{v_1, v_2, ..., v_{4k+1}\}$ and $D = \{v_1, v_2, v_5, v_6, ..., v_{4k+1}\}$. Then $|D| = 2k + 1$. Clearly D is a total c-a dominating set of C_n. Therefore, $\gamma^t_{c-a}(C_n) \leq 2k + 1$.

Let D_1 be a minimum total c-a dominating set of C_n. Let $v_i \in D_1$. Then either v_{i-1} or $v_{i+1} \in D_1$. If D_1 contains k pairs of adjacent vertices, then D_1 can totally dominate at most 4k vertices of C_n, a contradiction. Therefore, $|D_1| \geq 2k + 1$.
Hence, $\gamma^t_{c-a}(C_n) = 2k + 1 = \lceil \frac{n}{2} \rceil$.
Case (iv): $n \equiv 3(mod\ 4)$.
Let $n = 4k + 3$, $k \geq 0$.
Let $V(C_n) = \{v_1, v_2, ..., v_{4k+3}\}$ and $D = \{v_1, v_2, v_5, v_6, ..., v_{4k+1}, v_{4k+2}\}$. Then $|D| = 2k + 2$.

Clearly D is a total c-a dominating set of C_n. Therefore, $\gamma^t_{c-a}(C_n) \leq 2k+2$.

Let D_1 be a minimum total c-a dominating set of C_n. Let $v_i \in D_1$. Then either v_{i-1} or $v_{i+1} \in D_1$. If D_1 contains k pairs of adjacent vertices, then D_1 can totally dominate at most 4k vertices of C_n, a contradiction. Therefore, $|D_1| \geq 2k+2$.
Hence, $\gamma^t_{c-a}(C_n) = 2k+2 = \lceil \frac{n}{2} \rceil$. ■

Theorem 2.4. $\gamma^t_{c-a}(K_{m,n}) = min\{m, n\}$, $m, n \geq 2$.

Proof: Let $G = K_{m,n}$, $m, n \geq 2$.
Let $m \leq n$ Let V_1, V_2 be the bipartite sets of G.
Let $|V_1| = m$ and $|V_2| = n$.

Case (i): $m = 2$.

Choose a vertex u from V_1 and a vertex v from V_2. Then $\{u, v\}$ is a total c-a dominating set of G. Hence, $\gamma^t_{c-a}(G) = 2 = min\{m, n\}$.
Case (ii): $m \geq 3$.

Let $D = V_1 - \{u\} \cup \{v\}$, where $u \in V_1$ and $v \in V_2$. Then D is a total c-a dominating set of G. Therefore, $\gamma^t_{c-a}(G) \leq m$.

Let D_1 be a minimum total c-a dominating set of G. If D_1 contains $m-2$ or less points from V_1, then the remaining vertices of V_1 will form a cycle with any set of vertices of V_2 of cardinality greater than or equal to 2. Therefore, D_1 contains at least $m-1$ vertices from V_1. For total domination, D_1 contains at least one vertex from V_2 and hence $|D_1| \geq m = min\{m, n\}$. Therefore, $\gamma^t_{c-a}(G) = m = min\{m, n\}$. ■

Theorem 2.5. $\gamma^t_{c-a}(G) = 2$ iff G is obtained from an acyclic graph H of cardinality $n-2$ and adding two vertices to H and making them adjacent and dominating H.

Proof: Obvious. ■

Theorem 2.6. $\gamma^t_{c-a}(G) = n$ if and only if G = $\frac{n}{2}K_2$

Proof: Suppose $\gamma^t_{c-a}(G) = n$
Case(i): G is connected.

If G is of order 2, then $G = K_2$ and $\gamma^t_{c-a}(G) = 2$. Suppose G is of order greater than or equal to 3. Then for any u in $V(G)$, $V(G)$-$\{u\}$ is an acyclic dominating set of G. If $V(G)$-$\{u\}$ is independent, then u is adjacent with every vertex of $V(G)$-$\{u\}$. Therefore, G is a star of order greater than or equal to 3. In this case $\gamma^t_{c-a}(G) = 2 < |V(G)|$, a contradiction. Therefore, $V(G)$-$\{u\}$ is not independent. Therefore, $V(G)$-$\{u\}$ is a c-a dominating set of G. Let $N_i(u)$ be the set all independent neighbours of u. If $N_i(u) \geq 2$ and any vertex in $N_i(u)$ is not adjacent with any vertex in $V(G)$-$\{u\}$,then $(V(G) - N_i(u)) \cup \{u, u_1\}$ is a total c-a dominating set of G. If $N_i(u) = \{u_1\}$ and $|N(u)| \geq 3$, then there are atleast two adjacent neighbours of u say u_2, u_3. Then $V(G)$-$\{u_2, u_3\}$ is a total c-a dominating set of G.If $|N(u)|$ = 2, then $N_i(u) = \{u_1, u_2\}$, a contradiction. If $N(u) = 1$,then $\langle\{u, u_1\}\rangle$ is a component of G, a contradiction.In this case $\gamma^t_{c-a}(G) < n$, a contradiction.Therefore, G is of order 2 and hence $G = K_2$.

Case(ii): G is disonnected.

$\gamma^t_{c-a}(G) = \sum_{i=1}^{k} \gamma^t_{c-a}(G_i)$, where $G_1, G_2, ..., G_k$ are the components of G, $k \geq 2$. Since $\gamma^t_{c-a}(G) = n$, $\gamma^t_{c-a}(G_i) = |V(G_i|$, $1 \leq i \leq k$. Arguing as in Case(i), we get that $G_i = K_2$ for $1 \leq i \leq k$. Therefore, G is of even order and $G = \frac{n}{2}K_2$.

The converse is obvious. ∎

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