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Domination number of distance k-complement graph

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Abstract

In this paper the extremal graphs on the domination number of the distance k-complement graph G_k^c of a graph G are characterized and the relation between the domination number of G_k^c and the distance k-domination number of G are established.

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1 Introduction

Let $G = (V, E)$ be a simple graph of order p and size q. The distance $d_G(u, v)$ or $d(u, v)$ between two veritecs u and v in G is the length of a shortest path joining u and v in the graph G. For$u \in V(G)$, let $N(u) = \{v \in V(G) : uv \in E(G)\}$ and $d(u) = |N(u)|$. The eccentricity of a vertex u is given by $e(u)$ =max$\{d(u, v) : v \in V(G)\}$. The radius and diameter of of the graph G are given by $rad(G) = min\{e(u) : u \in V(G)\}$ and $diam(G) = max\{e(u) : u \in V(G)\}$ respectively. A vertex with eccentricity equal to $rad(G)$ is called central vertex. A vertex v is called an eccentric vertex of some vertex u in G if $d_G(u, v) = e(u)$. The set of all eccentric vertices of a vertex u is denoted by $C(u)$. The corona H^+ of a graph H is a graph obtained from H by adding new vertex u' corresponding to each vertex u in H and by joining u and u' by an edge. The complement graph $\overline{G}$ is a graph with vertex set $V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if $d_G(u, v) > 1$. The antipodal graph $A(G)$ of a graph G is a graph with vertex set $V(G)$ and two vertices u and v are adjacent in $A(G)$ if and only if $d_G(u, v) = diam(G)$. In [3] the authors introduced the concept of distance k-complement graph, a natural generalization of the complement graph and hence the antipodal graph and in this paper the extremal graphs on the domination number of the distance k-complement graph G_k^c of a graph G are characterized and the relation between the domination number of G_k^c and the distance k-domination number of G are established.

Definition 1.1. For an integer $k \geq 0$, the distance k-complement graph, denoted by G_k^c, of a graph $G = (V,\ E)$ is a graph whose vertex set is $V(G)$ and two vertices are adjacent in G_k^c if and only if $d_G(u, v) > k$. That is, the edge set is $\{uv : d_G(u,\ v) > k\}$.

Definition 1.2. For $k \geq 1$, a graph G is said to be a distance k-complete graph if $diam(G) \leq k$.

Theorem 1.3. [3] Let G be a graph. Then $k \leq rad(G) - 1$ if and only if G_k^c has no isolated vertices.

2 Domination in G_k^c

Definition 2.1. Let $G = (V, E)$ be a graph. A subset D of V is said to be a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$.

Definition 2.2. Let $G = (V, E)$ be a graph and $k \geq 1$ be an integer. A subset D of V is said to be a distance k-dominating set of G if for every vertex u in $V - D$, there is some vertex v in D such that $d_G(u, v) \leq k$. The minimum cardinality of a distance k-dominating set is called the distance k-domination number of G and is denoted by $\gamma_k(G)$.

Definition 2.3. A partition $V_1, V_2, \cdots, V_l$ of $V(G)$ of a graph G is said to be a domatic partition of G if each V_i is a dominating set of G. The maximum size of a such partition is called the domatic number of G and is denoted by $d(G)$.

By using the following theorem, we characterize the extremal graphs attaining the upper bound on the domination number of the distance k-complement graph.

Theorem 2.4. [2] For a graph G with even order p and no isolated vertices, $\gamma(G) = \frac{p}{2}$ if and only if the components of G are the cycle C_4 or the corona H^+ for any connected graph H.

Proposition 2.5. A graph G has an isolated vertex if and only if $\gamma(G_k^c) = 1$.

Proposition 2.6. Let G be a graph of order p. If $k \leq rad(G) - 1$, then $\gamma(G_k^c) \leq \lfloor \frac{p}{2} \rfloor$.

Proof: This follows from Theorem1.3 and the classical Ore's Theorem [2] on the bound on domination number of a graph. ∎

Theorem 2.7. Let $G = (V, E)$ be graph and $k \leq rad(G) - 1$ be an integer. Then the components of G_k^c are the cycle C_4 or the corona H^+ for some connected graph H if and only if there exists a partition V_1 and V_2 of $V(G)$ such that $|V_1| = |V_2|$ and at least one of V_1, V_2, say V_1 has the following property: Every vertex u in V_1 has either the unique eccentric vertex x in V_2 or has two eccentric vertices x_1, x_2 in V_2 with $c(x_1) = c(x_2) = \{u, v\}$ and $c(u) = c(v) = \{x_1, x_2\}$ for some vertex v in V_1 and $rad(G) = k+1$.

Proof: Let the components of G_k^c be the cycle C_4 or the corona H^+ for some connected graph H. Let V_1 be the set of all pendant vertices of H^+ if $H \neq K_1$, one pendant vertex of K_1^+ and the vertices in one partite set of C_4 and let $V_2 = V(G) - V_1$. Let $u \in V_1$. If u is a pendant vertex of some H^+ and x is its support vertex, then x is the unique eccentric vertex of u in G and $k = rad(G) - 1$. Let $d(u) = 2$ in G_k^c. Then there exist two vertices x_1 and x_2 such that $d(u, x_1) \geq k + 1$ and $d(u, x_2) \geq k + 1$. Since $d(u) = 2$ in G_k^c, $d(u, x_1) \leq k + 2$ and $d(u, x_2) \leq k + 2$. If $d(u, x_1) = k + 2$ and $d(u, x_2) = k + 2$, then there exists a vertex y such that $d_G(u, y) = k + 1$. Then $d(u) \geq 3$ in G_k^c which is a contradiction. Therefore, we have $d(u, x_1) = k + 1$ and $d(u, x_2) \leq k + 2$.

Suppose that $d(u, x_2) = k + 2$. If there exists a shortest $u - x_2$ path not containing x_1 in G, then there exists a vertex $y \neq x_1$ on the $u - x_2$ path such that $d_G(u, y) = k + 1$. Thus, $d(u) \geq 3$ in G_k^c which is a contradiction. Therefore, x_1 is on every shortest $u - x_2$ path in G. Let y be a vertex on a shortest $u-x_2$ path such that $d_G(u,y)=k$. Since G_k^c has no isolates, there exists a vertex z such that $d_G(y,z)=k+1$. If $w\in N(x_2)$ such that w lies on a shortest x_2-z path, then $d_G(u,w)\geq k+1$, since $d_G(u,x_2)=k+2$. If $x_1\neq w$, then $d(u)\geq 3$ in G_k^c which is a contradiction. Therefore, $w=x_1$. If $d_G(z,x_1)\leq k-1$, then the path containing the $z-x_1$ path followed by the edge x_1y is of length at most k which is a contradiction to $d_G(y,z)=k+1$. Thus, $d_G(z,x_1)\geq k$ and hence $d_G(z,x_2)\geq k+1$, since x_1 lies on every $z-x_2$ shortest path. Let u_1 be a vertex on the $u-x_2$ path such that $d_G(u_1,x_2)=k+1$. Since $d_G(u,y)=k$ and $d_G(z,y)=k+1$, $z\neq u_1$ and $z\neq u_1$. Therefore, $d(x_2)\geq 3$ in G_k^c which is a contradiction to x_2 lies on C_4. Thus, $d(u,x_1)=k+1$ and $d(u,x_2)=k+1$. Hence $c(u)=\{x_1,x_2\}$. If v is the other vertex of C_4, then $c(v)=\{x_1,x_2\}$. Hence $c(x_1)=c(x_2)=\{u,v\}$.

Conversely, let V_1 and V_2 be the partition of $V(G)$ having the given property and $k=rad(G)-1$. Then the vertices u,v,x_1 and x_2 induce the cycles C_4 in G_k^c. If u has the unique eccentric vertex x in G, then u is the pendant vertex and x is its support vertex in G_k^c. Thus, the components of G_k^c are the cycle C_4 or the corona H^+ for some connected graph H. ■

Corollary 2.8. Let $G=(V,E)$ be a graph of order p and $k\leq rad(G)-1$. Then $\gamma(G_k^c)=\frac{p}{2}$ if and only if there exists a partition V_1 and V_2 of $V(G)$ such that at least one of V_1, V_2, say V_1 has the following property: Every vertex u in V_1 has either the unique eccentric vertex x in V_2 or two eccentric vertices x_1,x_2 in V_2 with $c(x_1)=c(x_2)=\{u,v\}$ and $c(u)=c(v)=\{x_1,x_2\}$ for some v in V_1 and $rad(G)=k+1$.

Theorem 2.9. If a graph G has no isolated veritces and $diam(G)\geq 2k+1$, then $\gamma(G_k^c)=2$.

Proof: Since G has no full degree vertices, $\gamma(G_k^c)\geq 2$ by Proposition 2.5. Let u and v be two diametrical opposite vertices of G. Then $d_G(u,v)\geq 2k+1$ and for every $w\in V(G)-\{u,v\}$, $d_G(u,w)>k$ or $d_G(w,v)>k$. Thus $\{u,v\}$ is a dominating set of G_k^c. Hence, $\gamma(G_k^c)=2$. ■

Theorem 2.10. Let G be a graph of order p. Then for an integer $k\geq 1$, $\gamma_k(G).\gamma(G_k^c)\leq p$.

Proof: For $S\subset V(G)$, define $D_e(S)=\{y\in V-S : d_G(y,x)\leq k$ for every $x\in S\}$ and $D_i(S)=\{y\in S : d_G(y,x)\leq k$ for every $x\in S\}$. Let $d_i(S)=|D_i(S)|$ and let $d_e(S)=|D_e(S)|$. Let $D=\{x_1,x_2,\cdots,x_{\gamma_k}\}$ be a γ_k set of G and partition $V(G)$ into γ_k subsets $B_1,B_2,\cdots,B_{\gamma_k}$ such that $x_j\in B_j$ and all the vertices in B_j are adjacent to x_j, for $1\leq j\leq\gamma_k$. Let P be a such partition for which $\sum\limits_{j=1}^{\gamma_k} d_i(B_j)$ is maximum.

Suppose that $d_e(B_j)\geq 1$. Let $x\in D_e(Bj)$. Then $x\notin B_j$. Since B_j's partition $V(G)$, $x\in B_m$ for some $m\neq j$. If $x\in D_i(B_m)$, then $D-\{x_j,x_m\}\cup\{x\}$ is a distance k-dominating set of G, which is contradiction to D is a γ_k-set of G. Therefore, $x\notin D_i(B_m)$. Now let $B_l'=B_l$ for $l\neq m$ and $l\neq j$ and $B_j'=B_j\cup\{x\}$ and $B_m'=B_m-\{x\}$. Then $\{B_1',B_2',\cdots,B_{\gamma_K}'\}$ is a partition of V such that $\sum\limits_{j=1}^{\gamma_k} d_i(B_j')>\sum\limits_{j=1}^{\gamma_k} d_i(B_j)$, which is a contradiction to our choice of the partition P. Thus $d_e(B_j)=0$ and hence $D_e(B_j)=\phi$ for $1\leq j\leq\gamma_k$. Thus for every $y\notin B_j$, there exists $x\in B_j$ such that $d_G(x,y)\geq k+1$. Then x and y are adjacent in G_k^c. Thus, each B_j is a dominating set of G_k^c. Therefore $\gamma(G_k^c)\leq|B_j|$ and hence, $p=\sum\limits_{j=1}^{\gamma_k}|B_j|\geq\sum\limits_{j=1}^{\gamma_k}\gamma(G_k^c)=\gamma_k(G)\gamma(G_k^c)$. ■

In the proof of the above theorem, the partition P is a domatic partition of G_k^c. It gives the lower bound for the domatic number of G_k^c..

Corollary 2.11. For any graph G, $d(G_k^c) \geq \gamma_k(G)$.

The proof for the following theorems are straight forward as given in [2] for the domination number of G.

Proposition 2.12. Let G be a (p,q) graph and k be an integer. Then $\gamma_k(G) + \gamma(G_k^c) \leq p + 1$. Equality holds if and only if G is either a totally disconnected graph or a distance k-complete graph.

Theorem 2.13. Let G be a (p,q) graph and $k \geq 1$ be an integer. If the radius of each component of G is greater than or equal to $k + 1$, then $\gamma_k(G) + \gamma(G_k^c) \leq \lfloor \frac{p}{2} \rfloor + 2$. The bound is attained by $G = C_{2k+2}$.

Theorem 2.14. Let the radius of each component of G be greater than or equal to $k + 1$. Then for $k \geq 2$, $\gamma_k(G) + \gamma(G_k^c) = \lfloor \frac{p}{2} \rfloor + 2$ if and only if $\gamma(G_k^c) = \lfloor \frac{p}{2} \rfloor$.

Proof: Since $\gamma_k(G) \leq \lfloor \frac{p}{k+1} \rfloor$ and $k \geq 2$, the theorem follows. ■

Theorem 2.15. $G_k^c = H^+$ for some graph H if and only if G has at least $\frac{p}{2}$ central vertices having unique eccentric vertices and $k = rad(G) - 1$.

Proof: Assume that G has at least $\frac{p}{2}$ central vertices having unique eccentric vertices and $k = rad(G) - 1$. Since all the central vertices having unique eccentric verices in G are pendant vertices in G_k^c and the unique eccentric vertices in G are the support vertices in G_k^c , $G_k^c = H^+$ for some graph H.

Conversely, let $G_k^c = H^+$ for some graph H. Then each pendent verices in H^+ must be a central vertex and have unique eccentric vertex in G. Since H^+ has no isolates, $e(u) \geq k + 1$, for every $u \in V(G)$. Since $\delta(G_k^c) = 1$, $rad(G) = k + 1$. In H^+, some of the components may be K_2. Therefore, G has at least $\frac{p}{2}$ central vertices having unique eccentric vertices and $k = rad(G) - 1$. ■

Corollary 2.16. If G has at least $\frac{p}{2}$ central vertices having unique eccentric vertices and $k = rad(G) - 1$, then $\gamma(G_k^c) = \frac{p}{2}$.

References

[1] F. Haray , *Graph Theory*, Addison-Wesley Publishing Company. 1969.

[2] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc.. 1998.

[3] M. Kalanithi, V. Swaminathan and A. Selvam, *Distance k-complement graph of a graph*, Preprint.

[4] B. Rajendran, *Topics in Graph Theory : Antipodal Graphs*, Ph. D. Thesis, Madurai Kamaraj University, 1985.

[5] E. Sampathkumar, *K-dimensional Graph and Semi Graph Theory*, DST Project Report No. DST/MS/131/2K, 2005.