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Graceful and odd graceful labeling of some graphs

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Abstract

In this paper, we prove that the square graph of bistar $B_{n,n}$, the splitting graph of $B_{n,n}$ and the splitting graph of star $K_{1,n}$ are graceful graphs. We also prove that the splitting graph and the shadow graph of bistar $B_{n,n}$ admit odd graceful labeling.

Keywords: Graceful labeling, odd graceful labeling, shadow graph, splitting graph, square graph.
AMS Subject Classification(2010): 05C78.

1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with $|V(G)| = p$ and $|E(G)| = q$. For standard terminology and notation we follow Gross and Yellen [5]. We provide a brief summary of definitions and other information which serve as prerequisites for the present investigation.

Definition 1.1. *If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.*

Graph labelings is an active area of research in graph theory which has rigorous applications in coding theory, communication networks, optimal circuits layouts and graph decomposition problems. According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and the structure of graphs. For a dynamic survey of various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

Definition 1.2. *A function f is called graceful labeling of a graph G if $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. The graph which admits graceful labeling is called a graceful graph.*

Rosa [9] initially called this labeling a $\beta - valuation$ and later Golomb [4] named it as *graceful labeling* which is now the popular term. Several infinite families of graceful and non-graceful graphs have been studied. The famous Ringel-Kotzig tree conjecture [8] and many illustrious work on graceful graphs brought a tide of labeling scheme having graceful theme. Vaidya et al. [11] discussed gracefulness of union of two path graphs with grid graph and complete bipartite graph. Kaneria et al. [6] discussed gracefulness of some classes of disconnected graphs. Vaidya and Lekha [12] investigated graceful labeling of some cycle related graphs. Some variants of graceful labeling were also introduced in recent past, such as edge graceful labeling, Fibonacci graceful labeling, odd graceful labeling and the like.

Definition 1.3. *A function f is called odd graceful labeling of a graph G if $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. The graph which admits odd graceful labeling is called an odd graceful graph.*

The concept of odd graceful labeling was introduced by Gnanajothi [3]. It is possible to decompose the graph $K_{n,n}$ with suitable odd graceful labeling of tree T of order n. The splitting graphs of path P_n and even cycle C_n are proved to be odd graceful by Sekar [10] while ladders and graphs obtained from them by subdividing each step exactly once are shown odd-graceful by Kathiresan [7]. Vaidya and Lekha [13–15] proved many results on odd graceful labeling.

The following three types of problems are considered generally in the area of graph labeling.

1. How particular labeling is affected under various graph operations;
2. Investigation of new families of graphs which admit particular graph labeling;
3. Given a graph theoretic property P, characterizing the class/classes of graphs with property P that admit particular graph labeling.

From the literature survey, it is clear that the problems of second type are largely studied than the problems of first and third types. The present work is aimed to discuss some problems of the first kind in the context of graceful and odd graceful labeling.

Definition 1.4. *For a graph G the splitting graph S' of G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.*

Definition 1.5. *The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex v' in G''.*

Definition 1.6. *For a simple connected graph G the square of graph G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.*

2 Results on graceful labeling

Theorem 2.1. *$B^2_{n,n}$ is a graceful graph.*

Proof. Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i, v_i are the pendant vertices. Let G be the graph $B^2_{n,n}$ then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. We define the vertex labeling $f : V(G) \rightarrow \{0, 1, 2, \ldots, 4n + 1\}$ as follows.

$f(v) = 0,$
$f(u) = 4n + 1,$
$f(v_i) = i; \quad 1 \le i \le n$
$f(u_i) = f(v_n) + i; \quad 1 \le i \le n.$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \to \{1, 2, 3, \ldots, 4n + 1\}$. Thus f is graceful labeling of $G = B^2_{n,n}$. Hence, $B^2_{n,n}$ is a graceful graph. ∎

Illustration 2.2. *Graceful labeling of the graph $B^2_{7,7}$ is shown in Figure 1.*

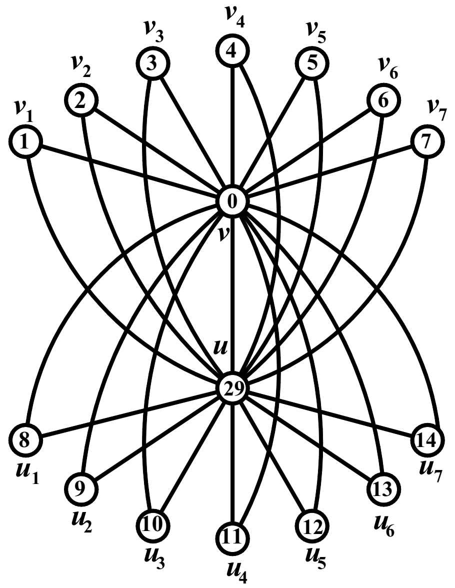

Figure 1: *Graceful labeling of $B^2_{7,7}$.*

Theorem 2.3. *$S'(B_{n,n})$ is a graceful graph.*

Proof. Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \le i \le n\}$ where u_i, v_i are the pendant vertices. In order to obtain $S'(B_{n,n})$, add u', v', u'_i, v'_i vertices corresponding to u, v, u_i, v_i where, $1 \le i \le n$. If $G = S'(B_{n,n})$ then $|V(G)| = 4(n+1)$ and $|E(G)| = 3(2n+1)$. To define the vertex labeling $f : V(G) \to \{0, 1, 2, 3, \ldots, 6n+3\}$, we consider the following two cases.

Case 1: n is even.

$$\begin{array}{lll}
 & f(u'_{1+i}) = n - 1 + i; & 1 \le i \le n-1 \\
f(u) = 2n - 1, & f(u_i) = 1 + 2i; & 1 \le i \le \dfrac{n-2}{2} \\
f(v) = 0, & f\left(u_{\frac{n}{2}-1+i}\right) = 5n + 4 - 2i; & 1 \le i \le \dfrac{n+2}{2} \\
f(u') = 2n, & & \\
f(v') = 1, & f(v'_{1+i}) = 6n + 3 - i; & 0 \le i \le n-1 \\
f(u'_1) = 4n, & f(v_1) = f(v'_n) - 1, & \\
 & f(v_{1+i}) = f(v_1) - 2i; & 1 \le i \le n-1
\end{array}$$

Case 2: n is odd.

$$
\begin{array}{lll}
 & f(u'_{1+i}) = n-1+i; & 1 \le i \le n-1 \\
f(u) = 2n-1, & f(u_i) = 2(i+1); & 1 \le i \le \dfrac{n-3}{2} \\
f(v) = 0, & & \\
f(u') = 2n, & f\left(u_{\frac{n-3}{2}+i}\right) = 5n+4-2i; & 1 \le i \le \dfrac{n+3}{2} \\
f(v') = 1, & f(v'_i) = 6n+4-i; & 1 \le i \le n \\
f(u'_1) = 2, & f(v_1) = f(v'_n) - 1, & \\
 & f(v_{1+i}) = f(v_1) - 2i; & 1 \le i \le n-1
\end{array}
$$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \rightarrow \{1, 2, 3, \ldots, 6n+3\}$. Thus, f is a graceful labeling of $G = S'(B_{n,n})$. Hence, $S'(B_{n,n})$ is a graceful graph. ■

Illustration 2.4. *Graceful labeling of the graph $S'(B_{5,5})$ is shown in Figure 2.*

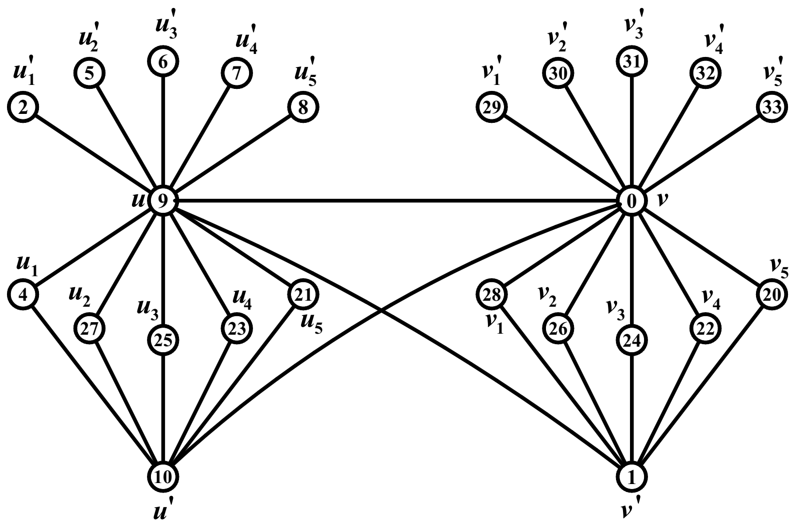

Figure 2: *Graceful labeling of $S'(B_{5,5})$.*

Illustration 2.5. *Graceful labeling of the graph $S'(B_{6,6})$ is shown in Figure 3.*

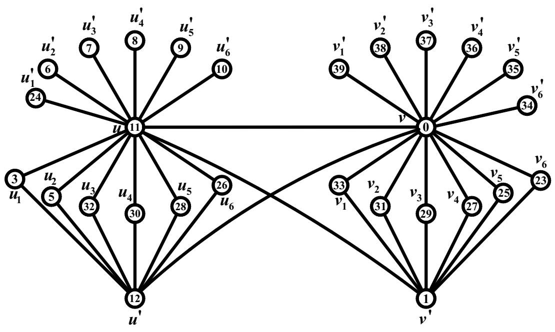

Figure 3: *Graceful labeling of $S'(B_{6,6})$.*

Theorem 2.6. *$S'(K_{1,n})$ is a graceful graph.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, u_3,$

$\ldots, u_n$ be added vertices corresponding to $v, v_1, v_2, v_3, \ldots, v_n$ to obtain $S'(K_{1,n})$. Let G be the graph $S'(K_{1,n})$ then $|V(G)| = 2n + 2$ and $|E(G)| = 3n$. We define vertex labeling $f : V(G) \to \{0, 1, 2, \ldots, 3n\}$ as follows.

$f(u) = 1, f(v) = 0,$

$f(v_i) = 2i; \quad 1 \leq i \leq n$

$f(u_i) = f(v_n) + i; \quad 1 \leq i \leq n$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \to \{1, 2, 3, \ldots, 3n\}$. Thus, f is a graceful labeling of $G = S'(K_{1,n})$. Hence, $S'(K_{1,n})$ is a graceful graph. ■

Illustration 2.7. *Graceful labeling of the graph $S'(K_{1,7})$ is shown in Figure 4.*

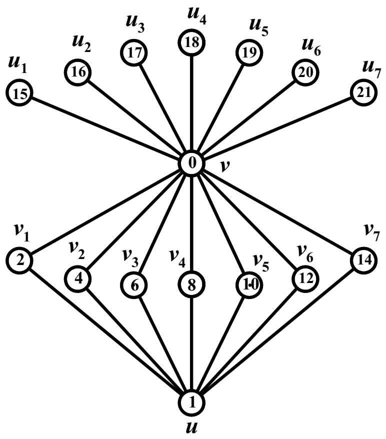

Figure 4: *Graceful labeling of the graph $S'(K_{1,7})$.*

3 Results on Odd Graceful labeling

Theorem 3.1. *$S'(B_{n,n})$ is an odd graceful graph.*

Proof. Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i, v_i are the pendant vertices. In order to obtain $S'(B_{n,n})$, add u', v', u'_i, v'_i vertices corresponding to u, v, u_i, v_i where, $1 \leq i \leq n$. If $G = S'(B_{n,n})$ then $|V(G)| = 4(n + 1)$ and $|E(G)| = 3(2n + 1)$. We define vertex labeling $f : V(G) \to \{0, 1, 2, 3, \ldots, 12n + 5\}$ as follows.

$f(u) = 0,$

$f(v) = 3,$

$f(u') = 2,$

$f(v') = 5,$

$f(u_i) = 5 + 4i; \quad 1 \leq i \leq n$

$f(u'_i) = 12n + 7 - 2i; \quad 1 \leq i \leq n$

$f(v'_1) = f(u'_n) + 1,$

$f(v'_{1+i}) = f(v'_1) - 2i; \quad 1 \leq i \leq n - 1$

$f(v_1) = f(v'_n) - 2,$

$f(v_{1+i}) = f(v_1) - 4i; \quad 1 \leq i \leq n - 1$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \to \{1, 3, \ldots, 12n + 5\}$. Thus, f is an odd graceful labeling of $G = S'(B_{n,n})$. Hence, $S'(B_{n,n})$ is an odd graceful graph. ■

Illustration 3.2. *Odd graceful labeling of the graph $S'(B_{6,6})$ is shown in Figure 5.*

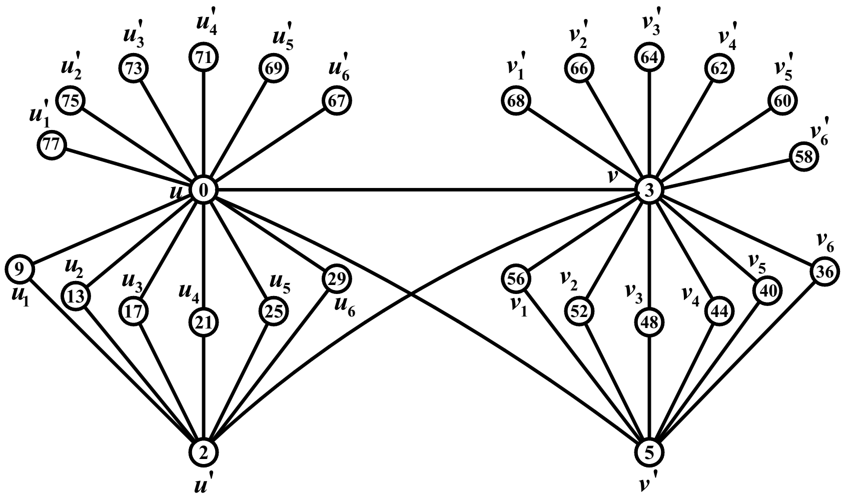

Figure 5: *Odd graceful labeling of $S'(B_{6,6})$.*

Theorem 3.3. *$D_2(B_{n,n})$ is an odd graceful graph.*

Proof. Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i : 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Let G be the graph $D_2(B_{n,n})$. Then $|V(G)| = 4(n + 1)$ and $|E(G)| = 4(2n + 1)$. We define vertex labeling $f : V(G) \to \{0, 1, 2, 3, \ldots, 16n + 7\}$ as follows.

$$f(u) = 2,$$
$$f(v) = 7,$$
$$f(u') = 0,$$
$$f(v') = 3,$$
$$f(u_i) = 16n + 11 - 4i; \quad 1 \leq i \leq n$$
$$f(u'_i) = f(u_n) - 4i; \quad 1 \leq i \leq n$$
$$f(v_{1+2i}) = 16 + 8i; \quad 0 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$$
$$f(v_{2+2i}) = 18 + 8i; \quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$$
$$f(v'_{1+2i}) = f(v_{n-1}) + 8(i + 1); \quad 0 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$$
$$f(v'_{2+2i}) = f(v_n) + 8(i + 1); \quad 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \to \{1, 3, \ldots, 16n + 7\}$. Thus, f is an odd graceful labeling for $G = D_2(B_{n,n})$. Hence, $D_2(B_{n,n})$ is an odd graceful graph. ■

Illustration 3.4. *Odd graceful labeling of the graph* $D_2(B_{n,n})$ *is shown in Figure 6.*

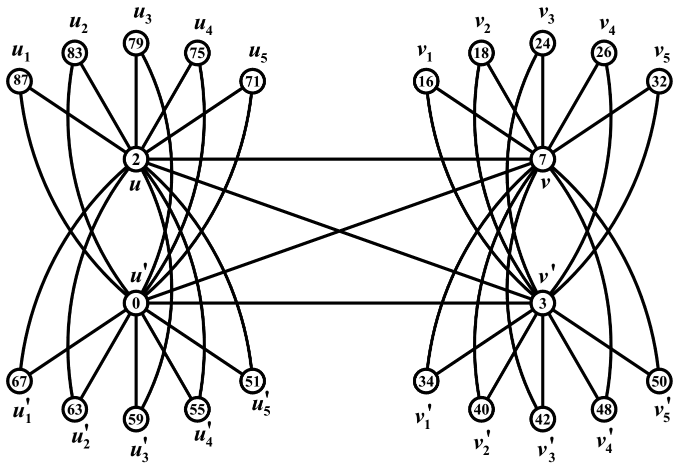

Figure 6: *Odd graceful labeling of the graph* $D_2(B_{n,n})$.

4 Concluding Remarks

Vaidya and Lekha [13] have proved that splitting graph of star $K_{1,n}$ is an odd graceful graph while we prove that it is also a graceful graph. We proved that splitting graph of bistar $B_{n,n}$ is both graceful and odd graceful. Moreover $B^2_{n,n}$ is a graceful graph but not an odd graceful graph as it contains odd cycles.

Rosa [9] proved that $K_{1,n}$ and $B_{n,n}$ are graceful graphs but we prove that the splitting graphs of star $K_{1,n}$ and bistar $B_{n,n}$ are also graceful graphs. Thus, gracefulness remains invariant for the splitting graphs of $K_{1,n}$ and $B_{n,n}$. It is also invariant for square graph of $B_{n,n}$. Moreover odd gracefulness is invariant for the splitting graph and the shadow graph of $B_{n,n}$.

References

[1] L. W. Beineke and S. M. Hegde, *Strongly multiplicative graphs,* Discuss. Math. Graph Theory , 21(2001), 63-75.

[2] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, 18(2011) $\#DS6$.

[3] R. B. Gnanajothi, *Topics in graph theory*, Ph. D Thesis, Madurai Kamaraj University, Madurai, (1991).

[4] S. W. Golomb, *How to number a graph in Graph Theory and Computing* R C Read, ed., Academic Press, New York (1972), 23-37.

[5] J. Gross and J. Yellen, *Graph theory and its applications*, CRC Press, (1999).

[6] V. J. Kaneria, S. K. Vaidya, G. V. Ghodasara and S. Srivastav, *Some classes of disconnected graceful graphs*, Proceedings of the First International Conference on Emerging Technologies and Applications in Engineering, Technology and Sciences, (2008), 1050 - 1056.

[7] K. M. Kathiresan, *Some classes of odd graceful graphs*, ANJAC Journal of Sciences, 7 (1) (2008), 5-10.

[8] G. Ringel, *Problem 25, Theory of graphs and its applications*, Proc. of Symposium Smolenice 1963, Prague, (1964), 162.

[9] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of graphs, International Symposium, Rome, July (1966), Gordon and Breach, New York and Dunod Paris (1967), 349-355.

[10] C. Sekar, *Studies in graph theory*, Ph. D Thesis, Madurai Kamaraj University, Madurai, (2002).

[11] S. K. Vaidya, V. J. Kaneria, S. Srivastav and N. A. Dani, *Gracefulness of union of two path graphs with grid graph and complete bipartite graph*, Proceedings of the First International Conference on Emerging Technologies and Applications in Engineering, Technology and Sciences,(2008), 616 - 619.

[12] S. K. Vaidya and Lekha Bijukumar, *Some new graceful graphs*, International Journal of Mathematics and Soft Computing, 1(1)(2011), 37-45.

[13] S. K. Vaidya and Lekha Bijukumar, *Odd graceful labeling of some new graphs*, Modern Applied Science, 4(10)(2010), 65-70.

[14] S. K. Vaidya and Lekha Bijukumar, *Some new odd graceful graphs*, Advances and Applications in Discrete Mathematics, 6(2)(2010), 101-108.

[15] S. K. Vaidya and Lekha Bijukumar, *New families of odd graceful graphs*, International Journal of Open Problems in Computer Science and Mathematics, 3(5)(2010), 166-171.