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Certain conditions for norm-attainability of elementary operators and derivations

N. B. Okelo
School of Mathematics and Actuarial Science,
Bondo University College, Box 210-40601, Bondo-KENYA.
E-mail: bnyaare@yahoo.com

J. O. Agure, P. O. Oleche
Department of Mathematics, Applied Statistics and Actuarial Science,
Maseno University, P.O Box 333, Maseno - KENYA.
E-mail: johnagure@yahoo.com, poleche@yahoo.com

Abstract

Let H be an infinite dimensional complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. In this paper, we study norm-attainability for elementary operators and generalized derivations. We give results on necessary and sufficient conditions for norm-attainability.

Keywords: Elementary operators, denseness, norm-attainable operators
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1 Introduction

Let H be an infinite dimensional complex Hilbert space and $B(H)$ the C^*-algebra of all bounded linear operators on H. $\mathcal{T} : B(H) \to B(H)$. $\mathcal{T}$ is called an elementary operator if it is represented as $\mathcal{T}(X) = \sum_{i=1}^{n} S_i X T_i,\ \forall\ X \in B(H)$, where $S_i,\ T_i$ are fixed in $B(H)$ or $\mathcal{M}(B(H))$ where $\mathcal{M}(B(H))$ is the multiplier algebra of $B(H)$. For $S,\ T \in B(H)$ we have the following elementary operators: (i) the left multiplication operator $L_S(X) = SX$, (ii) the right multiplication operator $R_T(X) = XT$, (iii) the inner derivation $\delta_S = SX - XS$, (iv) the generalized derivation $\delta_{S,T} = SX - XT$, (v) the basic elementary operator $M_{S,\,T}(X) = SXT$, (vi) the Jordan elementary operator and $\mathcal{U}_{S,\,T}(X) = SXT + TXS,\ \forall\ X \in B(H)$. Stampfli [4] characterized the norm of the generalized derivation by obtaining $\|\delta_{S,T}\| = \inf_{\beta\in\mathbb{C}}\{\|S - \beta\| + \|T - \beta\|\}$, where $\mathbb{C}$ is the complex plane. Further studies on derivations and elementary operators have been carried out in [2 and 3]. In this paper ,we prove that if δ_S^N is norm-attainable then the set $\{S : S \in B(H)\}$ is uniformly dense in $B(H)$. Also we prove that if $\delta_{S,T}^N$, $M_{S,T}^N$ and $\mathcal{U}_{S,\,T}^N$ are norm-attainable then the set $\{S, T : S, T \in B(H)\}$ is uniformly dense in $B(H) \times B(H)$.

2 Notations and Preliminaries

We denote by S^N, δ_S^N, $\delta_{S,T}^N$, $M_{S,T}^N$ and $\mathcal{U}_{S,\,T}^N$ a linear bounded operator, inner derivation, generalized derivation , basic elementary operators and a Jordan elementary operator respectively that are norm-attainable. We note that for $S,\ V \in B(H)$, S is said to be positive if $\langle Sx, x\rangle \geq 0,\ \forall\ x \in H$ and V an isometry(Co-isometry) if $V^*V = VV^* = I$ where I is an identity operator in $B(H)$. A subalgebra $\mathcal{M} \subset B(H)$ is a standard operator algebra on H if it contains $F(H)$ where $F(H)$ is the ideal of finite rank operators. Its clear that $F(H) \subset B(H)$. Moreover, $\mathcal{M}$ is dense in $B(H)$ if $\overline{\mathcal{M}} = B(H)$, where $\overline{\mathcal{M}}$ is the closure of $\mathcal{M}$.

Definition 2.1. An operator $S \in B(H)$ is said to be norm-attainable if there exists a unit vector $x_0 \in H$ such that $\|Sx_0\| = \|S\|$.

Definition 2.2. For an operator $S \in B(H)$ we define:

(i) Numerical range by $W(S) = \{\langle Sx, x\rangle : \ x \in H, \ \|x\| = 1\}$,

(ii) Maximal numerical range by $W_0(S) = \{\beta \in \mathbb{C} : \ \langle Sx_n, x_n\rangle \to \beta, \ where \ \|x_n\| = 1, \ \|Sx_n\| \to \|S\|\}$.

3 Main results

Lemma 3.1. *Let $S \in B(H)$. δ_S is norm-attainable if there exists a vector $\zeta \in H$ such that $\|\zeta\| = 1, \ \|S\zeta\| = \|S\|, \ \langle S\zeta, \zeta\rangle = 0$.*

Proof. For any x satisfying $x \perp \{\zeta, S\zeta\}$, define X as follows $X : \zeta \to \zeta, \quad S\zeta \to -S\zeta, \quad x \to 0$, because $S\zeta \perp \zeta$. Since X is a bounded operator on H and $\|X\zeta\| = \|\zeta\| = 1$, $\|SX\zeta - XS\zeta\| = \|S\zeta - (-S\zeta)\| = 2\|S\zeta\| = 2\|S\|$. Since $\langle S\zeta, \zeta\rangle = 0 \in W_0(S)$, it follows that $\|\delta_S\| = 2\|S\|$ by [4, Theorem 1]. Hence, $\|SX - XS\| = 2\|S\| = \|\delta_S\|$. Therefore, δ_S is norm-attainable. ∎

Theorem 3.2. *Let $S \in B(H), \ \beta \in W_0(S)$ and $\alpha > 0$. There exists an operator $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$. Furthermore, there exists a vector $\eta \in H, \ \|\eta\| = 1$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta\rangle = \beta$.*

Proof. Without loss of generality, we may assume that $\|S\| = 1$. Let $x_n \in H \ (n = 1, 2, ...)$ be such that $\|x_n\| = 1$, $\|Sx_n\| \to 1$ and $\lim\limits_{n\to\infty} \langle Sx_n, x_n\rangle = \beta$.

Consider a partial isometry G and $L = \int_0^1 \beta dE_\beta$, the spectral decomposition of L. Let $S = GL$, the polar decomposition of S. Since $\lim\limits_{n\to\infty} \|Sx_n\| = \|S\| = \|L\| = 1$, we have that $\|Lx_n\| \to 1$ as n tends to ∞ and $\lim\limits_{n\to\infty} \langle Sx_n, x_n\rangle = \lim\limits_{n\to\infty} \langle GLx_n, x_n\rangle = \lim\limits_{n\to\infty} \langle Lx_n, G^* x_n\rangle$.

Now for $H = \overline{Ran(L)} \oplus KerL$, we can choose x_n such that $x_n \in \overline{Ran(L)}$ for large n. Indeed, let $x_n = x_n^{(1)} \oplus x_n^{(2)}, \ n = 1, 2, ...$ Then we have that $Lx_n = Lx_n^{(1)} \oplus Lx_n^{(2)} = Lx_n^{(1)}$ and that $\lim\limits_{n\to\infty} \|x_n^{(1)}\| = 1, \ \lim\limits_{n\to\infty} \|x_n^{(2)}\| = 0$, since $\lim\limits_{n\to\infty} \|Lx_n\| = 1$.

Replacing x_n with $\frac{x_n^{(1)}}{\|x_n^{(1)}\|}$, we obtain $\lim\limits_{n\to\infty} \left\| L\frac{1}{\|x_n^{(1)}\|} x_n^{(1)} \right\| = \lim\limits_{n\to\infty} \left\| S\frac{1}{\|x_n^{(1)}\|} x_n^{(1)} \right\| = 1$ and $\lim\limits_{n\to\infty} \left(S\frac{1}{\|x_n^{(1)}\|} x_n^{(1)}, \frac{1}{\|x_n^{(1)}\|} x_n^{(1)} \right) = \beta$.

Now assume that $x_n \in \overline{RanL}$. Since G is a partial isometry from $\overline{RanL}$ onto $\overline{RanS}$, we have $\|Gx_n\| = 1$ and $\lim\limits_{n\to\infty} \langle Lx_n, G^* x_n\rangle = \beta$. For L is a positive operator, $\|L\| = 1$ and for any $x \in H$, $\langle Lx, x\rangle \le \langle x, x\rangle = \|x\|^2$.

Replacing x with $L^{\frac{1}{2}} x$, we get that $\langle L^2 x, x\rangle \le \langle Lx, x\rangle$, where $L^{\frac{1}{2}}$ is the positive square root of L. Therefore we have $\|Lx\|^2 = \langle Lx, Lx\rangle \le \langle Lx, x\rangle$. It is obvious that $\lim\limits_{n\to\infty} \|Lx_n\| = 1$ and that $\|Lx_n\|^2 \le \langle Lx_n, x_n\rangle \le \|Lx_n\|^2 = 1$. Hence, $\lim\limits_{n\to\infty} \langle Lx_n, x_n\rangle = 1 = \|L\|$.

Moreover, since $I - L \ge 0$, we have $\lim\limits_{n\to\infty} \langle (I - L)x_n, x_n\rangle = 0$. thus $\lim\limits_{n\to\infty} \|(I - L)^{\frac{1}{2}} x_n\| = 0$.

Indeed, $\lim_{n\to\infty}\|(I-L)x_n\| \leq \lim_{n\to\infty}\|(I-L)^{\frac{1}{2}}\|.\|(I-L)^{\frac{1}{2}}x_n\| = 0.$
For $\alpha > 0$, let $\gamma = [0, 1-\frac{\alpha}{2}]$ and let $\rho = [1-\frac{\alpha}{2}, 1]$. We have

$$\begin{aligned} L &= \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu \\ &= \int_0^{1-\frac{\alpha}{2}} \mu dE_\mu + \int_{1-\frac{\alpha}{2}+0}^{1} \mu dE_\mu \\ &= LE(\gamma) \oplus LE(\rho). \end{aligned}$$

Next we show that $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. If there exists a subsequence $x_{n_i}, (i = 1, 2, ...,)$ such that $\|E(\gamma)x_{n_i}\| \geq \epsilon > 0,\ (i = 1, 2, ...,)$, then since $\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\| = \lim_{i\to\infty}(I-L)x_{n_i} = 0$, it follows that,

$$\begin{aligned} \lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\| &= \lim_{i\to\infty}(\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 + \|E(\rho)x_{n_i} - LE(\rho)x_{n_i}\|^2) \\ &= 0. \end{aligned}$$

Hence, we have $\lim_{i\to\infty}(\|E\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 = 0.$

Now it is clear that,

$$\begin{aligned} \|E\gamma)x_{n_i} - LE(\gamma)x_{n_i}\| &\geq \|E\gamma)x_{n_i}\| - \|LE(\gamma)\|.\|E\gamma)x_{n_i}\| \\ &\geq (I - \|LE(\gamma)\|).\|E\gamma)x_{n_i}\| \\ &\geq \frac{\alpha}{2}\epsilon \\ &> 0. \end{aligned}$$

This is a contradiction. Therefore, $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0.$

Since $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1$, we have $\lim_{n\to\infty}\langle LE(\rho)x_n, E(\rho)x_n\rangle = 1$ and $\lim_{n\to\infty}\langle E(\rho)x_n, G^*E(\rho)x_n\rangle = \beta.$

It can be easily verified that $\lim_{n\to\infty}\|E(\rho)x_n\| = 1$, $\lim_{n\to\infty}\left(L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, \frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right) = 1$ and $\lim_{n\to\infty}\left(L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, G^*\frac{E(\rho)x_n}{\|E(\rho)x_n\|}\right) = \beta.$

Replacing x with $\frac{E(\rho)x_n}{\|E(\rho)x_n\|}$, we assume that $x_n \in E(\rho)H$ for each n and $\|x_n\| = 1$.
Let

$$\begin{aligned} J &= \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu \\ &= \int_0^{1-\frac{\alpha}{2}} \mu dE_\mu + \int_{1-\frac{\alpha}{2}+0}^{1} \mu dE_\mu \\ &= J_1 \oplus E(\rho). \end{aligned}$$

Then it is evident that, $\|J\| = \|S\| = \|L\| = 1, Jx_n = x_n,$ and $\|J - L\| < \frac{\alpha}{2}$.

If we can find a contraction V such that $V - G < \frac{\alpha}{2}$ and $\|Vx_n\| = 1$ for a large n, $\langle Vx_n, x_n\rangle = \beta$,

$Z = VJ$, we have $\|Zx_n\| = \|VJx_n\| = 1$, $\langle Zx_n, x_n\rangle = \langle VJx_n, x_n\rangle = \langle Vx_n, x_n\rangle = \beta$ and

$$\begin{aligned} \|S - Z\| &= \|GL - VJ\| \\ &\leq \|GL - GJ\| + \|GJ - VJ\| \\ &\leq \|G\|.\|L - J\| + \|G - V\|.\|J\| \\ &\leq \frac{\alpha}{2} + \frac{\alpha}{2} \\ &= \alpha. \end{aligned}$$

To complete the proof, we construct the desired contraction V. Clearly, $\lim_{n\to\infty}\langle x_n, G^*x_n\rangle = \beta$, as $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$ and $\lim_{n\to\infty}\|x_n - Lx_n\| = 0$. Let $Gx_n = \phi_n x_n + \varphi_n y_n, \quad (y_n \perp x_n,\ \|y_n\| = 1)$ then $\lim_{n\to\infty}\phi_n = \beta$, since $\lim_{n\to\infty}\langle Gx_n, x_n\rangle = \lim_{n\to\infty}\langle x_n, G^*x_n\rangle = \beta$. Also $\|Gx_n\|^2 = |\phi_n|^2 + |\varphi_n|^2 = 1$, implies that $\lim_{n\to\infty}|\varphi_n| = \sqrt{1 - |\beta|}^2$. Now for $\alpha > 0$, there exists an integer M such that $|\phi_M - \beta| < \frac{\alpha}{8}$. Choose φ_M^0 such that $|\varphi_M^0| = \sqrt{1 - |\beta|}^2, \quad |\varphi_M - \varphi_M^0| < \frac{\alpha}{8}$.

Now we have,

$$\begin{aligned} Gx_M &= \phi_M x_M + \varphi_M y_M - \beta x_M + \beta x_M - \varphi_M^0 y_M + \varphi_M^0 y_M \\ &= (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + \beta x_M + \varphi_M^0 y_M. \end{aligned}$$

Let $q_M = \beta x_M + \varphi_M^0 y_M$,. Then, $Gx_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + q_M$. Suppose that $y \perp x_M$, then

$$\begin{aligned} \langle Gx_M, Gy\rangle &= (\phi - \beta)\langle x_M, Gy\rangle + (\varphi_M - \varphi_M^0)\langle y_M, Gy\rangle + \langle q_M, Gy\rangle \\ &= 0, \end{aligned}$$

since G^*G is a projection from H to $RanL$.

It follows that, $|\langle q_M, Gy\rangle| \leq |\phi_M - \beta|.\|y\| + |\varphi_M - \varphi_M^0|.\|y\| \leq \frac{\alpha}{4}\|y\|$. If we suppose that $Gy = \phi q_M + y^0,\ (y^0 \perp q_M,)$ then y^0 is uniquely determined by y. Hence we define V as follows:

$$V : x_M \to q_M,\ y \to y^0,\ \phi x_M + \varphi_M y \to \phi q_M + \varphi_M y^0,$$

with both ϕ, φ being complex numbers. V is a linear operator. We prove that V is a contraction. Now,

$$\|Vx_M\|^2 = \|q_M\|^2 = |\beta|^2 = |\varphi_M^0|^2 = 1,$$

$$\|Vy\|^2 = \|Gy\|^2 - |\phi y|^2 \leq \|Gy\|^2 \leq \|y\|^2.$$

It follows that, $\|V\phi\|^2 = \|\phi\|^2\|Vx_M\|^2 + |\varphi|^2\|Vy\|^2 \leq |\phi|^2 + |\varphi|^2 = 1$, for each $x \in H$ satisfying $x = \phi x_M + \varphi_M y, \quad \|x\| = 1,\ x_M \perp y$, which is equivalent to that V is a contraction. From the definition of V, we show that $\|Gx_M - Vx_M\|^2 = |\phi - \beta|^2 + |\varphi_M - \varphi_M^0|^2 \leq \frac{2\alpha^2}{16} = \frac{1}{8}\alpha^2$. If $y \perp x_M,\ \|y\| \leq 1$ then we have, $\|Gy - Vy\| = |\phi|\|Vx_M\| = |\langle Gy, Vx_M\rangle| = |\langle q_M, Gy\rangle| < \frac{\alpha}{4}$.

Hence for any $x \in H,\ x = \phi x_M + \varphi_M y,\ \ \|x\| = 1,$

$$\begin{aligned} \|Gx - Vx\|^2 &= \|\phi(G-V)x_M + \varphi(G-V)y\|^2 \\ &= |\phi|^2\|(G-V)x_M\|^2 + |\varphi|^2\|(G-V)y\|^2 \\ &< |\phi|^2.\frac{\alpha^2}{16} + |\varphi|^2.\frac{\alpha^2}{16} \\ &< \frac{\alpha^2}{8}, \end{aligned}$$

which implies that $\|(G-V)x\| < \frac{\alpha}{2},\ \ \|x\| = 1$, and hence $\|(G-V)\| < \frac{\alpha}{2}$. Let $Z = VJ$. Then Z is the required operator and this completes the proof. ∎

Theorem 3.3. *The set of operators $S \in B(H)$ implementing a norm-attainable inner derivation, δ_S^N, is uniformly dense in $B(H)$.*

Proof. Let $\alpha > 0$ and $S \in B(H)$. It is clear that $\delta_S^N = \delta_{S-\beta}^N$ for any $\beta \in \mathbb{C}$. From [4], there is a $\beta \in \mathbb{C}$ such that $0 \in W_0(S-\beta)$ and $\|\delta_S^N\| = \|\delta_{S-\beta}^N\| = 2\|S-\beta\|$. From Theorem 3.2, there exists an operator $Z \in B(H)$ such that $\|S-\beta\| = \|Z\|,\ \ \|S-\beta-Z\| < \alpha$ and there exists a vector $\eta \in H$ such that $\|Z\eta\| = \|Z\|,\ \ \langle Z\eta, \eta\rangle = 0$. By Lemma 3.1, δ_Z^N is norm-attainable which is equivalent to that $\delta_{Z+\beta}^N$ is norm-attainable and implies that $\|S-(Z+\beta)\| = \|S-\beta-Z\| = 2\|S-\beta\| < \alpha$. ∎

Theorem 3.4. *Let $S, T \in B(H)$ If there exists vectors $\zeta, \eta \in H$ such that $\|\zeta\| = \|\eta\| = 1,\ \ \|S\zeta\| = \|S\|,\ \|T\eta\| = \|T\|$ and $\frac{1}{\|S\|}\langle S\zeta, \zeta\rangle = -\frac{1}{\|T\|}\langle T\eta, \eta\rangle$, then $\delta_{S,T}^N$ is norm-attainable.*

Proof. By linear dependence of vectors, if η and $T\eta$ are linearly dependent, that is,$T\eta = \phi\|T\|\eta$, then it is true that $|\phi| = 1$ and $|\langle T\eta, \eta\rangle| = \|T\|$. It follows that $|\langle S\zeta, \zeta\rangle| = \|S\|$ which implies $S\zeta = \varphi\|S\|\zeta$ and $|\varphi| = 1$. Hence $\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle = \varphi = -\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle = -\phi$. Defining X as $X : \eta \to \zeta,\ \ \{\zeta\}^\perp \to 0$, we have $\|X\| = 1$ and $(SX - XT)\eta = \varphi(\|S\| + \|T\|)\zeta$, which implies

$$\|SX - XT\| = \|(SX - XT)\eta\| = \|S\| + \|T\|.$$

By [4], it follows that $\|SX - XT\| = \|S\| + \|T\| = \|\delta_{S,T}^N\|$.

That is, $\delta_{S,T}^N$ is norm-attainable. If η and $T\eta$ are linearly independent, then $\left|\left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle\right| < 1$, which implies $\left|\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle\right| < 1$. Hence ζ and $S\zeta$ are also linearly independent. Let us redefine X as follows: $X : \eta \to \zeta,\ \ \frac{T\eta}{\|T\|} \to -\frac{S\zeta}{\|S\|},\ \ \ x \to 0$, where $x \in \{\eta, T\eta\}^\perp$. We show that X is a partial isometry. Let $\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle \eta + \tau h,\ \ \ \|h\| = 1,\ \ h \perp \eta$. Since η and $T\eta$ are linearly independent, $\tau \neq 0$. So we have, $X\frac{T\eta}{\|T\|} = \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle X\eta + \tau Xh = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta + \tau Xh$, which implies $\left\langle X\frac{T\eta}{\|T\|}, \zeta \right\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle + \tau\langle Xh, \zeta\rangle = -\left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle$.
Thn it follows that $\langle Xh, \zeta\rangle = 0$. That is, $Xh \perp \zeta(\zeta = X\eta)$. Hence we have

$$\left\| \left\langle \frac{S\zeta}{\|S\|}, \zeta \right\rangle \zeta \right\|^2 + \|\tau Xh\|^2 = \left\| X\frac{T\eta}{\|T\|} \right\|^2 = \left| \left\langle \frac{T\eta}{\|T\|}, \eta \right\rangle \right|^2 + |\tau|^2 = 1,$$

which implies $\|Xh\| = 1$. Now it is evident that X a partial isometry and $\|(SX - XT)\zeta\| = \|SX - XT\| = \|S\| + \|T\|$, which is equivalent to $\|\delta^N_{S,T}(X)\| = \|S\| + \|T\|$. By [4], $\|\delta^N_{S,T}\| = \|S\| + \|T\|$. Hence, $\delta^N_{S,T}$ is norm-attainable. ■

Lemma 3.5. *The set of operators $S, T \in B(H)$ which are implementing a norm-attainable generalized derivation, $\delta^N_{S,T}$, is uniformly dense in $B(H) \times B(H)$.*

Proof. Suppose either S or T is a scalar operator, say $T = \kappa I$, for some scalar κ, then $\delta^N_{S,T}(X) = SX - X\kappa I$. Clearly, $\|\delta^N_{S,T}\| = \|S-\kappa\|$. From Theorem 3.2, for $\alpha > 0$, there exists an operator $C \in B(H)$ and a vector $\zeta \in H$ such that $\|C\zeta\| = \|C\|$ and $\|S-\kappa-C\| < \alpha$. Let $Z = C+\kappa I,\ \ T_0 = \kappa I$. Therefore, we have that $\delta^N_{Z,T_0}(X) = CX,\ \ \forall\ X \in B(H)$. Let P be the projection on $\{\zeta\}$. Then $\|\delta^N_{Z,T_0}(P)\| = \|P\|$, which implies that δ^N_{Z,T_0} is norm-attainable and $\|\langle S,T\rangle - \langle Z - T_0\rangle\| < \alpha$. Suppose neither S nor T is a scalar operator, then there is a complex number β such that $\|\delta^N_{S,T}\| = \|S-\beta\| + \|T-\beta\|$ and $W_0\left(\frac{S-\beta}{\|S-\beta\|}\right) \cap W_0\left(\frac{T-\beta}{\|T-\beta\|}\right)$ is nonempty. Let $\nu \in W_0\left(\frac{S-\beta}{\|S-\beta\|}\right) \cap W_0\left(\frac{T-\beta}{\|T-\beta\|}\right)$. Then for $\alpha > 0$, from Theorem 3.2, there exists Z and T_0 and $\zeta, \eta \in H$ such that $\|Z\| = \|T_0\| = 1,\ \ \|\zeta\| = \|\eta\| = 1,\ \ \left\|\frac{S-\beta}{\|S-\beta\|} - Z\right\| < \frac{\alpha}{2}(\|\delta^N_{S,T}\| + 1)$ and $\left\|\frac{T-\beta}{\|T-\beta\|} - T_0\right\| < \frac{\alpha}{2}(\|\delta^N_{S,T}\| + 1)$, with

$$\|Z\zeta\| = \|Z\|,\ \ \|T_0\eta\| = \|T_0\|,\ \ \langle Z\zeta,\zeta\rangle = \nu,\ \ \langle T_0\eta,\eta\rangle = -\nu.$$

Let $Z_0 = \|S-\beta\|Z,\ \ \ T_{00} = \|T-\beta\|T_0$. Then $\left\langle \frac{Z_0\zeta}{\|Z_0\|}, \zeta\right\rangle = \nu = -\left\langle \frac{T_{00}\eta}{\|T_{00}\|}, \eta\right\rangle$. By Theorem 3.4, $\delta^N_{Z_0,T_{00}}$ is norm-attainable. But we have $\|S-\beta-Z_0\| < \frac{\alpha}{2}$ and $\|T-\beta-T_{00}\| < \frac{\alpha}{2}$. Hence, $\|\langle S,T\rangle - \langle Z_0+\beta, T_{00}+\beta\rangle\| < \alpha$, and $\delta^N_{\langle Z_0+\beta,T_{00}+\beta\rangle}$ is norm-attainable. ■

Theorem 3.6. *Let $S, T \in B(H)$. If both S and T are norm-attainable then the basic elementary operator $M^N_{S,\,T}$ is also norm-attainable.*

Proof. For any p$S, T \in B(H)$, $\|M^N_{S,\,T}\| = \|S\|\|T\|$. We assume that $\|S\| = \|T\| = 1$. If both S and T are norm-attainable, then there exists unit vectors ζ and η with $\|S\zeta\| = \|T\eta\| = 1$. Therefore, define an operator X by $X = \langle\cdot, T\eta\rangle\zeta$. Clearly, $\|X\| = 1$. Therefore, we have $\|SXT\| \geq \|SXT\eta\| = \|\|T\eta\|^2 S\zeta\| = 1$. Hence, $\|M^N_{S,\,T}(X)\| = \|SXT\| = 1$, that is $M^N_{S,\,T}$ is also norm-attainable. ■

Theorem 3.7. *Let $S, T \in B(H)$ If both S and T are norm-attainable then the Jordan elementary operator $\mathcal{U}^N_{S,\,T}$ is also norm-attainable.*

Proof. The proof is analogous to that of the previous Lemma. ■

As a remark, we give the following generalization on norm-attainable general elementary operator. An operator $\mathcal{T}_{\tilde{S},\tilde{T}}(X) = \sum_{i=1}^{n} S_i X T_i$, is said to be norm-attainable if there is a contraction X in the unit ball, $(B(H))_1$, such that $\|\mathcal{T}_{\tilde{S},\tilde{T}}(X)\| = \|\mathcal{T}_{\tilde{S},\tilde{T}}\|$, where $\tilde{S} = (S_1, ..., S_n)$ and $\tilde{T} = (T_1, ..., T_n)$ are n-tuples in $B(H)$.

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References

[1] P. R. Halmos, *A Hilbert space problem book*, second ed., Springer-Verlag, Berlin, 1982.

[2] N. B. Okelo, J .O. Agure and D. O. Ambogo, *Norms of elementary operators and characterization of norm - attainable operators*, Int. Journal of Math. Analysis, 24(2010), 687-693.

[3] A. Seddik, *On the numerical range and norm of elementary operators*, Linear Multilinear Algebra 52 (2004), 293-302.

[4] J. G Stampfli, *The norm of a derivation*, Pacific journal of mathematics, Vol. 33, 3 (1970), 737-747.