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Majority domination vertex critical graphs

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Abstract

This paper deals the graphs for which the removal of any vertex changes the majority domination number of the graph. In the following section, how the removal of a single vertex from G can change the majority domination number is surveyed for trees. Characterisation of $V_M^+(G)$ and $V_M^-(G)$ are determined.

Keywords: Majority Dominating Set, γ_M-critical vertex and graphs.
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1 Introduction

By a graph we mean a finite undirected graph without loops or multiple edges. Let $G = (V, E)$ be a finite graph with p vertices and q edges and v be a vertex in $V(G)$. The closed neighborhood of v is defined by $N[v] = N(v) \cup \{v\}$. The closed neighborhood of a set of vertices S is denoted as $N[S]$ and is $\bigcup\limits_{s \in S} N[s]$.

Definition 1.1. *[7]. A subset $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a Majority Dominating Set if at least half of the vertices of $V(G)$ are either in S or adjacent to elements of S. That is, $|N[S]| \geq \left\lceil \frac{|V(G)|}{2} \right\rceil$.*

A majority dominating set S is minimal if no proper subset of S is a majority dominating set.

The minimum cardinality of a minimal majority dominating set is called the majority domination number and is denoted by $\gamma_M(G)$.

The maximum cardinality of a minimal majority dominating set is denoted by $\Gamma_M(G)$. Majority dominating sets have super hereditary property .

Change of majority domination number in the case of removal of a single vertex is defined and studied. Here, CVR means the change in vertex removal of a graph G and UVR means unchanged vertex removal of a graph G . The graphs in CVR were characterisied by Bauer et al. and H. B. Waliker and B. D. Acharya [8].

Definition 1.2. *[4]. For any graph G, CVR and UVR with respect to majority domination is defined by*
$CVR_M: \quad \gamma_M(G-u) \neq \gamma_M(G), \quad$ *for all* $u \in V(G)$.
$UVR_M: \quad \gamma_M(G-u) = \gamma_M(G), \quad$ *for all* $u \in V(G)$.

Definition 1.3. *For any graph G, the vertex set can be partitioned with respect to majority domination into three sets $V_M^0(G)$, $V_M^-(G)$and $V_M^+(G)$ and is defined by*

$V_M^0(G) = \{u \in V(G) : \gamma_M(G-u) = \gamma_M(G)\}$
$V_M^-(G) = \{u \in V(G) : \gamma_M(G-u) < \gamma_M(G)\}$
$V_M^+(G) = \{u \in V(G) : \gamma_M(G-u) > \gamma_M(G)\}$.

2 Change of $\gamma_M(G)$ for Trees

Proposition 2.1. *Let T be a tree with even numbers of vertices. Then $\gamma_M(T-u) \geq \gamma_M(T)$ for every $u \in V(T)$.*

Proof. Suppose $\gamma_M(T-u) < \gamma_M(T)$. Let $D = \{w_1, w_2, ..., w_s\}$ be a γ_M-set of $T-u$. (i. e.), $\left\lceil \frac{p-1}{2} \right\rceil \leq |N[D]| < \lceil \frac{p}{2} \rceil$. Given $p = 2r$, then $r \leq |N[D]| < r$, a contradiction. ∎

Proposition 2.2. *Let T be a tree with odd number of vertices. Let D be a γ_M-set of T . Let $u \in V - D$. Then $\gamma_M(T-u) \leq \gamma_M(T)$.*

Proposition 2.3. *Let T be a tree. Let D be a γ_M-set of T such that there exists $u \in V(T)$ with $u \notin N[D]$. Then $\gamma_M(T-u) \leq \gamma_M(T)$.*

Corollary 2.4. *Let T be a tree with even number of vertices. Let D be a γ_M-set of T such that there exists a vertex $u \in V(T)$ with $u \notin N[D]$.*
Then $\gamma_M(T-u) = \gamma_M(T)$.

Proposition 2.5. *Let T be a tree. Let $u \in V(T)$. Suppose there exists a γ_M-set of D of T such that $u \notin N[D]$ and there exists a γ_M-set D_1 of $T-u$ with $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$. Then $\gamma_M(T-u) = \gamma_M(T)$.*

Proof. Since $|N[D_1]| \geq \lceil \frac{p}{2} \rceil$, D_1 is a majority dominating set of T. Therefore, $\gamma_M(T) \leq |D_1| = \gamma_M(T-u)$. Since there exists a γ_M-set D of T such that $u \notin N[D]$, $\gamma_M(T-u) \leq \gamma_M(T)$. ∎

Definition 2.6. *Let $S \subseteq V(G)$. Let $x \in V(G)$. If $x \in S$, then $pn[x,S] = N[x] - N[S-x]$. If $x \notin S$, then $pn[x,S] = N[x] - N[S]$.*

Theorem 2.7. *Let T be a tree . Let u be a pendent vertex and v be its support. Let D be a γ_M-set of $T-u$ such that there exists a vertex $v \in V - D$ with the property that $|pn[v,D]| > |pn[x,D]|$ for some $x \in D$.*
Then $\gamma_M(T) = \gamma_M(T-u)$.

Proof. Let S be a γ_M-set of T . If $u, v \notin S$, then $S - \{u\}$ is a majority dominating set of $T - \{u\}$. Therefore, $\gamma_M(T-u) \leq |S - \{u\}| = |S| = \gamma_M(T)$.
Let $v \in S$. Then $u \notin S$. Therefore, $\gamma_M(T-u) \leq |S - \{u\}| = |S| = \gamma_M(T)$. Suppose $u \in S$ and $v \notin$

S. Let $S_1 = (S - \{u\}) \cup \{v\}$. Therefore, S_1 is a majority dominating set of T of cardinality $\gamma_M(T)$ and $v \in S_1$. By the previous argument, $\gamma_M(T-u) \leq \gamma_M(T)$ (1)
Given D is a γ_M-set of $T - \{u\}$ such that there exists a vertex $v \in V - D$ with the property that $|pn[v, D]| > |pn[x, D]|$ for some $x \in D$. Therefore, $(D - \{x\}) \cup \{v\}$ is a majority dominating set of T. Hence, $\gamma_M(T) \leq |(D - \{x\}) \cup \{v\}| = \gamma_M(T-u)$ (2).
Then, $\gamma_M(T) = \gamma_M(T-u)$. ∎

Proposition 2.8. *Let T be a tree with even number of vertices. Then there exists a pendent vertex $u \in V(T)$ such that $\gamma_M(T-u) = \gamma_M(T)$.*

3 Change of $\gamma_M(G)$ for Graphs

Theorem 3.1. *In any graph G with an isolate, there exists a γ_M-set of G not containing that isolate.*

Remark 3.2.

1. Let v be an isolate and D be a γ_M-set of G not containing v. Then $D - v$ majority dominates $G - v$. Therefore, $\gamma_M(G-v) \leq |D - v| = \gamma_M(G)$.
2. Let $v \in V_M^+(G)$. Then v is not an isolate of G.

Theorem 3.3. *Let G be a graph of odd order. Let $v \in V_M^+(G)$. Then v belongs to every γ_M-set of G.*

Proof. Let $v \in V_M^+(G)$. Suppose there exists a γ_M-set D such that $v \notin D$. Let $v \notin N[D]$. Then $|N[D]| \geq \lceil \frac{p}{2} \rceil \geq \left\lceil \frac{p-1}{2} \right\rceil$ in $G - v$.

Therefore, $\gamma_M(G-v) \leq |D| = \gamma_M(G)$ implies $v \notin V_M^+(G)$,a contradiction. Hence $v \in N[D]$. If $|N[D]| > \lceil \frac{p}{2} \rceil$ in G, then $|N[D]| \geq \lceil \frac{p}{2} \rceil$ in $G - v$. Therefore, D is a majority dominating set of $G - v$. It implies $v \notin V_M^+(G)$, a contradiction. Hence $|N[D]| = \lceil \frac{p}{2} \rceil$ in G. Then $|N[D]| = \lceil \frac{p}{2} \rceil - 1$ in $G - v$. Given p is odd, D is a majority dominating set in $G - v$, a contradiction to $v \in V_M^+(G)$. Thus v belongs to every γ_M-set of G. ∎

Theorem 3.4. *Let G be a graph. Let $v \in V_M^+(G)$ Then v belongs to $N[D]$ for every γ_M-set D of G.*

Remark 3.5. *1. If G is of odd order, then any vertex in $V_M^+(G)$ belongs to every γ_M-set of G.*

2. If G is of even order, then any vertex in $V_M^+(G)$ belongs to every γ_M-set D of G such that $|N[D]| > \frac{p}{2}$ and belongs to $N[D]$ if $|N[D]| = \frac{p}{2}$.

Proposition 3.6. *Let G be any graph. Let $v \in V_M^+(G)$. Then there exists a vertex $w \in V(G)$ such that $\gamma_M(G-w) = \gamma_M(G)$.*

Proof. Let $O(G) = 2n + 1$. Let $D = \{v, u_1, u_2, ..., u_{k-1}\}$ be a γ_M-set of G. Let $w \in V - D$ (note that $V - D \neq \phi$. Then $D \subset V - w$. Now, $|N_G[D]| \geq \lceil \frac{p}{2} \rceil = n + 1$. Therefore, $|N_{G-w}[D]| \geq n$. Then $\gamma_M(G-w) \leq |D| = \gamma_M(G)$. If $\gamma_M(G-w) < \gamma_M(G)$, then $w \in V_M^-(G)$. Then $V = V_M^-(G)$, a

contradiction since $V_M^+(G) \neq \phi$.
Let G be a graph of order $2n$. Let D be a γ_M-set of G.
Case(i): Let $N[D] \neq V(G)$. Then there exists $w \notin N[D]$. Therefore, $w \notin D$. Then $D \subseteq V - w$ and $|N_G[D]| \geq \lceil \frac{p}{2} \rceil = n$. Therefore, $|N_{G-w}[D]| \geq n$ (since $w \notin N[D]$) $\Rightarrow$ $|N_{G-w}[D]| = \left\lceil \frac{p-1}{2} \right\rceil$. Therefore, D is a majority dominating set of $G-w$. Then $\gamma_M(G-w) \leq |D| = \gamma_M(G)$. If $\gamma_M(G-w) < \gamma_M(G)$, then $w \in V_M^-(G)$. Then $V(G) = V_M^-(G)$, a contradiction since $V_M^+(G) \neq \phi$.
Case(ii): Let $N[D] = V(G)$. Now, $|N_G[D]| = 2n$. Therefore, $|N_{G-w}[D]| = 2n - 1 \geq n = \left\lceil \frac{p-1}{2} \right\rceil$, for all $n \geq 1$. Then $\gamma_M(G - w) \leq |D| = \gamma_M(G)$. If $\gamma_M(G - w) < \gamma_M(G)$, then $w \in V_M^-(G)$. Then $V(G) = V_M^-(G)$, a contradiction since $V_M^+(G) \neq \phi$. Hence $\gamma_M(G - w) = \gamma_M(G)$. ∎

4 Characterisation of $V_M^+(G)$ and $V_M^-(G)$

Theorem 4.1. *A vertex $v \in V_M^+(G)$ if and only if $v \in V(G)$ satisfies the following conditions:*
(i) v is not an isolate and v belongs to every γ_M-set D of G if G is of odd order. If G is of even order, v belongs to every γ_M-set D of G such that $|N[D]| > \frac{p}{2}$ and belongs to $N[D]$ if $|N[D]| = \frac{p}{2}$.
(ii) No subset of $V(G) - \{v\}$ of cardinality $\gamma_M(G)$ majority dominates
$G - \{v\}$.

Proof. Suppose $v \in V_M^+(G)$. Then by remarks 3.2, the theorem 3.3 and remarks 3.5, the condition (i) holds.
Let $D \subseteq V(G) - \{v\}$ and $|D| = \gamma_M(G)$. Suppose D majority dominates $G - \{v\}$. Then $\gamma_M(G - \{v\}) \leq |D| = \gamma_M(G)$,a contradiction to $v \in V_M^+(G)$. Hence the condition (ii) holds.
Conversely suppose the condition (i) and (ii) hold. Let D be a $\gamma_M(G)$ subset of $G - \{v\}$. If $|D| = \gamma_M(G)$, then we get a contradiction to (ii). Therefore, $|D| \neq \gamma_M(G)$. Suppose $|D| < \gamma_M(G)$. Then $|D| \leq \gamma_M(G) - 1$. Suppose $|N[D]| > \left\lceil \frac{p-1}{2} \right\rceil$. Then $|N[D]| \geq \lceil \frac{p}{2} \rceil$. Therefore, D is a majority dominating set of G, which is a contradiction to $|D| \leq \gamma_M(G) - 1$. Hence $|N[D]| = \left\lceil \frac{p-1}{2} \right\rceil$. Let $p \geq 3$, then there exists a vertex $u \neq v$, $u \notin N[D]$. $|N[D] \cup \{u\}| = \left\lceil \frac{p-1}{2} \right\rceil + 1 \geq \lceil \frac{p}{2} \rceil$. That implies $D \cup \{u\}$ is a majority dominating set of G. Hence $|D \cup \{u\}| = \gamma_M(G)$ and $|D| = \gamma_M(G) - 1$, $u \notin D$.
Suppose p is odd. By hypothesis, $v \in D \cup \{u\}$. But $v \neq u$. Therefore, $v \in D$. But D is a γ_M-set of $G - v$. It implies that $v \notin D$, a contradiction. Hence $|D| \geq \gamma_M(G)$.
Suppose p is even. $D \cup \{u\}$ is a γ_M-set of G and $|N[D \cup \{u\}]| \geq \left\lceil \frac{p-1}{2} \right\rceil + 1 > \frac{p}{2}$. Therefore, $v \in D \cup \{u\}$ but $v \neq u$. $v \in D$, which is a contradiction to $D \subseteq V(G) - \{v\}$. Hence $|D| \geq \gamma_M(G)$. Since $|D| \neq \gamma_M(G)$, $|D| > \gamma_M(G)$. Therefore, $v \in V_M^+(G)$. ∎

Theorem 4.2. *A vertex $v \in V_M^-(G)$ if and only if (i) order of the graph is odd. (ii) $\gamma_M(G) = \gamma_M(G - v) + 1$.*

Proof. Let $v \in V_M^-(G)$. Then $\gamma_M(G - v) < \gamma_M(G)$. Let D be γ_M-set of $G - \{v\}$. Then $|N_{G-v}[D]| \geq \left\lceil \frac{p-1}{2} \right\rceil$ and $|N_G[D]| < \lceil \frac{p}{2} \rceil$. Let $p = 2n$. Then $|N_{G-v}[D]| \geq n$ and $|N_G[D]| < n$, a contradiction. Therefore, p is odd. Then $|N_{G-v}[D]| \geq n$ and $|N_G[D]| < n + 1$. Therefore, $|N_{G-v}[D]| = n$.
Case(i): Let $v \notin N_G[D]$. Then $|N_G[D \cup \{v\}]| \geq n + 1 = \lceil \frac{p}{2} \rceil$. Therefore, $D \cup \{v\}$ is a majority

dominating set of G. Then $\gamma_M(G) \leq |D \cup \{v\}| = \gamma_M(G-v)+1$. By hypothesis, $\gamma_M(G-v) < \gamma_M(G) \leq \gamma_M(G-v)+1$.

Case(ii): Let $v \in N_G[D]$. Let $u \in V(G)$ such that $u \notin N_G[D]$. Then $|N_G[D \cup \{u\}]| \geq n+1 = \lceil \frac{p}{2} \rceil$. Therefore, $D \cup \{u\}$ is a majority dominating set of G. Proceeding as in case(i), $\gamma_M(G) = \gamma_M(G-v)+1$. The converse is obvious. ∎

Remark 4.3.

1. For a graph with even number of vertices $V_M^-(G) = \phi$.
2. Let $V_M^-(G) \neq \phi$. Then $V_M^-(G) = V(G)$.
3. $G \in CVR_M$ if and only if $V(G) = V_M^-(G)$.

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