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Multipliers of Hypersemilattices

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Abstract

The notion of multipliers is introduced in a hypersemilattice and some properties of multipliers are studied. In addition, a set of equivalent conditions are established for two multipliers of a hypersemilattice to be equal in the sense of mappings. Further, the properties of invariance subsets and invariance congruences are studied with respect to multipliers.

Keywords: Hypersemilattice, multiplier, ideal, fixed element, f-invariant subset, f-invariant congruence.
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1 Introduction

The theory of hyper algebras (multi-algebras) was introduced by F. Marty [3] in 1934. Hyper algebras has many applications to several branches of both pure Mathematics and applied Mathematics such as hypergroups [1], hyperrings [4], hyper BCI-algebras [6] and so on. In 2011, K.H. Kim [2] introduced and studied the properties of multipliers in BE-algebras.

The aim of this paper is to introduce and study the properties of multipliers in hypersemilattices. The images and inverse images of ideals under a multiplier are studied. The multipliers of direct products of hypersemilattices are studied. A set of equivalent conditions are derived for two multipliers to be equal. Finally, the concept of multiplier invariant congruences θ is introduced and obtained a necessary condition for the existence of multiplier of the quotient algebra L/θ.

For the notions and notations, the reader is referred to [7]. However, some of the preliminary definitions and results are presented for the ready reference of the readers. Throughout the rest of this note, L stands for a hypersemilattice unless otherwise mentioned.

Definition 1.1. [4] Let L be a non-empty set and $P(L)$ denotes the power set of L, $P^*(L) = P(L) - \emptyset$. A binary hyperoperation $\circ$ on L is a function from $L \times L$ into $P^*(L)$. The image of pair (x, y) is denoted by $x \circ y$.

Let A, B be two non-empty subsets of L and $x \in L$. Then the sets $A \circ B$, $A \circ c$ and $c \circ C$ can be considered as follows:

(1) $x \circ A = \bigcup\limits_{a \in A} (x \circ a)$

(2) $A \circ x = \bigcup\limits_{a \in A} (a \circ x)$

(3) $A \circ B = \bigcup\limits_{a \in A, b \in B} (a \circ b)$

Definition 1.2. [4] Let L be a non-empty set with a binary hyperoperation $\otimes$ on L satisfying the following conditions, for all $a, b, c \in L$,

(1) $a \in a \otimes a$
(2) $a \otimes b = b \otimes a$
(3) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

Then $(L, \otimes)$ is called a hypersemi-lattice.

Definition 1.3. [4] Let $(L, \otimes)$ be a hypersemi-lattice. An element $a \in L$ is called an absorbent element of L if $c \in a \otimes c$ for all $c \in L$. An element $b \in L$ is called a fixed element of L if $b \otimes c = \{b\}$ for all $c \in L$.

Proposition 1.4. *[7] Let L be a non-empty set and $\otimes$ a binary hyperoperation on L. Then $(L, \otimes)$ is a hypersemilattice if and only if for all $A, B, C \in P^*(L)$, the following conditions hold*

(a) $A \subseteq A \otimes A$
(b) $A \otimes B = B \otimes A$
(c) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

Definition 1.5. [4] Let $(L, \otimes)$ be a hypersemilattice and N a nonempty subset of L. We say that N is an ideal of $(L, \otimes)$ if $a \otimes N \subseteq N$ for all $a \in L$. An ideal N of L is called proper if $N \neq L$.

Theorem 1.6. *[7] Let $(L, \otimes)$ be hypersemilattice and N a nonempty subset of L. Then the following conditions are equivalent:*

(1) N is an ideal of L
(2) $a \otimes n \in P^*(N)$ for all $a \in L$ and $n \in N$
(3) $L \otimes N \subseteq N$

Definition 1.7. [7] Let $(L, \otimes)$ be a hypersemilattice and $a, b \in L$. We say that $a \leq_L b$ if $a \otimes c \subseteq b \otimes c$ for all $c \in L$. In this case, we call $\leq_L$ the hyperorder on hypersemilattice L.

Proposition 1.8. *[7] Let N be an ideal of a hypersemi-lattice L and $a \in L$. If $b \leq_L a$, then $b \in N$.*

Definition 1.9. [4] Let $(L, \otimes)$ be a hypersemilattice and θ an equivalence relation on L. We say that, $(A, B) \in \theta$ if and only if to each $a \in A$, there exists $b \in B$ such that $(a, b) \in \theta$ for any $A, B \subseteq L$.

Definition 1.10. [4] Let $(L, \otimes)$ be a hypersemilattice. An equivalence relation θ on L is said to be a congruence relation on L if for any $a, b, c, d \in L$, $(a, b) \in \theta$ and $(c, d) \in \theta$ imply $(a \otimes c, b \otimes d) \in \theta$. We denote the equivalence class of any $x \in L$ by $[x]_\theta = \{y \in L \mid (x, y) \in \theta\}$

2 Multipliers of hypersemilattices

In this section, the notion of multipliers is introduced in hypersemilattices and their properties are studied. The invariant properties of a subset and a congruence under a multiplier are studied in a hypersemilattice.

Definition 2.1. Let $(L, \otimes)$ be a hypersemilattice. Then a self mapping $f : L \longrightarrow L$ is called a multiplier of L if for all $a, b \in L$ it satisfies

$$f(a \otimes b) = a \otimes f(b)$$

Example 2.2. *Let $L = \{a, b, c\}$ and $\otimes$ a binary hyperoperation on L defined as follows:*

$\otimes$	a	b	c
a	{a}	{a}	{a}
b	{a}	{a,b}	{a}
c	{a}	{a}	{c}

Then $(L, \otimes)$ is a hypersemilattice. Now define $f : L \longrightarrow L$ as follows:

$$f(x) = \begin{cases} a & \text{if } x = a, b \\ c & \text{if } x = c \end{cases}$$

It can be easily observed that f is a multiplier of L.

Lemma 2.3. *Let $(L, \otimes)$ be a hypersemilattice and f a multiplier of L. Then for any $a, b, c \in L$, we have the following:*

(1) if a is a fixed element then $f(a) = a$
(2) $a \otimes f(b) = f(a) \otimes b$
(3) $a \leq_L b$ implies that $f(a) \leq_L f(b)$

Proof. (1) Let a be a fixed element of L. Then $a \otimes c = \{a\}$ for all $c \in L$. Since $a \in a \otimes a$, we get $f(a) \in f(a \otimes a) = a \otimes f(a) = \{a\}$. Hence $f(a) = a$.
(2) For any $a, b \in L$, $a \otimes f(b) = f(a \otimes b) = f(b \otimes a) = b \otimes f(a) = f(a) \otimes b$.
(3) Suppose $a \leq_L b$. For any $c \in L$, we get that $f(a) \otimes c = f(a \otimes c) \subseteq f(b \otimes c) = f(b) \otimes c$. Therefore $f(a) \leq_L f(b)$. ■

Proposition 2.4. *Let $(L, \otimes)$ be a hypersemilattice. Then every idempotent multiplier of L is a homomorphism.*

Proof. Let f be an idempotent multiplier of L. Then $f^2(x) = f(x)$ for all $x \in L$. Let $a, b \in L$. Then

$$f(a \otimes b) = f^2(a \otimes b) = f(f(a \otimes b)) = f(a \otimes f(b)) = f(a) \otimes f(b)$$

Therefore, f is a homomorphism of L. ■

Proposition 2.5. *Let $(L, \otimes)$ be a hypersemilattice. For any $a \in L$, define the self mapping $f_a : L \longrightarrow L$ by $f_a(x) = t$ if and only if $t \in a \otimes x$ for all $x \in L$. Then f_a is a multiplier of L.*

Proof. Fix $a \in L$. Let $x, y \in L$. Suppose $f_a(x \otimes y) = t$. Then

$$\begin{aligned} t &\in a \otimes (x \otimes y) \\ &= (a \otimes x) \otimes y \\ &= (x \otimes a) \otimes y \\ &= x \otimes (a \otimes y) \end{aligned}$$

Hence $t = x \otimes c$ for some $c \in a \otimes y$. Now $c \in a \otimes y$ implies that $f_a(y) = c$. Therefore $t = x \otimes c = x \otimes f_a(y)$. Thus $f_a(x \otimes y) = t = x \otimes f_a(y)$. Therefore f_a is a multiplier of L. ∎

Remark 2.6. The composition of two multipliers f and g of a hypersemilattice L is also a multiplier of L where $(f \circ g)(x) = f(g(x))$ for all $x \in L$.

Let L_1 and L_2 be two hypersemilattices. Then $L_1 \times L_2$ is also a hypersemilattice [5] with respect to the operation given by:

$$(a, b) \otimes (c, d) = \{(x, y) \mid x \in a \otimes c, y \in b \otimes d\}$$

Now, we introduce the multipliers of direct products.

Proposition 2.7. *Let $(L_1, \otimes)$ and $(L_2, \otimes)$ be two hypersemilattices and k a fixed element of L_2. Define a self mapping $f : L_1 \times L_2 \longrightarrow L_1 \times L_2$ by $f(x, y) = (x, k)$ for all $(x, y) \in L_1 \times L_2$. Then f is a multiplier of the direct product $L_1 \times L_2$.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in L_1 \times L_2$. Then we get

$$\begin{aligned} f((x_1, x_2) \otimes (y_1, y_2)) &= f(\{(x, y) \mid x \in x_1 \otimes y_1, y \in x_2 \otimes y_2\}) \\ &= \{(x, k) \mid x \in x_1 \otimes y_1\} \\ &= \{(x, k) \mid x \in x_1 \otimes y_1, k \in x_2 \otimes k\} \\ &= (x_1, x_2) \otimes (y_1, k) \\ &= (x_1, x_2) \otimes f(y_1, y_2) \end{aligned}$$

Therefore, f is a multiplier of the direct product $L_1 \times L_2$. ∎

Proposition 2.8. *Let f be a multiplier of a hypersemilattice $(L, \otimes)$. Then we have the following:*

(1) If a is a fixed element of L, then $f^{-1}(a)$ is an ideal of L.

(2) If M is an ideal of L, then $f(M)$ is an ideal of L.

(3) If M is an ideal of L, then $f^{-1}(M)$ is an ideal of L.

Proof. (1) Let a be a fixed element of L. Let $x \in f^{-1}(a)$. Then $f(x) = a$. For any $m \in L$, we have $f(m \otimes x) = m \otimes f(x) = m \otimes a = \{a\}$, because of a is a fixed element. Hence $m \otimes x \subseteq f^{-1}(a)$. Thus $f^{-1}(a)$ is an ideal of L.

(2) Let M be an ideal of L. Let $x \in L$ and $y \in f(M)$. Then $y = f(a)$ for some $a \in M$. Since M is an ideal, we get that $x \otimes a \subseteq M$. Now $x \otimes f(a) = f(x \otimes a) \subseteq f(M)$. Therefore $f(M)$ is an ideal of L.

(3) Let M be an ideal of L. Let $x \in L$ and $a \in f^{-1}(M)$. Then $f(a) \in M$. Since M is an ideal, we get that $f(x \otimes a) = x \otimes f(a) \subseteq x \otimes M \subseteq M$. Thus $x \otimes a \subseteq f^{-1}(M)$. Hence $f^{-1}(M)$ is an ideal of L. ∎

Definition 2.9. Let f be a multiplier of $(L, \otimes)$. Define $\triangle_f(L) = \{x \in L \mid f(x) = x\}$.

Clearly every fixed element of L is a member of $\triangle_f(L)$. If f is an idempotent multiplier, then clearly $f(x) \in \triangle_f(L)$ for all $x \in L$.

Theorem 2.10. *Let f and g be two idempotent multipliers of a hypersemilattice L such that $f \circ g = g \circ f$. Then the following conditions are equivalent.*

(1) $f = g$.

(2) $f(L) = g(L)$.

(3) $\triangle_f(L) = \triangle_g(L)$.

Proof. $(1) \Rightarrow (2)$: It is obvious.

$(2) \Rightarrow (3)$: Assume that $f(L) = g(L)$. Let $x \in \triangle_f(L)$. Then we get $x = f(x) \in f(L) = g(L)$. Hence $x = g(y)$ for some $y \in L$. Now $g(x) = g^2(y) = g(y) = x$. Thus $x \in \triangle_g(L)$. Therefore, $\triangle_f(L) \subseteq \triangle_g(L)$. Similarly, we can obtain that $\triangle_g(L) \subseteq \triangle_f(L)$. Therefore $\triangle_f(L) = \triangle_g(L)$.

$(3) \Rightarrow (1)$: Assume that $\triangle_f(L) = \triangle_g(L)$. Let $x \in L$. Since $f(x) \in \triangle_f(L) = \triangle_g(L)$, we get $g(f(x)) = f(x)$. Also we have $g(x) \in \triangle_g(L) = \triangle_f(L)$. Hence, $f(g(x)) = g(x)$. Thus we have

$$f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x)$$

Therefore, $f = g$. ∎

Definition 2.11. Let f be a multiplier of a hypersemilattice L. A subset S of L is called f-invariant if $x \in S$ implies $f(x) \in S$.

Note that $\emptyset$ and L are the f-invariant subsets of L. Also $\triangle_f(L)$ is an f-invariant subset of L. Let us denote the set of all f-invariant subsets of a hypersemilattice by $I_f(L)$.

Theorem 2.12. *Let f be a multiplier of a hypersemilattice $(L, \otimes)$. Then $(I_f(L), \otimes)$ is a hypersemilattice.*

Proof. Let $A, B \in I_f(L)$. Let $x \in A \otimes B$. Then $x = a \otimes b$ for some $a \in A$ and $b \in B$. Hence $f(x) = f(a \otimes b) = a \otimes f(b) \in A \otimes B$. Therefore $A \otimes B$ is f-invariant. It is clear that $A \subseteq A \otimes A$ for any $A \subseteq L$. Again, we have $x = a \otimes b = b \otimes a$. Hence, $A \otimes B = B \otimes A$. In a similar way, we can prove that $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for any $A, B, C \in I_f(L)$. Therefore, $(I_f(L), \otimes)$ is a hypersemilattice. ∎

We now introduce a congruence on L in terms of multipliers.

Proposition 2.13. *Let f be a multiplier of a hypersemilattice $(L, \otimes)$. Define a relation θ_f on L by $(x, y) \in \theta_f$ if and only if $f(x) = f(y)$ for all $x, y \in L$. Then θ_f is a congruence on L.*

Proof. Clearly θ_f is an equivalence relation on L. Let $(a, b), (c, d) \in \theta_f$. Then $f(a) = f(b)$ and $f(c) = f(d)$. Let $x \in a \otimes c$. Then $f(x) \in f(a \otimes c) = a \otimes f(c) = a \otimes f(d) = f(a \otimes d) = f(a) \otimes d = f(b) \otimes d = f(b \otimes d)$. Hence $f(x) = f(y)$ for some $y \in b \otimes d$. Therefore $(a \otimes c, b \otimes d) \in \theta_f$. ∎

Definition 2.14. Let f be a multiplier of L. A congruence θ on L is called f-invariant if it satisfies the following property:

$$(x, y) \in \theta \text{ if and only if } (f(x), f(y)) \in \theta \text{ for all } x, y \in L$$

In [5], the author proved that $L_{/\theta}$ is a hyperlattice for any congruence θ on a hyperlattice L. Hence the following result is a similar observation.

Theorem 2.15. *If θ is a congruence on L, then L/θ is a hypersemilattice.*

In view of Theorem 2.15, we conclude this article with the existence of an injective multiplier on the quotient lattice $L_{/\theta}$, where θ is an f-invariant congruence on L.

Theorem 2.16. *Let f be an injective multiplier of an a hypersemilattice L. If θ is an f-invariant congruence on L, then there exists an injective multiplier on L/θ.*

Proof. By Theorem 2.16, L/θ is a hypersemilattice. Define $h : L/\theta \longrightarrow L/\theta$ by $h([x]_\theta) = [f(x)]_\theta$ for all $[x]_\theta \in L/\theta$. Clearly h is well defined. Let $[x]_\theta, [y]_\theta \in L/\theta$ be such that $h([x]_\theta) = h([y]_\theta)$. Then $[f(x)]_\theta = [f(y)]_\theta$. Hence $(f(x), f(y)) \in \theta$. Since θ is invariant under f, we get $(x, y) \in \theta$. Thus $[x]_\theta = [y]_\theta$. Therefore, h is injective. Again

$$\begin{aligned} h([x]_\theta \otimes [y]_\theta) &= h(\{[t]_\theta \mid t \in x \otimes y\}) \\ &= \{[f(t)]_\theta \mid t \in x \otimes y\} \\ &= \{[f(t)]_\theta \mid f(t) \in f(x \otimes y)\} \\ &= \{[f(t)]_\theta \mid f(t) \in x \otimes f(y)\} \\ &= [x]_\theta \otimes [f(y)]_\theta \\ &= [x]_\theta \otimes h([y]_\theta) \end{aligned}$$

Therefore, h is an injective multiplier of L/θ. ∎

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