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Total edge irregularity strength of the disjoint union of sun graphs

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Abstract

An edge irregular total k-labeling $\varphi : V \cup E \to \{1, 2, \dots, k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of G in such a way that for any different edges uv and $u'v'$ their weights $\varphi(u) + \varphi(uv) + \varphi(v)$ and $\varphi(u') + \varphi(u'v') + \varphi(v')$ are distinct. The total edge irregularity strength, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling. In this paper, we consider the total edge irregularity strength of the disjoint union of *p* isomorphic sun graphs, $tes(pM_n)$, disjoint union of *p* consecutive non-isomorphic sun graphs, $tes(\bigcup_{j=1}^{p} M_{n_j})$.

Keywords: irregularity strength, total edge irregularity strength, edge irregular total labeling, disjoint union of sun graphs.
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1 Introduction

The graphs in this paper are simple, finite and undirected. In [5] Bača, Jendroľ, Miller and Ryan defined the notion of an *edge irregular total k-labeling* of a graph $G = (V, E)$ to be a labeling of vertices and edges of G $\psi : V \cup E \to \{1, 2, \dots, k\}$ such that the *edge weights* $wt_\psi(uv) = \psi(u) + \psi(uv) + \psi(v)$ are different for all the edges, that is, $wt_\psi(uv) \neq wt_\psi(u'v')$ for all the edges $uv, u'v' \in E$. The minimum k for which the graph G has an edge irregular total k-labeling is called *the total edge irregularity strength of* G, $tes(G)$.

The definition of total edge irregularity strength came from irregular assignments and the irregularity strength of graphs introduced by Chartrand et.al. [8]. An *irregular assignment* is a k-labeling of the edges $\psi : E \to \{1, 2, \dots, k\}$ such that the sum of the labels of edges incident with a vertex is different for all the vertices of G and the smallest k for which there is an irregular assignments is the *irregularity strength*, $s(G)$.

Computing the irregularity strength of a graph is difficult even for graphs with simple structure [6], [9], [12] and [17]. Karoński et.al conjectured that the edges of every connected graph of order at least 3 can be assigned labels from $\{1, 2, 3\}$, such that for all pairs of adjacent vertices, the sums of the labels of the incident edges are different.

Bača et.al [5] found a lower bound on the total edge irregularity strength of a graph:

$$tes(G) \geq \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}, \tag{1}$$

where $\Delta(G)$ is the maximum degree of G. They also determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs.

Recently Ivančo and Jendroľ [11] posed the following conjecture:

Conjecture 1.1. [11] *Let G be an arbitrary graph different from K_5. Then*

$$tes(G) = \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}. \tag{2}$$

Conjecture 1 has been verified for trees in [11], for complete graphs and complete bipartite graphs in [13] and [14], for the Cartesian product of two paths $P_n \Box P_m$ in [16], for corona product of a path with certain graphs in [18], for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [7], for the categorical product of two paths $P_n \times P_m$ in [3] ,for the categorical product a cycle and a path $C_n \times P_m$ in [1, 19], for strong product in [4] and for Subdivision of star in [20].

Motivated by [2], we investigate the total edge irregularity strength of the disjoint union of sun graphs.

A *sun graph* M_n is defined as the graph obtained by adding a pendent edge to every vertex of the cycle C_n. In this paper, we determine the total edge irregularity strength of the disjoint union of p isomorphic sun graphs and the disjoint union of p consecutive non isomorphic sun graphs.
The results in this paper add further support to Conjecture 1 by demonstrating that the disjoint union sun graphs has total edge irregularity strength equal to $\left\lceil\frac{|E(\bigcup_{j=1}^{p} M_{n_j})|+2}{3}\right\rceil$.

2 Main Results

First we determine the total edge irregularity strength of the disjoint union of p isomorphic sun graphs with the vertex set $V(pM_n)$ and the edge set $E(pM_n)$, where

$$V(pM_n) = \left\{x_i^j,\ y_i^j : 1 \leq i \leq n, 1 \leq j \leq p\right\}$$

$$E(pM_n) = \left\{x_i^j y_i^j,\ x_i^j x_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq p\right\}$$

where the subscript i is taken modulo n.

Lemma 2.1. *Let $n \geq 3$. Then $tes(2M_n) = \lceil\frac{4n+2}{3}\rceil$.*

Proof. Since $|E(2M_n)| = 4n$, then from (1) it follows that $tes(2M_n) \geq \lceil\frac{4n+2}{3}\rceil$. Let $k = \left\lceil\frac{4n+2}{3}\right\rceil$. To prove the converse inequality, we construct the labeling $\varphi_1 : V \cup E \to \{1, 2, \ldots, k\}$ for $1 \leq i \leq n$ as follows:

$$\varphi_1(x_i^1 y_i^1) = i+1, \quad \varphi_1(x_i^2 y_i^2) = i,$$
$$\varphi_1(x_i^1 x_{i+1}^1) = i, \quad \varphi_1(x_i^2 x_{i+1}^2) = 3n+2-2k+i,$$
$$\varphi_1(x_i^1) = 1, \quad \varphi_1(x_i^2) = k, \quad \varphi_1(y_i^1) = n, \quad \varphi_1(y_i^2) = 2n+2-k.$$

Under the labeling φ_1, the weights of the edges are described as follows:

(i) The edges $(x_i^1 x_{i+1}^1)$ admits the consecutive integers from 3 to $n+2$,

(ii) The edges $(x_i^1 y_i^1)$ admits the consecutive integers from $n+3$ to $2n+2$,
(iii) The edges $(x_i^2 y_i^2)$ admits the consecutive integers from $2n+3$ to $3n+2$,
(iv) The edges $(x_i^2 x_{i+1}^2)$ admits the consecutive integers from $3n+3$ to $4n+2$.

It is easy to see that the weight of the edges are pair wise distinct. Thus, φ_1 is the desired edge irregular k-labeling. This concludes the proof. ∎

Theorem 2.2. *Let p, $n \geq 3$ be two integers. Then the total edge irregularity strength of the disjoint union of p isomorphic sun graphs is* $\left\lceil \frac{2(pn+1)}{3} \right\rceil$.

Proof. As $|E(pM_n)| = 2pn$, (1) implies that $tes(pM_n) \geq \left\lceil \frac{2(pn+1)}{3} \right\rceil$. We suppose that $k = \left\lceil \frac{2(pn+1)}{3} \right\rceil$. To prove the reverse inequality, we define the total edge irregular k-labeling ψ_1 for $1 \leq i \leq n$ and $1 \leq j \leq p$ as follows:

$$\psi_1(x_i^j) = \psi_1(y_i^j) = \min\left\{(j-1)n+1,\ k\right\}$$

- For $1 \leq j \leq p$ such that $(j-1)n+1 < k$,

$$\psi_1(x_i^j y_i^j) = i, \quad \psi_1(x_i^j x_{i+1}^j) = n+i$$

- For $1 \leq j \leq p$ such that $(j-1)n+1 \geq k$,

$$\psi_1(x_i^j y_i^j) = \begin{cases} 2(l-k+n)+i, & \text{if } j = w \\ 2(l-k)+2n(j-w+1)+i, & \text{if } w+1 \leq j \leq p \end{cases}$$

$$\psi_1(x_i^j x_{i+1}^j) = \begin{cases} 2(l-k)+3n+i, & \text{if } j = w \\ 2(l-k)+n(2j-2w+3)+i, & \text{if } w+1 \leq j \leq p \end{cases}$$

where

$$w = \min\left\{j\ ;\quad 1 \leq j \leq p \ \textit{such that}\ (j-1)n+1 \geq k\right\}$$
$$l = \max\left\{t_j\ ;\quad 1 \leq j \leq p \ \textit{such that}\ (j-1)n+1 < k\right\}$$
$$t_j = \min\left\{(j-1)n+1,\ k\right\}\ \textit{for}\ 1 \leq j \leq p\ \textit{such that}\ (j-1)n+1 < k$$

Under the labeling ψ_1, the total weights of the edges are described as follows:

(i) The edges $(x_i^1 y_i^1)$ for $1 \leq i \leq n$ receive the consecutive integers from the interval $[3,\ n+2]$.

(ii) The edges $(x_i^j y_i^j)$ for $1 \leq i \leq n$ and $2 \leq j \leq p$ receive the consecutive integers from the interval $[2n(j-1)+3,\ (2j-1)n+2]$.

(iii) The edges $(x_i^1 x_{i+1}^1)$ for $1 \leq i \leq n$ receive the consecutive integers from the interval $[n+3,\ 2n+2]$.

(iv) The edges $(x_i^j x_{i+1}^j)$ for $1 \leq i \leq n$ and $2 \leq j \leq p$ receive the consecutive integers from the

interval $[n(2j-1)+3,\ 2nj+2]$.

It cab be easily verified that all the vertex and edge labels are at most k and the edge-weights of the edges $(x_i^j y_i^j)$ and $(x_i^j x_{i+1}^j)$ are pairwise distinct. Thus, the resulting total labeling is the desired edge irregular k-labeling. This concludes the proof. ∎

Now we determine the total edge irregularity strength of the disjoint union of p consecutive non-isomorphic sun graphs with the vertex set $V(\bigcup_{j=1}^{p} M_{n_j})$ and the edge set $E(\bigcup_{j=1}^{p} M_{n_j})$, where

$$V(\bigcup_{j=1}^{p} M_{n_j}) = \left\{x_i^j,\ \ y_i^j : 1 \le i \le n_j, 1 \le j \le p\right\}$$

$$E(\bigcup_{j=1}^{p} M_{n_j}) = \left\{x_i^j y_i^j,\ \ x_i^j x_{i+1}^j : 1 \le i \le n_j, 1 \le j \le p\right\}$$

where the subscript i is taken modulo n.

Lemma 2.3. $tes(M_{n_1} \cup M_{n_2}) = \left\lceil \frac{2(n_1+n_2+1)}{3} \right\rceil$.

Proof. Let $k = \left\lceil \frac{2(n_1+n_2+1)}{3} \right\rceil$, then from (1) it follows that $tes(M_{n_1} \cup M_{n_2}) \ge k$. Now to prove the converse inequality we define an edge irregular k-labeling as follows:

$$\varphi_2(x_1^2 x_2^2) = 3n_1 - 2k + n_2 + 2,$$

$$\varphi_2(x_i^1 x_{i+1}^1) = \varphi_2(x_i^1 y_i^1) = i, \quad for \quad 1 \le i \le n_1$$

$$\varphi_2(x_i^2 x_{i+1}^2) = 2n_1 - 2k + n_2 + i, \quad for \quad 2 \le i \le n_2$$

$$\varphi_2(x_i^2 y_i^2) = 2n_1 - 2k + 2n_2 + 2, \quad for \quad 1 \le i \le n_2$$

$$\varphi_2(x_1^2) = k - n_1, \quad \varphi_2(x_i^2) = k + 2 - i, \quad for \quad 2 \le i \le n_2$$

$$\varphi_2(x_i^1) = 1, \quad \varphi_2(y_i^1) = 1 + n_1, \quad \varphi_2(y_i^2) = k. \quad for \quad 1 \le i \le n_1$$

It is easy to check that all the edge weights are pair wise distinct. Thus the labeling φ_2 is the desired edge irregular total k-labeling. ∎

Lemma 2.4. $tes(M_{n_1} \cup M_{n_2} \cup M_{n_3}) = \left\lceil \frac{2(n_1+n_2+n_3+1)}{3} \right\rceil$.

Proof. From (1) it follows that $tes(M_{n_1} \cup M_{n_2} \cup M_{n_3}) \ge \left\lceil \frac{2(n_1+n_2+n_3+1)}{3} \right\rceil$. Let $k = \left\lceil \frac{2(n_1+n_2+n_3+1)}{3} \right\rceil$. The existence of the optimal labeling φ_3 below proves the reverse inequality.

$$\varphi_3(x_i^1) = \varphi_3(y_i^1) = 1, \quad for \quad 1 \le i \le n_1$$

$$\varphi_3(x_i^2) = \varphi_3(y_i^2) = 1 + n_1, \quad for \quad 1 \le i \le n_2$$

$$\varphi_3(x_i^3) = \varphi_3(y_i^3) = \begin{cases} 2(1+n_1), & \text{if } 1 \le i \le 2 \\ k, & \text{if } 3 \le i \le n_3 \end{cases}$$

$$\varphi_3(x_i^1 x_{i+1}^1) = i + n_1, \quad \varphi_3(x_i^1 y_i^1) = i, \quad for \quad 1 \le i \le n_1$$

$$\varphi_3(x_i^2 x_{i+1}^2) = i + n_2, \quad \varphi_3(x_i^2 y_i^2) = i, \quad for \quad 1 \le i \le n_2$$

$$\varphi_3(x_i^3 x_{i+1}^3) = 1 + i, \quad \varphi_3(x_i^3 y_i^3) = i + n_1, \quad for \quad 3 \le i \le n_3$$

$$\varphi_3(x_1^3 y_1^3) = 1, \ \varphi_3(x_1^3 x_2^3) = \varphi_3(x_2^3 x_3^3) = 4, \ \varphi_3(x_2^3 y_2^3) = \varphi_3(x_{n_3}^3 x_1^3) = 2.$$

Obviously all the vertex and edge labels are at most k and the edge-weights are pairwise distinct. Thus, the resulting total labeling is the desired edge irregular k-labeling. ∎

Theorem 2.5. *Let $p \ge 4$. Then the total edge irregularity strength of the disjoint union of p consecutive non isomorphic sun graphs is $\left\lceil \frac{2(\sum_{j=1}^{p} n_j + 1)}{3} \right\rceil$.*

Proof. As $|E(\bigcup_{j=1}^{p} M_{n_j})| = 2\sum_{j=1}^{p} n_j$, from (1) it follows that $tes(\bigcup_{j=1}^{p} M_{n_j}) \ge \left\lceil \frac{2(\sum_{j=1}^{p} n_j + 1)}{3} \right\rceil$. Let $k = \left\lceil \frac{2(\sum_{j=1}^{p} n_j + 1)}{3} \right\rceil$. To prove the converse inequality, we define the total edge irregular labeling ψ_2 for $1 \le i \le n_j$ and $1 \le j \le p$ as follows:

$$\psi_2(x_i^1) = \psi_2(y_i^1) = 1 \ \ for \ \ 1 \le i \le n_1$$

$$\psi_2(x_i^j) = \psi_2(y_i^j) = \min\left\{ \sum_{s=1}^{j-1} n_s + 1, k \right\} \ for \ \ 2 \le j \le p$$

$$\psi_2(x_i^1 y_i^1) = i, \quad \psi_2(x_i^1 x_{i+1}^1) = n_1 + i \ \ for \ \ 1 \le i \le n_1$$

- For $2 \le j \le p$ such that $\sum_{s=1}^{j-1} n_s + 1 < k$

$$\psi_2(x_i^j y_i^j) = i, \quad \psi_2(x_i^j x_{i+1}^j) = n_j + i$$

- For $2 \le j \le p$ such that $\sum_{s=1}^{j-1} n_s + 1 \ge k$

$$\psi_2(x_i^j y_i^j) = \begin{cases} 2\left(\sum_{s=1}^{w-1} n_s - k + 1\right) + i, & \text{if } j = w \\ 2\left(\sum_{s=1}^{w-1} n_s - k + 1\right) & \text{if } w+1 \le j \le p \\ +2n_{j-1} + i, & \end{cases}$$

$$\psi_2(x_i^j x_{i+1}^j) = \begin{cases} 2\left(\sum\limits_{s=1}^{w-1} n_s - k + 1\right) + n_w + i, & \text{if } j = w \\ 2\left(\sum\limits_{s=1}^{w-1} n_s - k + 1\right) + n_w & \text{if } w+1 \le j \le p \\ +n_{j-1} + n_j + i, & \end{cases}$$

where

$$w = \min\Big\{j\ ;\quad 2 \le j \le p \ such\ that\ \sum_{s=1}^{j-1} n_s + 1 \ge k\Big\}$$

Under the labeling ψ_2, the total weights of the edges are described as follows:

(i) The edges $(x_i^1 y_i^1)$ for $1 \le i \le n_1$ receive the consecutive integers from the interval $[3, 2+n_1]$.

(ii) The edges $(x_i^j y_i^j)$ for $1 \le i \le n_j$ and $2 \le j \le p$ receive the consecutive integers from the interval $[2(\sum_{s=1}^{j-1} n_s + 1) + 1,\ 2(\sum_{s=1}^{j-1} n_s + 1) + n_j]$.

(iii) The edges $(x_i^1 x_{i+1}^1)$ for $1 \le i \le n_1$ receive the consecutive integers from the interval $[3 + n_1, 2n_1 + 2]$.

(iv) The edges $(x_i^j x_{i+1}^j)$ for $1 \le i \le n_j$ and $2 \le j \le p$ receive the consecutive integers from the interval $[2(\sum_{s=1}^{j-1} n_s + 1) + n_j + 1,\ 2(\sum_{s=1}^{j-1} n_s + 1)]$.

It can be easily verified that all the vertex and edge labels are at most k and the edge-weights of the edges $(x_i^j y_i^j)$ and $(x_i^j x_{i+1}^j)$ are pairwise distinct. Thus, the resulting total labeling is the desired edge irregular k-labeling. ∎

Open Problem: In this paper we determine the total edge irregularity strength of the disjoint union of *p* isomorphic sun graphs and disjoint union of *p* consecutive non isomorphic sun graphs. We tried to find an edge irregular total labeling for disjoint union of any sun graphs but so far without success. So, we conclude with the following open problem.

Find the total edge irregularity strength of disjoint union of any sun graphs.

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