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Prime labeling in the context of duplication of graph elements

S K Vaidya
Saurashtra University, Rajkot - 360005,
Gujarat, INDIA.
E-mail: samirkvaidya@yahoo.co.in

U M Prajapati
St. Xavier's College, Ahmedabad - 380009,
Gujarat, INDIA.
E-mail: udayan64@yahoo.com

Abstract

In this paper, we investigate prime labeling for some graphs obtained by duplication of graph elements and also we derive some result for P_n, $K_{1,n}$ and C_n in this context.

Keywords: Graph labeling, prime labeling, prime graph.
AMS Subject Classification(2010): 05C78.

1 Introduction

We begin with finite, undirected and non-trivial graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The elements of $V(G)$ and $E(G)$ are commonly termed as graph elements. Throughout this work C_n denotes the cycle with n vertices and P_n denotes the path on n vertices. The star $K_{1,n}$ is a graph with a vertex of degree n called apex and n vertices of degree one called pendant vertices. Throughout this paper $|V(G)|$ and $|E(G)|$ denote the cardinality of the vertex set and edge set respectively. For various graph theoretic notation and terminology we follow West [14] and for number theory we follow Burton [1]. We give brief summary of definitions and other information which are useful for the present investigation.

Definition 1.1. *If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.*

Definition 1.2. *A prime labeling of a graph G is an injective function $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for every pair of adjacent vertices u and v, $gcd(f(u), f(v)) = 1$.*

The graph which admits a prime labeling is called a *prime graph*.

The notion of a prime labeling was originated by Entringer and it was discussed by Tout et al [6]. Fu and Huang [3] proved that P_n and $K_{1,n}$ are prime graphs. Lee et al [5] proved that W_n is a prime graph if and only if n is even. Deretsky et al [2] proved that C_n is a prime graph. Vaidya and Kanani [9] discussed prime labeling of some cycle related graphs. A stronger concept of k-prime labeling is also introduced by Vaidya and Prajapati [10]. The switching invariance of various graphs is discussed by Vaidya and Prajapati [11] and the same authors introduced the concept of strongly prime graph in [12]. A variant of prime labeling known as vertex-edge prime labeling is also introduced by Venkatachalam and Antoni Raj in [13].

Definition 1.3. *[14]*Duplication *of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.*

Definition 1.4. *[7]* Duplication *of a vertex v_k by a new edge $e = v'_k v''_k$ in a graph G produces a new graph G' such that $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$.*

Definition 1.5. *[7]* Duplication *of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.*

Definition 1.6. *[8]* Duplication *of an edge $e = uv$ of a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$.*

Bertrand's Postulate 1.7 [1]: For every positive integer $n > 1$ there is a prime p such that $n < p < 2n$.

2 Duplication of Graph Elements in P_n

Theorem 2.1. *The graph obtained by duplication of a vertex in P_n is a prime graph.*

Proof. The result is obvious for $n = 1, 2$. Therefore we start with $n \geqslant 3$. Let $v_1, v_2, \dots, v_n$ be the consecutive vertices of P_n and G be the graph obtained by duplication of the vertex v_j by a new vertex v'_j. Depending upon the $deg(v_j)$ we have the following cases:
Case (i) If $deg(v_j) = 1$ then v_j is either v_1 or v_n. Without loss of generality let $v_j = v_1$. Then define $f : V(G) \to \{1, 2, \dots, n, n+1\}$ as $f(v_i) = i - 1, \forall i = 2, 3, \dots, n$, $f(v_1) = n$ and $f(v'_1) = n + 1$.
Case (ii) If $deg(v_j) \neq 1$ then $j \in \{2, 3, \dots, n-2, n-1\}$. Define $f : V(G) \to \{1, 2, \dots, n, n+1\}$ as $f(v_j) = 4$, $f(v_{j-1}) = 3$, $f(v_{j+1}) = 1$, $f(v'_j) = 2$, $f(v_k) = j - k + 3, \forall k = j-2, j-3, \dots, 1$ and $f(v_k) = k + 1, \forall k = j+2, j+3, \dots, n$.
In both the cases f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ∎

Theorem 2.2. *The graph obtained by duplication of a vertex by an edge in P_n is a prime graph.*

Proof. We start with $n \geqslant 3$ as the result is obvious for $n = 1, 2$. Let $v_1, v_2, \dots, v_n$ be the consecutive vertices of P_n and G be the graph obtained by duplication of a vertex v_j by an edge $v'_j v''_j$. Then G contains a cycle C_3 whose vertices are v_j, v'_j and v''_j, which is a prime graph with an assignment of labels in such a way that the label 1 is assigned to v_j. Consequently G is a graph in which at the most two paths are attached at v_j. Then such a graph is a prime graph as proved by Vaidya and Prajapati in [10]. ∎

Theorem 2.3. *The graph obtained by duplication of every vertex by an edge in P_n is not a prime graph.*

Proof. We start with $n \geqslant 3$ as the result is obvious for $n = 1, 2$. Let $v_1, v_2, \dots, v_n$ be the consecutive vertices of P_n and G be the graph obtained by duplication of every vertex v_j by an edge $v'_j v''_j$ for $j = 1, 2, \dots, n$. Then G is a graph with $3n$ vertices and having n vertex disjoint cycles each of length three. Any prime labeling of G must contains at the most one even label in each of these n cycles as it is not possible to assign even labels for two adjacent vertices. Consequently at the most n vertices will receive even labels out of $3n$ vertices. Hence the number of even integers which are left to be used as labels is at least $\left\lfloor \frac{3n}{2} \right\rfloor - n = \left\lfloor \frac{3n}{2} - n \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \geqslant 1$. That means at least one even integer from $\{1, 2, \ldots, 3n\}$ is left for label assignment. This is not possible as prime labeling is bijective. Hence G is not a prime graph. ■

Theorem 2.4. *The graph obtained by duplication of an edge by a vertex in P_n is a prime graph.*

Proof. We start with $n \geqslant 3$ because the result is obvious for $n = 1, 2$. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of P_n and G be the graph obtained by duplication of an edge $v_j v_{j+1}$ by a vertex v'_j. Define $f : V(G) \to \{1, 2, \ldots, n, n+1\}$ by $f(v_j) = 1, f(v'_j) = 2, f(v_k) = k - j + 2, \forall k = j+1, j+2, \ldots, n$, $f(v_k) = n + k + 2 - j, \forall k = 1, 2, \ldots, j-1$.

Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ■

Theorem 2.5. *The graph obtained by duplication of an edge in P_n is a prime graph.*

Proof. We start with $n \geqslant 4$ because the result is obvious for $n = 1, 2, 3$. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of P_n and G be the graph obtained by duplication of an edge $e = v_j v_{j+1}$ by an edge $v'_j v'_{j+1}$ in P_n. Depending upon the types of edges of P_n we have the following two cases:
Case (i): If the edge e is not a pendant edge of P_n then G contains a cycle $C_6 = v_{j+2} v'_{j+1} v'_j v_{j-1} v_j v_{j+1} v_{j+2}$. The vertices $v_{j+2}, v'_{j+1}, v'_j, v_{j-1}, v_j, v_{j+1}$ can be labeled as 1,6,5,2,3,4 respectively. Consequently C_6 is a prime graph. Now G can be considered as a graph with two paths attached to C_6 at v_{j-1} and at v_{j+2} respectively. The vertices of the path attached to v_{j-1} can be labeled consecutively with $7, 8, \ldots, t-1$. On the other hand the vertices of the remaining path attached to v_{j+2} can be labeled consecutively as $t, t+1, \ldots, n+1, n+2$. Such a labeling is a prime labeling of G. Hence G admits a prime labeling.
Case (ii): If the edge e is a pendant edge of P_n say $e = v_{n-1} v_n$ then G can be considered as a graph in which two paths $v_{n-2} v'_{n-1} v'_n$ and $v_{n-2} v''_{n-1} v''_n$ each of length 2 attached to the path $v_{n-2} v_{n-3} \ldots v_2 v_1$ at v_{n-2}. Such a graph admits a prime labeling as proved in [10]. ■

3 Duplication of Graph Elements in $K_{1,n}$

Theorem 3.1. *The graph obtained by duplication of a vertex in $K_{1,n}$ is a prime graph.*

Proof. Let v_0 be the apex vertex and $v_1, v_2, \ldots, v_n$ be the consecutive pendant vertices of $K_{1,n}$. Let G be the graph obtained by duplication of a vertex v_j by a vertex v'_j in $K_{1,n}$. Depending upon the $deg(v_j)$ in $K_{1,n}$ we have the following cases:
Case (i): If $deg(v_j) = n$ then $v_j = v_0$. Define $f : V(G) \to \{1, 2, \ldots, n+1, n+2\}$ as $f(v_i) = i + 1, \forall i = 0, 1, 2, 3, \ldots, n$, $f(v'_0) = n + 2$. Let p be the largest prime in the set $\{1, 2, \ldots, n+1, n+2\}$. Using the Bertrand's Postulate $\left\lfloor \frac{n+2}{2} \right\rfloor \leqslant p \leqslant n+2$ and p is relatively prime to every integer from the

set $\{1,2,\ldots,n+1,n+2\}-\{p\}$. As f is onto there exists $k \in \{1,2,\ldots,n\}$ such that $f(v_k)=p$. Using the function f define $h : V(G) \to \{1,2,\ldots,n+1,n+2\}$ as $h(x)=f(x)$ if $x \in V(G)-\{v_k, v_0'\}$, $h(v_k)=n+2$, $h(v_0')=p$. In this case h is an injection and it admits a prime labeling for G.
Case (ii): If $v \neq v_0$ then we may assume that $v=v_n$. $G=K_{1,n+1}$, which is again a star graph. Hence it is a prime graph as proved in [3]. ■

Theorem 3.2. *The graph obtained by duplication of a vertex by an edge in $K_{1,n}$ is a prime graph.*

Proof. Let v_0 be the apex vertex and $v_1, v_2, \ldots, v_n$ be the consecutive pendant vertices of $K_{1,n}$. Let G be the graph obtained by duplication of a vertex v_j in $K_{1,n}$ by an edge $v_j'v_j''$. Depending upon to the $deg(v_j)$ in $K_{1,n}$ we have the following cases:

Case (i): If $deg(v_j)=n$ in $K_{1,n}$ then $v_j=v_0$. Define $f : V(G) \to \{1,2,\ldots,n+2,n+3\}$ as $f(v_i)=i+1, \forall i=0,1,2,\ldots,n$, $f(v_0')=n+2$ and $f(v_0'')=n+3$. Then f is an injection and it admits a prime labeling for G.

Case (ii): If $deg(v_j) \neq n$ in $K_{1,n}$ then $v_j \neq v_0$. Without loss of generality we assume that $v_j = v_n$. Then in G we have a cycle of length three having vertices v_n, v_n' and v_n''. Label the vertices v_0, v_n, v_n', v_n'' by $1, 3, 4$ and 5 respectively. And label the remaining vertices $v_1, v_2, \ldots, v_{n-1}$ arbitrarily with $2, 6, 7, 8, \ldots, n+3$. This labeling admits a prime labeling for G.
Hence G is a prime graph in both the cases. ■

Theorem 3.3. *The graph obtained by duplication of every vertex by an edge in $K_{1,n}$ is not a prime graph if $n \geqslant 2$.*

Proof. Let v_0 be the apex vertex and $v_1, v_2, \ldots, v_n$ be the consecutive pendant vertices of $K_{1,n}$. Let G be the graph obtained by duplicating each of the vertices v_j in $K_{1,n}$ by an edge $v_j'v_j''$. Then G is a graph with $3n+3$ vertices and having $n+1$ vertex disjoint cycles each of length three. Any prime labeling of G contains at the most one even label in each of these $n+1$ cycles as it is not possible to assign even labels for two adjacent vertices. Consequently at the most $n+1$ vertices receive even labels out of $3n+3$ vertices. Hence the number of even integers which are left to be used as labels is at least $\left[\dfrac{3n+3}{2}\right]-(n+1) = \left[\dfrac{3n+3}{2}-(n+1)\right] = \left[\dfrac{n+1}{2}\right] \geqslant 1$. That means at least one even integer from $\{1,2,\ldots,3n+3\}$ is left for label assignment. This is not possible as prime labeling is bijective. Hence G is not a prime graph. ■

Theorem 3.4. *The graph obtained by duplication of an edge by a vertex in $K_{1,n}$ is a prime graph.*

Proof. Let v_0 be the apex vertex and $v_1, v_2, \ldots, v_n$ be the consecutive pendant vertices of $K_{1,n}$. Let G be the graph obtained by duplication of the edge v_0v_n in $K_{1,n}$ by a vertex v_n'. Define $f : V(G) \to \{1,2,\ldots,n+1,n+2\}$ as $f(v_i)=i+1, \forall i=0,1,2,\ldots,n$, $f(v_n')=n+2$. Then obviously f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ■

Theorem 3.5. *The graph obtained by duplication of every edge by a vertex of $K_{1,n}$ is a prime graph.*

Proof. Let v_0 be the apex vertex and $v_1, v_2, \ldots, v_n$ be the consecutive pendant vertices of $K_{1,n}$. Let G be the graph obtained by duplication of each of the edges v_0v_j by a vertex v'_j. In G, $|V(G)| = 2n + 1$ and $|E(G)| = 3n$. So G being a one point union of n cycles, is a prime graph as reported by Gallian in [4]. ∎

Theorem 3.6. *The graph obtained by duplication of an edge in $K_{1,n}$ is a prime graph.*

Proof. Let v_0 be the apex vertex and $v_1, v_2, \ldots, v_n$ be the consecutive pendant vertices of $K_{1,n}$. Let p be the largest prime less than or equal to $n+1$. Then p is relatively prime to every integer in $\{1, 2, \ldots, n+1\} - \{p\}$. Let G be the graph obtained by duplication of the edge $e = v_0v_{p-1}$ by a new edge $e' = v'_0v'_{p-1}$. Hence in G, $deg(v_0) = n, deg(v'_0) = n-1$, $deg(v_{p-1}) = 1$, $deg(v'_{p-1}) = 1$ and $deg(v_i) = 2, \forall i \in \{1, 2, \ldots, n\} - \{p-1\}$. The integers $n+2$ and $n+3$ are relatively prime being consecutive integers. So p cannot divide both. Therefore we have the following cases:

Case (i): If p does not divide $n+2$ then define $g : V(G) \to \{1, 2, \ldots, n+2, n+3\}$ as

$$g(x) = \begin{cases} i+1 & \text{if } i \in \{0, 1, 2, \ldots, n\} - \{p-1\}\,; \\ n+3 & \text{if } x = v_{p-1}; \\ n+2 & \text{if } x = v'_{p-1}; \\ p & \text{if } x = v'_0. \end{cases}$$

Case (ii): If p does not divide $n+3$ then define $g : V(G) \to \{1, 2, \ldots, n+2, n+3\}$ as

$$g(x) = \begin{cases} i+1 & \text{if } i \in \{0, 1, 2, \ldots, n\} - \{p-1\}\,; \\ n+2 & \text{if } x = v_{p-1}; \\ n+3 & \text{if } x = v'_{p-1}; \\ p & \text{if } x = v'_0. \end{cases}$$

In both the cases g is an injection and it admits a prime labeling for G. Hence G is a prime graph. ∎

Illustration 3.7. *A prime labeling of the graph obtained by duplication of an edge e in $K_{1,8}$ is shown in the Figure 1.*

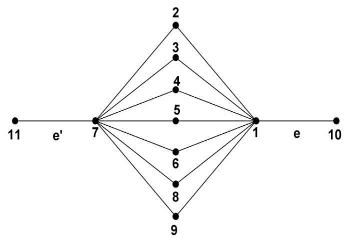

Figure 1: *The graph obtained by duplication of edge e in $K_{1,8}$ and its prime labeling.*

4 Duplication of Graph Elements in C_n

Theorem 4.1. *The graph obtained by duplication of a vertex by an edge in C_n is a prime graph.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of C_n. Let G be the graph obtained by duplication the vertex v_1 in C_n by new edge $v_1'v_1''$. Define $f : V(G) \to \{1, 2, \ldots, n+1, n+2\}$ as $f(v_i) = i, \forall i = 1, 2, \ldots, n$, $f(v_1') = n+1$ and $f(v_1'') = n+2$. Then obviously f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ∎

Theorem 4.2. *The graph obtained by duplication of every vertex by an edge in C_n is not a prime graph.*

Proof. We start with $n \geqslant 4$ as the result is obvious for $n = 1, 2, 3$. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of C_n and G be the graph obtained by duplication of each of the vertices v_j in C_n by a new edge $v_j'v_j''$ for $j = 1, 2, \ldots, n$. Then G is a graph with $3n$ vertices and having n vertex disjoint cycles each of length three. Any prime labeling of G contains at the most one even label in each of these n cycles as it is not possible to assign even labels for two adjacent vertices. Consequently at the most n vertices can receive even labels out of $3n$ vertices. Hence the number of even integers which are not assigned is at least $\left\lfloor \frac{3n}{2} \right\rfloor - n = \left\lfloor \frac{3n}{2} - n \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \geqslant 1$. That means there is at least one even integer from $\{1, 2, \ldots, 3n\}$ which is not assigned to any vertex. This is not possible as prime labeling is bijective. Hence G is not a prime graph. ∎

Theorem 4.3. *The graph obtained by duplication of an edge by a vertex in C_n is a prime graph.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of C_n. Let G be the graph obtained by duplication of the edge $v_{n-1}v_n$ by a vertex w. Define $f : V(G) \to \{1, 2, \ldots, n, n+1\}$ as $f(v_i) = i, \forall i = 1, 2, \ldots, n$, $f(w) = n+1$. Then obviously f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ∎

Theorem 4.4. *The graph obtained by duplication of every edge by a vertex in C_n is a prime graph.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of C_n. Let G be the graph obtained by duplication of all the edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1$ by new vertices $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ respectively. Then $|V(G)| = 2n$ and $|E(G)| = 3n$. Define $f : V(G) \to \{1, 2, \ldots, 2n\}$ as $f(v_i) = 2i - 1, \forall i = 1, 2, \ldots, n$ and $f(u_i) = 2i, \forall i = 1, 2, \ldots, n$. Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ∎

Illustration 4.5. *A prime labeling of the graph obtained by duplication of each edge by a vertex in C_6 is shown in Figure 2.*

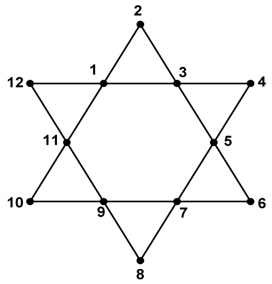

Figure 2: *The graph obtained by duplication of each edge by a vertex in C_6 and its prime labeling.*

Theorem 4.6. *The graph obtained by duplication of an edge in C_n is a prime graph if $n \geqslant 3$.*

Proof. We start with $n \geqslant 5$ as the result is obvious for $n = 3, 4$. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of C_n. Let G be the graph obtained by duplication of the edge v_1v_n in C_n by a new edge $v_1'v_n'$. Define $f : V(G) \to \{1, 2, \ldots, n+1, n+2\}$ as $f(v_{n-1}) = 1, f(v_2) = 2, f(v_1') = 3, f(v_n') = 4, f(v_1) = 5, f(v_n) = 6$ and $f(v_i) = i + 4$, $\forall i = 3, 4 \ldots, n-2$. Then f is an injection and it admits a prime labeling for G. Hence G is a prime graph. ∎

5 Concluding Remarks

We have investigated some results on prime labeling for the graphs resulted from the duplication of graph elements. Extending the study to other families of graph is an open area of research.

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