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Some results on (*k*, *d*) - even mean labeling

S. Kalaimathy
Department of Mathematics,
VHNSN College, Virudhunagar – 626001, INDIA.
E-mail: ranimathy@yahoo.co.in

Abstract

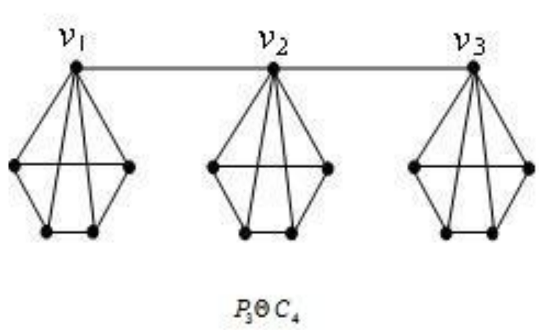
Let $G(V, E)$ be a graph with p vertices and q edges. A labeling is an assignment of numbers to vertices. For every labeling $f : V(G) \rightarrow \{0, 1, 2,\ldots, q\}$, an induced edge labeling $f^*: E(G) \rightarrow \{1,2,\ldots,q\}$ is defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$, if $f(u)$ and $f(v)$ are of same parity and $f^*(uv) = \frac{f(u)+f(v)+1}{2}$, otherwise. If the resulting edge labels are distinct, then f is called a mean labeling of G. If for a labeling $f : V(G) \rightarrow \{0, 1, 2,\ldots, 2k+2(q-1)d\}$, $f^*(E) = \{2k, 2k+2d, \ldots, 2k+2(q-1)d\}$, then f is called a (k, d) - even mean labeling of G. In this paper, we prove some results on (k, d) - even mean labeling of some graphs.

Keywords: *k*- even mean labeling, *k*- even mean graph, (*k*, *d*) - even mean labeling, (*k*, *d*) - even mean graph.

AMS Subject Classification(2010): 05C78.

1 Introduction

In this paper, we consider only finite, simple and undirected graphs. For notations and terminology, we follow [3]. The disjoint union of m copies of a graph G is denoted by mG. Let G_1 be a graph with vertices $v_1, v_2,\ldots, v_p$ and G_2 be any graph. The corona $G_1 \Theta G_2$ is the graph obtained from one copy of G_1 and p copies of G_2 by joining the vertex v_i of G_1 to every vertex in the i^{th} copy of G_2. The example for the corona graphs $P_3 \Theta C_4$ and $C_4 \Theta P_3$ are shown in Figure 1.



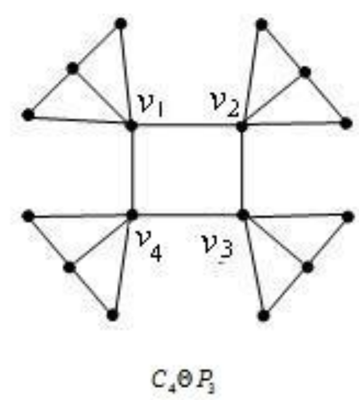


Figure 1.

Let G_1 and G_2 be two graphs with fixed vertices u and v respectively. Denote the graph $(G_1 \cup G_2) + uv$ by $G_1 \bullet G_2\,(uv)$.

Tthe graphs G_1, G_2 and $G_1 \bullet G_2\,(uv)$ are given in Figure 2.

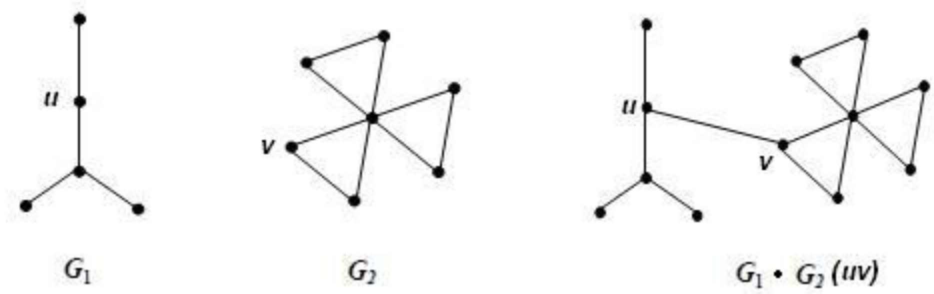


Figure 2.

For a graph G with a fixed vertex v, the graph $(P_m: G)$ is obtained from m copies of G and a path P_m: $u_1u_2...u_m$ by joining each u_i with the vertex corresponding to v in the i^{th} copy of G by means of an edge, for $1 \le i \le m$. Throughout this paper, k and d denote positive integers greater than or equal to 1.

The concept of mean *labeling* was initiated and studied by Somasundaram and Ponraj [10] in 2003. For more results on mean labeling, one can refer to [1], [2], [11], [12] and [13]. Recently, a notion of *odd mean labeling* was introduced in [9]. *k - odd mean labeling* and (*k, d*) - *odd mean labeling* have been introduced and discussed in [4], [5] and [6]. *k - even mean labeling* has been introduced in [7]. In 2011, Gayathri and Gopi [8] introduced the concept of (*k, d*) - even mean labeling.

A (p, q) graph G is said to have a (k, d) - *even mean labeling* [(k, d) – *EML*] if there exists an injection f: $V(G) \to \{0, 1, 2,\ldots, 2k+2(q-1)d\}$ such that the induced map f^* defined on E by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd}, \end{cases}$$

is a bijection from E onto $\{2k, 2k+2d, 2k+4d, \ldots, 2k+2(q-1)d\}$. A graph that admits a (k, d) - even mean labeling is called a (k, d) - *even mean graph.*

A (3, 3) - EML of a graph G and a (4, 3) - EML of the path P_5 are shown in Figure 3.

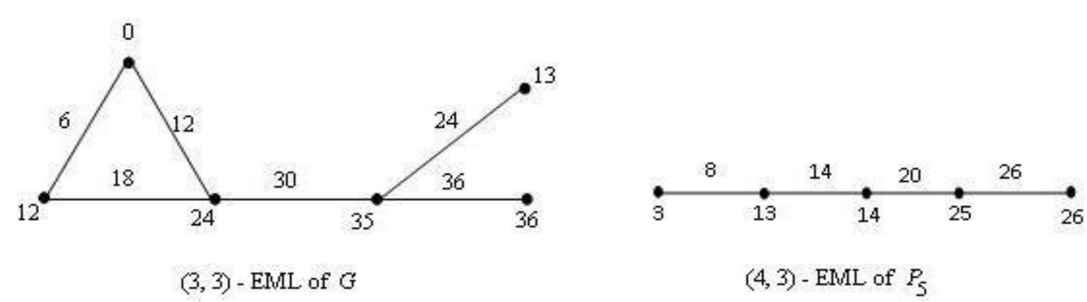


Figure 3.

In [8] it was proved that the graph $P_m \Theta\, nK_1$ $(m \ge 3, n \ge 2)$ is a (k, d) - even mean graph if

(i) m is even , $n \ge 1$, $k \ge 1$ and $d \ge 1$.

(ii) both m and n are odd, $k \ge 1$ and $d \ge 1$

and (iii) m is odd, n is even and $k \ge d \ge 1$.

In this paper, we prove some results on (k, d) - EML of some graphs.

2 (*k*, *k*) Even Mean Labeling

In this section, we prove some results on $(k, 1)$ and (k, k) - EML. Also, we prove that $G_1 \bullet G_2\,(uv)$ is a (1, 1) - even mean graph.

Lemma 2.1. *If a graph G has a mean labeling in which either*
(*i*) *no two adjacent vertices receive odd labels when q is even or*
(*ii*) *no two adjacent vertices receive even labels when q is odd*
then G has a (1, 1) - *EML.*

Proof. Let f: $V(G) \rightarrow \{0, 1, 2,\ldots, q\}$ be a mean labeling of G. Then $f^*(E) = \{1, 2,\ldots, q\}$. Let $u_1, u_2 \ldots u_p$ be the vertices of G.
When q is even, define g: $V(G) \rightarrow \{0, 1, 2,\ldots, 2q\}$ by

$$g(u_i) = \begin{cases} 2f(u_i), \text{when } f(u_i) \text{ is even} \\ 2f(u_i)+1, \text{when } f(u_i) \text{ is odd}. \end{cases}$$

Then for $1 \le i, j \le p$, $g^*(u_iu_j) = 2f^*(u_iu_j)$. Thus $g^*(E) = \{2, 4, \ldots, 2q\}$ and hence g is a (1, 1) - EML.
When q is odd, define g: $V(G) \rightarrow \{0, 1, 2,\ldots, 2q\}$ by

$$g(u_i) = \begin{cases} 2f(u_i)+1, \text{when } f(u_i) \text{ is even} \\ 2f(u_i), \quad \text{when } f(u_i) \text{ is odd}. \end{cases}$$

Then for $1 \le i, j \le p$, $g^*(u_iu_j) = 2f^*(u_iu_j)$. Thus $g^*(E) = \{2, 4, \ldots, 2q\}$ and g is a (1, 1) - EML. ■

The example for the mean labelings of the cube Q_3 (q is even) and the comb $P_5 \Theta K_1$ (q is odd) and the corresponding (1, 1) - EMLs are shown in Figure 4.

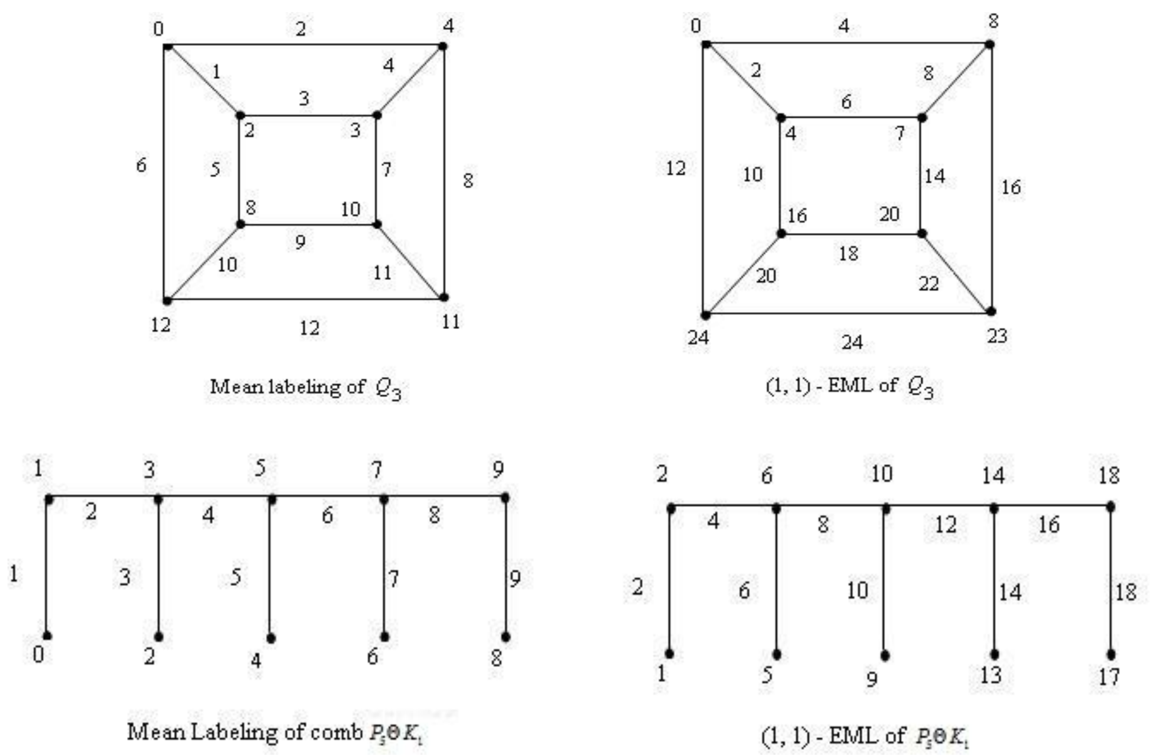


Figure 4.

Lemma 2.2. *If a graph G has a* (1, 1) - *EML in which no two adjacent vertices receive odd labels, then G has a* (k, k) - *EML for any* $k \geq 1$.

Proof. Let f: $V(G) \rightarrow \{0, 1, 2,\ldots, 2q\}$ be a (1, 1) - EML of G. Then $f^*(E) = \{2, 4,\ldots, 2q\}$.

Let $u_1, u_2, \ldots, u_p$ be the vertices of G and let $k \geq 1$. Define g: $V(G) \rightarrow \{0, 1, 2, \ldots, 2kq\}$ by

$$g(u_i) = \begin{cases} kf(u_i), & \text{if } f(u_i) \text{ is even} \\ kf(u_i) + k - 1, & \text{if } f(u_i) \text{ is odd}. \end{cases}$$

Then for $1 \leq i, j \leq p$, $g^*(u_iu_j) = kf^*(u_iu_j)$. Thus $g^*(E) = \{2k, 2k+2, \ldots, 2kq\}$ and hence g is a (k, k) - EML.

■

The example for a (1, 1) - EML and the corresponding (4, 4) - EML of a graph G are shown in Figure 5.

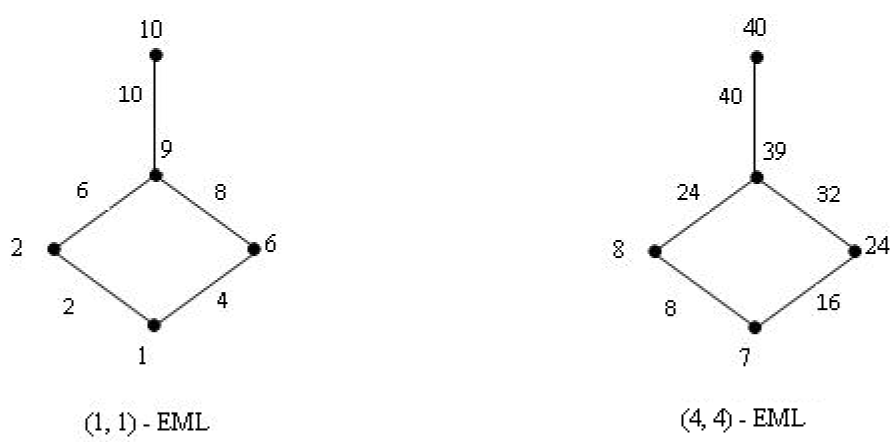


Figure 5.

Theorem 2.3. *If a graph G has a mean labeling in which*

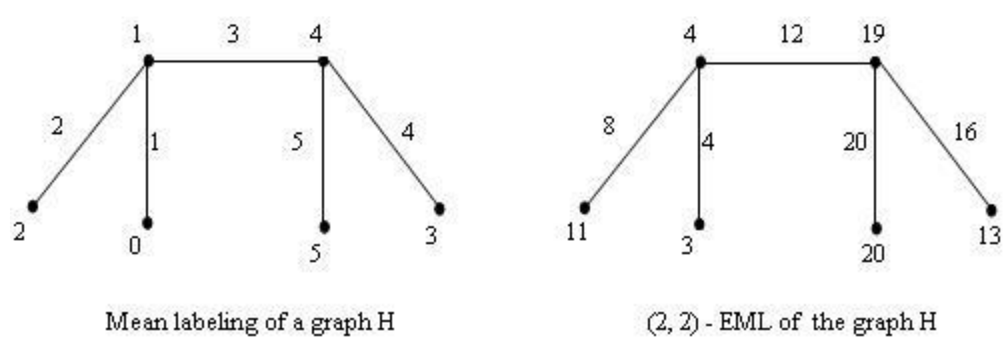
(i) *no two adjacent vertices receive odd labels when q is even*

or (*ii*) *no two adjacent vertices receive even label when q is odd,*

then G has a (k, k) - *EML for any* $k \geq 1$.

Proof. The theorem follows from Lemma 2.1 and Lemma 2.2. ■

The labelings of the two graphs *H* and *G* shown in Figure 6 illustrate Theorem 2.3.



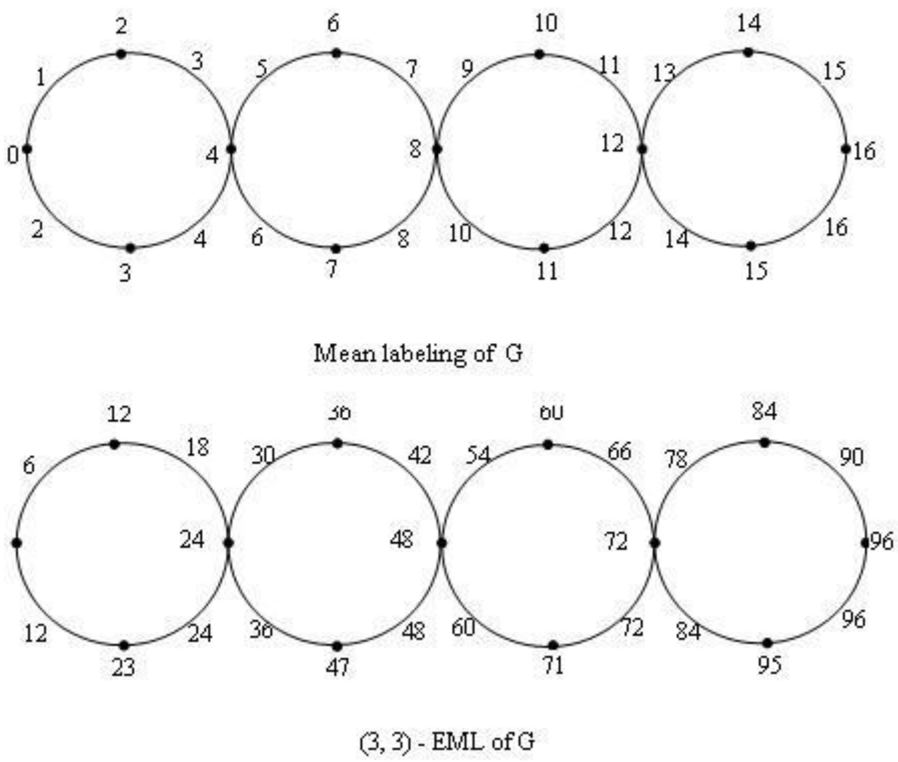


Figure 6.

Lemma 2.4. *If a graph G has a* (k, k) *-EML f in which* $\left\{ \left\lfloor \frac{f(v)}{k} \right\rfloor / v \in V \right\} = \{0, 1, \ldots 2q\}$, *then G has a* $(1, 1)$ *- EML.*

Proof. Let f: $V(G) \rightarrow \{0, 1, 2, \ldots, 2kq\}$ be a (k, k) - EML of G. Then $f^*(E) = \{2k, 2k+2, \ldots, 2kq\}$. Let $u_1, u_2, \ldots, u_p$ be the vertices of G.

Define g: $V(G) \rightarrow \{0, 1, 2, \ldots, 2q\}$ by $g(u_i) = \left\lfloor \frac{f(u_i)}{k} \right\rfloor$, for $1 \leq i \leq p$.

Thus $g^*(E) = \{2, 4, \ldots, 2q\}$. Hence g is a $(1, 1)$ - EML. ■

The example for a $(3, 3)$ - EML of a graph G and the corresponding $(1, 1)$ - EML of G are shown in Figure 7.

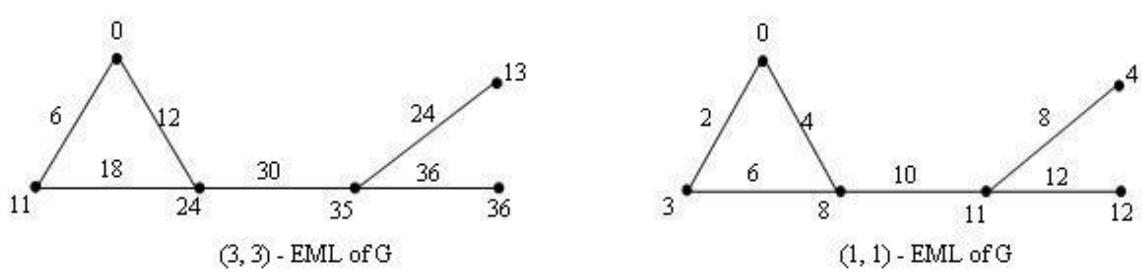


Figure 7.

Lemma 2.5. *If a graph G has a* (1, 1) *-EML f in which* $\left\{ \left\lfloor \frac{f(v)}{2} \right\rfloor / v \in V \right\}$ = {*0,* 1,...*q*}*, , then G has a mean labeling.*

Proof. Let *f*: *V*(*G*) → {0, 1, 2, …, 2*q*} be a (1, 1) - EML of *G*. Then f^*(E) = {2, 4,…, 2*q*}. Let $u_1, u_2, \ldots, u_p$ be the vertices of *G*.

Define *g*: *V*(*G*) → {0, 1, 2,…, *q*} by $g(u_i) = \left\lfloor \frac{f(u_i)}{2} \right\rfloor$, for $1 \le i \le p$.

Thus $g^*(E)$ = {1, 2,…, *q*}. Hence *g* is a mean labeling of *G*. ■

The example for a (1, 1) - EML of a graph *G* and the corresponding mean labeling of *G* are shown in Figure 8.

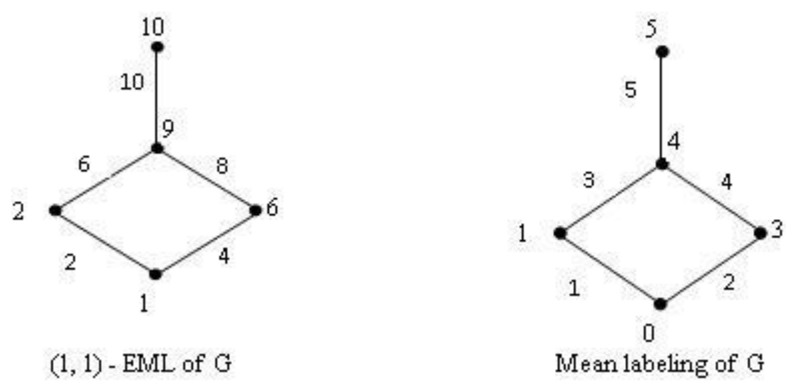


Figure 8.

Theorem 2.6. *If a graph G has a* (*k, k*) *-EML f in which* $\left\{ \left\lfloor \frac{f(v)}{k} \right\rfloor / v \in V \right\}$ = {*0,* 1,...*2q*}*, then G has a mean labeling.*

Proof. The theorem follows from Lemma 2.4 and Lemma 2.5. ■

The example for a (3, 3) - EML of a graph *G* and the corresponding mean labeling of *G* are shown in Figure 9.

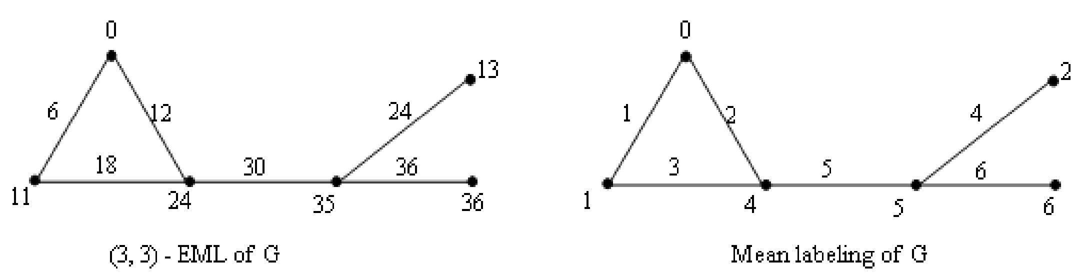


Figure 9.

Theorem 2.7. *A graph G has a* (1, 1) *- EML if and only if it has a* (*k*, 1) *- EML g such that g:V*(*G*) → {*2k- 2, 2k*-1*,…, 2k+2*(*q*-1)} *for any* $k \ge 1$.

Proof. Let f: $V(G) \rightarrow \{0, 1, 2,\ldots, 2q\}$ be a (1, 1) - EML of G. Then $f^*(E) = \{2, 4,\ldots, 2q\}$. Let $k \geq 1$ be an integer. Now, define g: $V(G) \rightarrow \{0, 1, 2\ldots 2k+2(q\text{-}1)\}$ by $g(v) = f(v) + 2(k\text{-}1)$. We claim that g is a $(k, 1)$ - EML of G.

Let x and y be two adjacent vertices in G. We consider the following two cases.

Case (i): $g(x) + g(y)$ is even.

$$\text{Then, } g^*(xy) = \frac{g(x)+g(y)}{2}$$
$$= \frac{f(x)+2(k-1)+f(y)+2(k-1)}{2}$$
$$= \frac{f(x)+f(y)}{2}+2(k-1)$$
$$= f^*(xy)+2(k-1).$$

Case (ii): $g(x) + g(y)$ is odd.

$$\text{Then, } g^*(xy) = \frac{g(x)+g(y)+1}{2}$$
$$= \frac{f(x)+2(k-1)+f(y)+2(k-1)+1}{2}$$
$$= \frac{f(x)+f(y)+1}{2}+2(k-1)$$
$$= f^*(xy)+2(k-1).$$

Thus $g^*(E) = \{f^*(xy) + 2(k\text{-}1) \,/\, xy \in E(G)\} = \{2k, 2k+1, \ldots, 2k+2(q\text{-}1)\}$ and hence g is a $(k, 1)$ - EML for $k \geq 1$.

Conversely, let g: $V(G) \rightarrow \{2k\text{-}2, \ldots, 2k+2(q\text{-}1)\}$ be a $(k, 1)$ - EML of G. Then $g^*(E) = \{2k, 2k+2,\ldots, 2k+2(q\text{-}1)\}$.

Now, define f: $V(G) \rightarrow \{0, 1, 2, \ldots, 2q\}$ by $f(v) = g(v) + 2(1\text{-}k)$ for $k \geq 1$ and $g(v) \geq 2(k\text{-}1)$.

We prove that f is a (1, 1) -EML of G.

Let x and y be the two vertices in G. We consider the following two cases.

Case (i): $f(x) + f(y)$ is even.

$$\text{Now, } f^*(xy) = \frac{f(x)+f(y)}{2}$$
$$= \frac{g(x)+2(1-k)+g(y)+2(1-k)}{2}$$
$$= \frac{g(x)+g(y)}{2}+2(1-k)$$
$$= g^*(xy)+2(1-k).$$

Case (ii): $f(x) + f(y)$ is odd.

$$\text{Now, } f^*(xy) = \frac{f(x)+f(y)+1}{2}$$
$$= \frac{g(x)+2(1-k)+g(y)+2(1-k)+1}{2}$$
$$= \frac{g(x)+g(y)+1}{2}+2(1-k)$$
$$= g^*(xy)+2(1-k).$$

Thus, $f^*(E) = \{ g^*(xy) + 2(1\text{-}k) \,/\, xy \in E(G) \} = \{2, 4, \ldots, 2q)\}$ and hence f is a $(k, 1)$ - EML for $k \geq 1$. ■

The example for a (1, 1) - EML and the corresponding (5, 1) - EML of a graph G are shown in Figure 10.

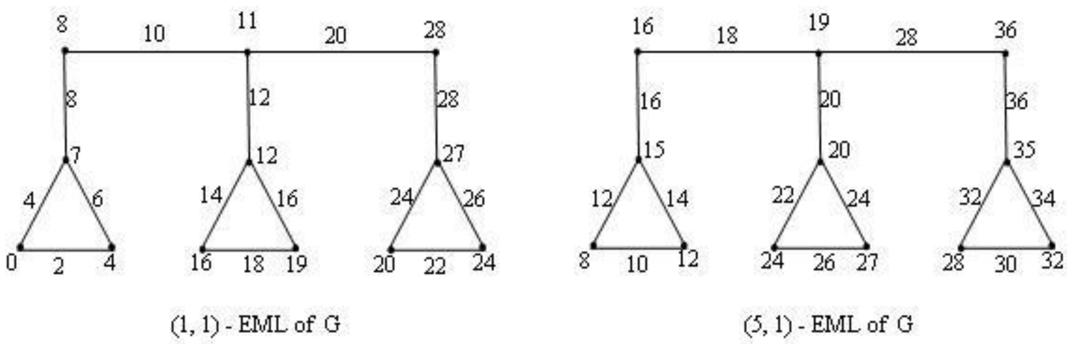


Figure 10.

Also, a (4, 1) - EML and the corresponding (1, 1) - EML of a graph G are shown in Figure 11.

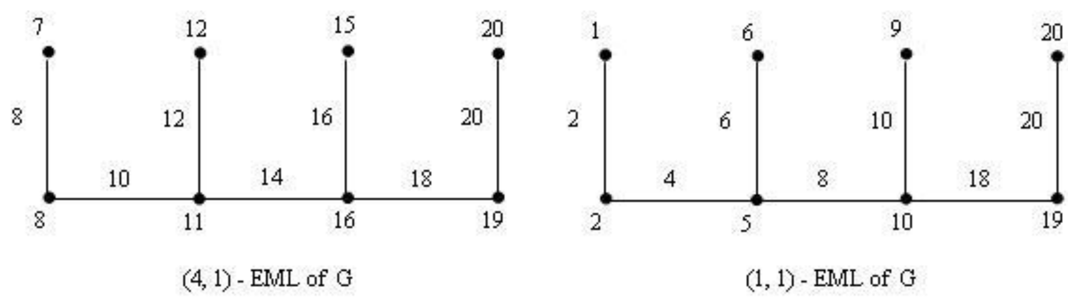


Figure 11.

Theorem 2.8. *Let* $G_1(p_1, q_1)$ *be a* (1, 1) - *even mean graph and let* $G_2(p_2, q_2)$ *be a* (1, 1) - *even mean graph with EML g such that* $g^*(vw) = 2$ *and* $g(v) = 1$ *or 3, then* $G_1 \bullet G_2(uv)$ *is a* (1, 1) - *even mean graph for some vertex u in* G_1.

Proof. Let G_1 be a (1, 1) - even mean graph with EML f. Let $u_1, u_2, \ldots, u_{p_1}$ and $v_1, v_2, \ldots, v_{p_2}$ be the vertices of graphs G_1 and G_2 respectively.
Let $h: V(G_1 \bullet G_2(uv)) \rightarrow \{0, 1, 2, \ldots, 2(q_1+q_2)\}$ be defined by $h(u_i) = f(u_i)$ for $1 \le i \le p_1$ and $h(v_j) = f(v_j) + 2q_1 + 2$ for $1 \le i \le p_2$.
Fix some vertex u in G_1, such that

$$f(u) = \begin{cases} 2q_1, \text{ when } g(v) = 1 \\ 2q_1 - 1, \text{ when } g(v) = 3. \end{cases}$$

The labels of the edges of $G_1 \bullet G_2(uv)$ are $h^*(e) = \begin{cases} f^*(e), & \text{if } e \in G_1 \\ g^*(e) + 2q_1 + 2, & \text{if } e \in G_2 \end{cases}$

Thus h is a (1, 1) - EML. Hence $G_1 \bullet G_2(uv)$ is a (1, 1) - even mean graph. ■

The example for (1, 1) - EML's of G_1 and G_2 and a (1, 1) - EML of $G_1 \bullet G_2(uv)$ are shown in Figure 12.

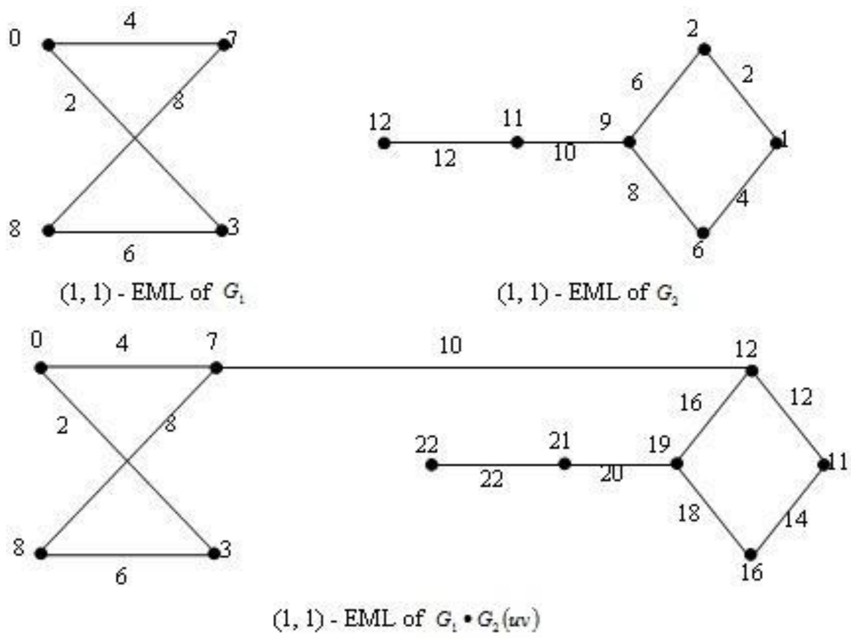


Figure 12.

Theorem 2.9. *If a tree T has a* (1, 1) *-EML f in which* $\left\{ \left\lfloor \frac{f(v)}{2} \right\rfloor / v \in V \right\}$ = *{0, 1,…q}, then T ΘK*$_1$ *has a mean labeling.*

Proof. Let f: $V(T) \rightarrow \{0, 1, 2, \ldots, 2q\}$ be a (1, 1) - EML of T. Then $f^*(E) = \{2, 4, \ldots, 2q\}$. Let $u_1, u_2, \ldots, u_p$ be the vertices of T and $v_1, v_2, \ldots, v_p$ be the corresponding new vertices in $T\Theta K_1$.
Define g: $V(T\,\Theta K_1) \rightarrow \{0, 1, 2, \ldots, 2q+1\}$ by $g(u_i) = f(u_i)$, for $1 \le i \le p$

$$g(v_i) = \begin{cases} f(u_i)+1, \text{ when } f(u_i) \text{ is even} \\ f(u_i)-1, \text{ when } f(u_i) \text{ is odd}. \end{cases}$$

If both x and y are in T, then $g^*(xy) = f^*(xy) = \{2, 4, 6, \ldots, 2q\}$.
If not, then $g(x)$ and $g(y)$ are consecutive integers and therefore, $g^*(xy) \in \{1, 3, 5, \ldots, 2q+1\}$.
Thus $g^*(E) = \{1, 2, \ldots, 2q+1\}$. Hence g is a mean labeling of $T\,\Theta K_1$ and $T\Theta K_1$ is a mean graph. ■

The example for a (1, 1) - EML of a tree T and the mean labeling of $T\,\Theta K_1$ are shown in Figure 13.

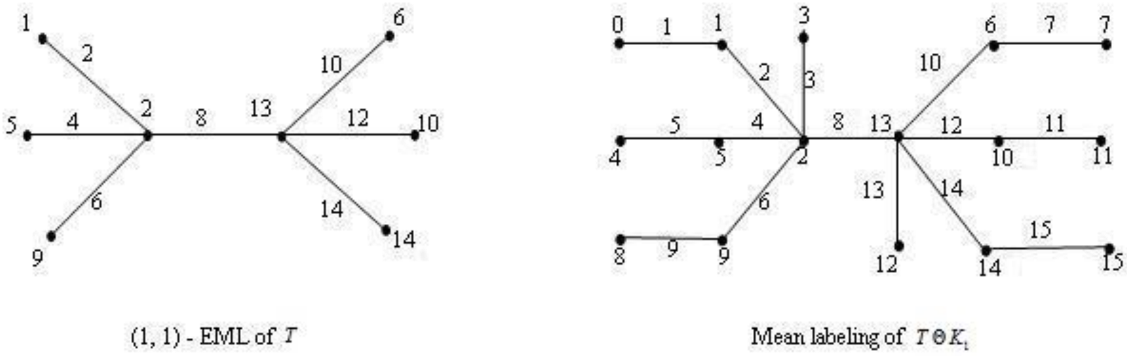


Figure 13.

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