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On semi*- open sets

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Abstract

In this paper we define the semi*-border, semi*-frontier and semi*-exterior of a subset of a topological space and study some of their properties.

Keywords: semi*-open, semi*-closed, semi*-closure, semi*-interior point, semi*-interior, semi*-limit point, semi*-derived set, semi*-border, semi*-frontier, semi*-exterior.

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1 Introduction

In 1963 Levine[8] introduced the concept of semi-open sets in topological spaces. Biswas[1] defined semi-closed sets. Crossley and Hildebrand [2] defined semi-closure of sets and irresolute functions. Levine [9] defined generalized closed sets in 1970. Das [4] defined semi-interior point and semi-limit point of a subset. The semi-derived set of a subset of a topological space was also defined and studied by him. Dunham [6] introduced the concept of generalized closure using Levine's generalized closed sets and defined a new topology τ^* and studied their properties. Di Maio and Noiri [5] defined and studied semi-regular subsets.

We [10, 11] introduced the concepts namely semi*-open sets, semi*-closed sets, semi*-closure and semi*-derived set of a subset and also studied their fundamental properties. In this paper, we define the semi*-border, semi*-frontier and semi*-exterior of a subset of a topological space and study some of their properties.

2 Preliminaries

Throughout this paper (X, τ) denotes a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of a space (X, τ), $Cl(A)$, $Int(A)$ and $D[A]$ denote the closure, the interior and the derived set of A respectively.

Definition 2.1. *A subset A of a topological space (X, τ) is called*
(i) *semi-open* [8] (*respectively semi*-open* [10]) *if there is an open set U in X such that* $U \subseteq A \subseteq \text{Cl}(U)$ (*respectively* $U \subseteq A \subseteq \text{Cl}^*(U)$) *or equivalently if* $A \subseteq \text{Cl}(Int(A))$ (*respectively* $A \subseteq \text{Cl}^*(Int(A))$).
(ii) *semi-closed* [1] (*respectively semi*-closed* [11]) *if* $X \backslash A$ *is semi-open* (*respectively semi*-open*) *or equivalently if* $Int(\text{Cl}(A)) \subseteq A$ (*respectively* $Int^*(\text{Cl}(A)) \subseteq A$).
(iii) *semi-regular* [5] (*respectively semi*-regular* [11]) *if it is both semi-open and semi-closed* (*respectively both semi*-open and semi*-closed*).

(iv) *generalized closed* (*g-closed*)[9] if $Cl(A)\subseteq U$ *whenever* $A\subseteq U$ *and U is open in X*.
(v) *generalized open* (*g-open*) [9] *if* $X\backslash A$ *is g-closed in X*.
The class of all semi-open (respectively semi-closed, semi*-open, semi*-closed) sets is denoted by $SO(X, \tau)$ (respectively $SC(X, \tau)$, $S^*O(X, \tau)$, $S^*C(X, \tau)$).

Definition 2.2. *Let A be a subset of X.*
(i) *The semi-interior* [2] (*respectively generalized interior and semi*-interior* [10]) *of A is defined as the union of all semi-open* (*respectively g-open and semi*-open*) *subsets of A and is denoted by* $sInt(A)$ (*respectively* $Int^*(A)$ *and* $s^*Int(A)$).
(ii) *The semi-closure* [2] (*respectively generalized closure* [6] *and semi*-closure* [11]) *of A is defined as the intersection of all semi-closed* (*respectively g-closed and semi*-closed*) *sets containing A and is denoted by* $sCl(A)$(*respectively* $Cl^*(A)$ *and* $s^*Cl(A)$).
(iii) *The border of A is defined as* $Bd(A)=A\backslash Int(A)$.
(iv) *The frontier of A is defined by* $Fr(A)=Cl(A)\backslash Int(A)$.
(v) *The exterior of A is defined by* $Ext(A)=Int(X\backslash A)$**.**

Remark 2.3. The ***semi-border*** $sBd(A)$, the ***semi-frontier*** $sFr(A)$ and the ***semi-exterior*** $sExt(A)$ of a subset A are defined analogously by replacing $Int(A)$ and $Cl(A)$ respectively by $sInt(A)$ and $sCl(A)$ in the definitions of the border, the frontier and the exterior of A.

Definition 2.4. *Let A be a subset of X. A point x in X is*
(i) *a semi-interior point* [4] (*respectively semi*-interior point* [10])*of A if A contains a semi-open set* (*respectively a semi*-open set*)*containing x*.
(ii) *a semi-limit point* [4] (*respectively semi*-limit point* [11]) *of A if every semi-open set* (*respectively semi*-open set*) *containing x intersects A in a point different from x*.

Definition 2.5. *The set of all semi-limit* (*respectively semi*-limit*) *points of A is called the semi-Derived set* [4] (*respectively semi*-Derived set* [11]) *of A and is denoted by* $Ds[A]$ (*respectively* $Ds^*[A]$).

3 Main Results

Theorem 3.1. *In any topological space (X, τ), the following hold.*
If A and B are subsets of X,
(i) $D_s[A]\subseteq D_{s^*}[A]\subseteq D[A]$.
(ii) $A\subseteq B\Longrightarrow D_{s^*}[A]\subseteq D_{s^*}[B]$.
(iii) $D_{s^*}[D_{s^*}[A]]\backslash A\subseteq D_{s^*}[A]$.
(iv) $D_{s^*}[A\cup B]\supseteq D_{s^*}[A]\cup D_{s^*}[B]$.
(v) $D_{s^*}[A\cap B]\subseteq D_{s^*}[A]\cap D_{s^*}[B]$.
(vi) $D_{s^*}[A\cup D_{s^*}[A]]\subseteq A\cup D_{s^*}[A]$.
(vii) $s^*Int(A)=A\backslash D_{s^*}[X\backslash A]$.

Proof. (i) follows from Theorem 3.17 and Theorem 3.24 [10].
(ii) follows from Definition 2.4.
(iii). Let $x\in D_{s^*}[D_{s^*}[A]]\backslash A$ and U be a semi*-open set containing x. Then $U\cap(D_{s^*}[A]\backslash\{x\})\neq\phi$. Let $y\in U\cap(D_{s^*}[A]\backslash\{x\})$. Then $x\neq y\in U$ and $y\in D_{s^*}[A]$. Hence $U\cap(A\backslash\{y\})\neq\phi$. If $z\in U\cap(A\backslash\{y\})$, then $z\neq x$ and $U\cap(A\backslash\{x\})\neq\phi$ which shows that $x\in D_{s^*}[A]$ and hence $D_{s^*}[D_{s^*}[A]]\backslash A\subseteq D_{s^*}[A]$.
(iv) and (v) follow from (ii) and set theoretic properties. From Theorem 3.49[11], we have $A\cup D_{s^*}[A]=s^*Cl(A)$ which is semi*-closed by Theorem 3.16[11] and hence (vi) follows from Corollary3.50[11].

(vii). Let $x \in A \setminus D_{s*}[X \setminus A]$. Then $x \notin D_{s*}[X \setminus A]$ which implies that there is a semi*-open set U containing x such that $U \cap [(X \setminus A) \setminus \{x\}] = \phi$. Since $x \in A$, $U \cap (X \setminus A) = \phi$ which implies $U \subseteq A$. Then $x \in U \subseteq A$ and hence $x \in s^*Int(A)$.

On the other hand if $x \in s^*Int(A)$ then $s^*Int(A)$ is a semi*-open set containing x and contained in A. Hence $s^*Int(A) \cap (X \setminus A) = \phi$ which implies that $x \notin D_{s*}[X \setminus A]$.Thus $s^*Int(A) \subseteq A \setminus D_{s*}[X \setminus A]$. ■

Remark 3.2. *The inclusions in* (i) *of Theorem* 3.1, *may be strict and equality may hold as well. This can be seen from the following examples.*

Example 3.3. Consider the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a, b\}, X\}$.

Let $A = \{a, c\}$. Then $D_s[A] = D_{s*}[A] = D[A] = \{b, c, d\}$.
Let $B = \{c, d\}$. Then $D_s[B] = \phi$; $D_{s*}[B] = D[B] = \{c, d\}$. Here $D_s[B] \subsetneq D_{s*}[B] = D[B]$.

Example 3.4. Consider the space (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Let $A = \{a, c, d\}$.Then $D_s[A] = D_{s*}[A] = \phi$; $D[A] = \{d\}$. Here $D_s[A] = D_{s*}[A] \subsetneq D[A]$.

Example 3.5. Consider the space (X, τ) where $X = \{a, b, c, d, e, f, g\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, e\}, \{a, c, d\}, \{b, f, g\}, \{a, b, c, d\}, \{a, b, f, g\}, \{a, b, c, d, e\}, \{a, b, e, f, g\}, \{a, b, c, d, f, g\}, X\}$. Let $A = \{a, f, g\}$.Then $D_s[A] = \{c, d\}$; $D_{s*}[A] = \{c, d, f, g\}$; $D[A] = \{c, d, e, f, g\}$.

Here $D_s[A] \subsetneq D_{s*}[A] \subsetneq D[A]$.

Theorem 3.6. A subset D of X is dense in X if and only if $s^*Cl(D) = X$.

Proof. Let D be dense in X , $x \in X$ and A be a semi*-open set in X containing x. Then there exists an open set U such that $U \subseteq A \subseteq Cl^*(U)$. Since D is dense in X, $U \cap D \neq \phi$ and hence $A \cap D \neq \phi$. Thus every semi*-open set in X containing x intersects D. Therefore $x \in s^*Cl(D)$. Hence $s^*Cl(D) = X$.
Conversely, suppose $s^*Cl(D) = X$. Then by (vi) of Theorem 3.40[11], $X = s^*\mathrm{Cl}(D) \subseteq Cl(D) \subseteq X$. Hence $Cl(D) = X$. That is, D is dense in X. ■

Definition 3.7. *If A is a subset of X , then the* ***semi*-border of A*** *is defined by* $s^*Bd(A) = A \setminus s^*Int(A)$.

Theorem 3.8. *In any topological space* (X, τ) *the following hold.*

(i) $s^*Bd(\phi) = \phi$.

(ii) $s^*Bd(X) = \phi$.

If A is a subset of X,

(iii) $s^*Bd(A) \subseteq A$.

(iv) $A = s^*Int(A) \cup s^*Bd(A)$.

(v) $s^*Int(A) \cap s^*Bd(A) = \phi$.

(vi) $sBd(A) \subseteq s^*Bd(A) \subseteq Bd(A)$.

(vii) $s^*Int(s^*Bd(A)) = \phi$.

(viii) A is *semi*-open if and only if* $s^*Bd(A) = \phi$.

(ix) $s^*Bd(s^*Int(A)) = \phi$.

(x) $s^*Bd(s^*Bd(A)) = s^*Bd(A)$.

(xi) $s^*Bd(A) = A \cap s^*Cl(X \setminus A)$.

(xii) $s^*Bd(A) = A \cap D_{s*}[X \setminus A]$.

Proof. (i), (ii), (iii), (iv) and (v) follow from Definition 3.7. (vi) follows from Theorem 3.52(vi)[10].
(vii) If possible, let $x \in s^*Int(s^*Bd(A))$. Then $x \in s^*Bd(A)$. Since $s^*Bd(A) \subseteq A$, $x \in s^*Int(s^*Bd(A)) \subseteq s^*Int(A)$. This implies $x \in s^*Int(A) \cap s^*Bd(A)$ which contradicts (v).
(viii) follows from Definition 3.7 and Theorem 3.14 [10].

(ix) follows from (viii) and the fact that $s^*Int(A)$ is semi*-open. (x) follows from (vii). (xi) follows from Definition 3.7 and Theorem 3.48(i)[11].

(xii).From (xi) and Theorem 3.49 of [11] we have $s^*Bd(A)=A\cap s^*Cl(X\backslash A)= A\cap((X\backslash A)\cup D_{s^*}[X\backslash A])= A\cap D_{s^*}[X\backslash A]$. ■

Remark 3.9. *The inclusions in (vi) and (xii) of Theorem 3.8 may be strict and equality may hold as well. This can be seen from the following examples.*

Example 3.10. Consider the space given in Example 3.5.
Let $A=\{b, d, f, g\}$.Then $sBd(A)=s^*Bd(A)=Bd(A)=\{d\}$.
Let $B=\{b, c, d, e\}$.Then $sBd(B)=s^*Bd(B)=\{c, d\}$; $Bd(B)=\{c, d, e\}$. Here, $sBd(B)=s^*Bd(B)\subsetneq Bd(B)$.
Let $C=\{a, d, f\}$.Then $sBd(C)=\{f\}$; $s^*Bd(C)=Bd(C)=\{d, f\}$. Here, $sBd(C)\subsetneq s^*Bd(C)=Bd(C)$.
Let $D=\{b, c, d, e, f\}$. Then $sBd(D)=\{c, d\}$; $s^*Bd(D)=\{c, d, f\}$; $Bd(D)=\{c, d, e, f\}$ Here, $sBd(D)\subsetneq s^*Bd(D)\subsetneq Bd(D)$.

Definition 3.11. *If A is a subset of X then the* ***semi*-frontier*** *of A is defined by* $s^*Fr(A) = s^*Cl(A)\backslash s^*Int(A)$.

Theorem 3.12. *If A is a subset of a space X, the following hold.*
(i) $s^*Fr(\phi)=\phi$.
(ii) $s^*Fr(X)=\phi$.
(iii) $s^*Cl(A)=s^*Int(A)\cup s^*Fr(A)$.
(iv) $s^*Int(A)\cap s^*Fr(A)=\phi$.
(v) $s^*Bd(A)\subseteq s^*Fr(A)\subseteq s^*Cl(A)$.
(vi) A is semi*-closed if and only if $A=s^*Int(A)\cup s^*Fr(A)$.
(vii) $sFr(A)\subseteq s^*Fr(A)\subseteq Fr(A)$.
(viii) $s^*Fr(A)=s^*Cl(A)\cap s^*Cl(X\backslash A)$.
(ix) $s^*Fr(A)$ is semi*-closed and hence $s^*Cl(s^*Fr(A))=s^*Fr(A)$.
(x) $s^*Fr(A)=s^*Fr(X\backslash A)$.
(xi) $s^*Fr(A)\subseteq s^*Bd(A)\cup D_{s^*}[A]$.
(xii) *A is semi*-closed if and only* if $s^*Fr(A)=s^*Bd(A)$. *Hence A is semi*-closed if and only if A contains its semi*-frontier.*
(xiii) *A is semi*-regular if and only if* $s^*Fr(A)=\phi$.
(xiv) If A is semi*-open, then $s^*Fr(A)\subseteq D_{s^*}[A]$.
(xv) $s^*Fr(s^*Int(A))\subseteq s^*Fr(A)$.
(xvi) $s^*Fr(s^*Cl(A))\subseteq s^*Fr(A)$.
(xvii) $s^*Fr(s^*Fr(A))\subseteq s^*Fr(A)$.
(xviii) $X=s^*Int(A)\cup s^*Int(X\backslash A)\cup s^*Fr(A)$.

Proof. (i), (ii), (iii), (iv) and (v) follow from Definition 3.11. (vi) follows from (iii) and Theorem 3.17[11]. (vii) follows from Theorem 3.52(vi)[10] and Theorem 3.40(vi)[11]. (viii) follows from Definition 3.11and Theorem 3.48(i)[11]. (ix) follows from the fact that $s^*Cl(A)$ is semi*-closed, Theorem 3.11[11] and (viii) above. (*x*) follows from (viii) and Theorem 3.11[11]. $s^*Cl(A) = A\cup D_{s^*}[A] = s^*Int(A)\cup s^*Bd(A)\cup D_{s^*}[A]$. This proves (xi). (xii) follows from the fact that A is semi*-closed if and only if $s^*Cl(A)=A$ and from $s^*Bd(A)\subseteq A$. (xiii) follows from Definition 3.11, Theorem 3.14[10] and Theorem 3.17[11]. (xiv) follows from Theorem 3.49 [11]. (xv) follows from Definition 3.11 and Theorem 3.52(v)[10]. (xvi) follows from Definition 3.11 and Theorem 3.40(v)[11]. (xvii) follows from Definition 3.11 and the fact that $s^*Fr(A)$ is semi*-closed. (xviii) follows from (iii) and Theorem 3.48(ii)[11]. ■

Remark 3.13. *In* (v) *and* (vii) *of Theorem 3.12, the inclusions may be strict and equality may hold as well. This can be seen from the following example.*

Example 3.14. Consider the space given in Example 3.5.
Let $A=\{b, c, d, e\}$.Then $s^*Bd(A)=\{c, d\}$; $s^*Fr(A)=\{c, d, f, g\}$. Here $s^*Bd(A)\subsetneq s^*Fr(A)$.
Let $B=\{a, c, d, f, g\}$. Then $s^*Bd(B)=s^*Fr(B)=\{f, g\}$.
Let $C=\{a, b, c, d\}$.Then $sFr(C)= s^*Fr(C)=Fr(C) =\{e, f, g\}$.
Let $D=\{a, e\}$. Then $sFr(D)=s^*Fr(D)=\{c, d\}$; $Fr(D)=\{c, d, e\}$. Here $sFr(D)=s^*Fr(D)\subsetneq Fr(D)$.
Let E=$\{a, b, c, d, e, g\}$.Then $sFr(\mathrm{E})=\{f\}$; $s^*Fr(\mathrm{E})= Fr(\mathrm{E})=\{f, g\}$. Here $sFr(\mathrm{E})\subsetneq s^*Fr(\mathrm{E})=Fr(\mathrm{E})$.
Let F=$\{ b, d, f, g\}$.Then $sFr(\mathrm{F})=\{d\}$; $s^*Fr(\mathrm{F})=\{c, d\}$ $Fr(\mathrm{F})=\{c, d, e\}$.
Here $sFr(\mathrm{F})\subsetneq s^*Fr(\mathrm{F})\subsetneq Fr(\mathrm{F})$.

Remark 3.15. *In* (xi) *and* (xiv) *of Theorem 3.12, the inclusions may be strict and equality may hold as well. This can be seen from the following example.*

Example 3.16. Consider the space given in Example 3.5.
Let $A=\{a, d, f\}$.Then $s^*Fr(A)=\{c, d, f, g\}$; $s^*Bd(A)=\{d, f\}$; $D_{s^*}[A]=\{c, d, g\}$.
Here, $s^*Fr(A)= s^*Bd(A)\cup D_{s^*}[A]$.
Let $B=\{b, d, f, g\}$.Then $s^*Fr(B)=\{c, d\}$; $s^*Bd(B)=\{d\}$; $D_{s^*}[B]=\{c, f, g\}$.
Here, $s^*Fr(B)\subsetneq s^*Bd(B)\cup D_{s^*}[B]$.
Let $C=\{a, b, c, d\}$.Then C is semi*-open and $s^*Fr(C)=\{e, f, g\}$; $D_{s^*}[C]=\{c, d, e, f, g\}$.
Here, $s^*Fr(C)\subsetneq D_{s^*}[C]$.
Let $D=\{a, e\}$.Then D is semi*-open and $s^*Fr(D)=\{c, d\}$; $D_{s^*}[D]=\{c, d\}$. Here $s^*Fr(D)=D_{s^*}[D]$.

Remark 3.17. *In* (xv), (xvi) *and* (xvii) *of Theorem 3.12, the inclusions may be strict and equality may hold as well. This can be seen from the following example.*

Example 3.18. Consider the space (X, τ) given in Example 3.5.
Let $A=\{a, c, e\}$. Then $s^*Fr(A)=\{c, d\}$; $s^*Fr(s^*Int(A))=\{c, d\}$; $s^*Fr(s^*Fr(A))=\{c, d\}$.
Here $s^*Fr(A)= s^*Fr(s^*Int(A))$ and $s^*Fr(s^*Fr (A))=s^*Fr(A)$.
Let $B=\{b, c, d, e\}$. Then $s^*Fr(B)=\{c, d, f, g\}$; $s^*Fr(s^*Int(B))=\{f, g\}$. Here, $s^*Fr(s^*Int(B))\subsetneq s^*Fr(B)$.
Let $C=\{a, d, f\}$. Then $s^*Fr(C)=\{c, d, f, g\}$; $s^*Fr(s^*Cl(C))=\{f, g\}$. Here $s^*Fr(s^*Cl(C))\subsetneq s^*Fr(C)$.

Theorem 3.19. *If A is a subset of X, then $s^*Fr(s^*Fr(s^*Fr(A)))=s^*Fr(s^*Fr(A))$.*

Proof. From Theorem 3.12((viii) and (ix)) we have

$$s^*Fr(s^*Fr(s^*Fr(A))) =s^*Cl(s^*Fr(s^*Fr(A)))\cap s^*Cl(X \setminus s^*Fr(s^*Fr(A)))$$
$$=s^*Fr(s^*Fr(A))\cap s^*Cl(X\setminus s^*Fr(s^*Fr(A))). \quad (1)$$

Now, $X\setminus s^*Fr(s^*Fr(A))$ $=X\setminus[s^*Cl(s^*Fr(A))\cap s^*Cl(X\setminus s^*Fr(A))]$ (by Theorem 3.12(viii))
$=X\setminus[s^*Fr(A)\cap s^*Cl(X\setminus s^*Fr(A))]$(by Theorem 3.12(ix))
$=[X\setminus s^*Fr(A)]\cup[X\setminus(s^*Cl(X\setminus s^*Fr(A)))]$.
$s^*Cl(X\setminus s^*Fr(s^*Fr(A)))$ $=s^*Cl\{[X\setminus s^*Fr(A)]\cup[X\setminus(s^*Cl(X\setminus s^*Fr(A)))]\}$
$\supseteq s^*Cl[X\setminus s^*Fr(A)]\cup s^*Cl[X\setminus(s^*Cl(X\setminus s^*Fr(A)))]$
$=D\cup s^*Cl(X\setminus D)$ where $D=s^*Cl[X\setminus s^*Fr(A)]$
$\supseteq D\cup(X\setminus D)=X$.
Hence from (1), we have $s^*Fr(s^*Fr(s^*Fr(A)))=s^*Fr(s^*Fr(A))\cap X=s^*Fr(s^*Fr(A))$. ■

Theorem 3.20. *If a subset A is semi*-open or semi*-closed in X, then $s^*Fr(s^*Fr(A))=s^*Fr(A)$).*

Proof. $s^*Fr(s^*Fr(A))=s^*Cl(s^*Fr(A))\cap s^*Cl(X\setminus s^*Fr(A))$ (by Theorem 3.12(viii))
$=s^*Fr(A)\cap s^*Cl(X\setminus s^*Fr(A))$ (by Theorem 3.12(ix))
$=s^*Cl(A)\cap s^*Cl (X\setminus A)\cap s^*Cl(X\setminus s^*Fr(A))$.
If A is semi*-open in X and $s^*Fr(A)\cap A=\phi$. This implies that
$A\subseteq X\setminus s^*Fr(A)\Rightarrow s^*Cl(A) \subseteq s^*Cl(X\setminus s^*Fr(A))\Rightarrow s^*Cl(A)\cap s^*Cl(X\setminus s^*Fr(A))=s^*Cl(A)$.
If A is semi*-closed in X, $s^*Fr(A) \subseteq A$.
Hence, $X\setminus A\subseteq X\setminus s^*Fr(A)$ $\Rightarrow s^*Cl(X\setminus A) \subseteq s^*Cl(X\setminus s^*Fr(A))$
$\Rightarrow s^*Cl(X\setminus A)\cap s^*Cl(X\setminus s^*Fr(A))=s^*Cl(X\setminus A)$.

From the above, we get $s^*Fr(s^*Fr(A))=s^*Cl(A)\cap s^*Cl(X\backslash A)=s^*Fr(A)$. ■

Remark 3,21. *Since* $s^*Fr(A)$ *is semi*-closed, Theorem 3.19 can also be deduced from Theorem 3.20.*

Theorem 3.22. *If A and B are subsets of X such that* $A \subseteq B$ *and* B is semi*- closed in X, then $s^*Fr(A) \subseteq B$.

Proof. $s^*Fr(A)=s^*Cl(A)\backslash s^*Int(A) \subseteq s^*Cl(B)\backslash s^*Int(A)=B\backslash s^*Int(A) \subseteq B$. ■

Lemma 3.23. If A and B are subsets of X such that $A\cap B=\phi$ and A is semi*-open in X, then $A\cap s^*Cl(B)=\phi$.

Proof. On the contrary, suppose that $x\in A\cap s^*Cl(B)$. Then A is a semi*-open set containing x and $x\in s^*Cl(B)$. This implies $A\cap B\neq \phi$, which is a contradiction to the assumption.

Theorem 3.24. *If A and B are subsets of X such that* $A\cap B=\phi$ *and A is semi*-open in X, then* $A\cap s^*Fr(B)=\phi$.

Proof. Follows from the fact $s^*Fr(B)\subseteq s^*Cl(B)$ and Lemma 3.23. ■

Definition 3.25. *If A is a subset of X then* ***semi*-exterior*** *of A is defined by* $s^*\text{Ext}(A)=s^*Int(X\backslash A)$.

Theorem 3.26. *If A, B are subsets of a topological space X, the following hold.*

(i) $s^*\text{Ext}(X)=\phi$.
(ii) $s^*\text{Ext}(\phi)=X$.
(iii) $A\subseteq B\Rightarrow s^*\text{Ext}(A)\supseteq s^*\text{Ext}(B)$.
(iv) $s^*\text{Ext}(A)$ is semi*-open.
(v) $\text{Ext}(A)\subseteq s^*\text{Ext}(A)\subseteq s\text{Ext}(A)\subseteq X\backslash A$.
(vi) *A is semi*-closed if and only if* $s^*\text{Ext}(A)=X\backslash A$.
(vii) $s^*\text{Ext}(A)=X\backslash[s^*Cl(A)]$.
(viii) $s^*\text{Ext}(s^*\text{Ext}(A))=s^*Int(s^*Cl(A))$.
(ix) $s^*\text{Ext}(A)=s^*\text{Ext}(X\backslash s^*\text{Ext}(A))$.
(x) $s^*Int(A)\subseteq s^*\text{Ext}(s^*\text{Ext}(A))$.
(xi) $X=s^*Int(A)\cup s^*\text{Ext}(A)\cup s^*Fr(A)$.
(xii) $s^*\text{Ext}(A\cup B)\subseteq s^*\text{Ext}(A)\cap s^*\text{Ext}(B)$.
(xiii) $s^*\text{Ext}(A\cap B)\supseteq s^*\text{Ext}(A)\cup s^*\text{Ext}(B)$.

Proof. (i), (ii) and (iii) follow from Definition 3.25. (iv) follows from Definition 3.25 and Theorem 3.13 [10]. (v) follows from Theorem 3.52(vi) of [10]. (vi) follows from Theorem 3.14 [10]. (vii) follows from Theorem 3.48(ii)[11]. (viii) follows from Theorem 3.48(i)[11]. (ix) follows from $s^*\text{Ext}(X\backslash s^*\text{Ext}(A)) = s^*\text{Ext}(X\backslash s^*Int(X\backslash A)) = s^*Int(X\backslash(X\backslash s^*Int(X\backslash A))) = s^*Int(X\backslash A)=s^*\text{Ext}(A)$. ($x$) follows from (viii) above. (xi) follows from Definition 3.25 and Theorem 3.12(xviii). (xii) and (xiii) follow from (iii) above and set theoretic properties. ■

Remark 3.27. *In* (v) *of Theorem 3.26, the inclusions may be strict and equality may hold as well. This can be seen from the following examples.*

Example 3.28. Consider the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau=\{\phi, \{a\}, \{a, b, c\}, X\}$.
Let $A=\{d\}$.Then $\text{Ext}(A)=s^*\text{Ext}(A)=s\text{Ext}(A)=\{a, b, c\}=X\backslash A$.
Let $B=\{b, d\}$. Then $\text{Ext}(B)=s^*\text{Ext}(B)=\{a\}$; $s\text{Ext}(B)=\{a, c\}=X\backslash B$.
Here, $\text{Ext}(B)=s^*\text{Ext}(B)\subsetneq s\text{Ext}(B)=X\backslash B$.
Let $C=\{b, c\}$.Then $\text{Ext}(C)=\{a\}$; $s^*\text{Ext}(C)=s\text{Ext}(C)=\{a, d\}=X\backslash C$.
Here $\text{Ext}(C)\subsetneq s^*\text{Ext}(C)=s\text{Ext}(C)=X\backslash C$.
Let $D=\{c\}$.Then $\text{Ext}(D)=\{a\}$; $s^*\text{Ext}(D)=\{a ,d\}$; $s\text{Ext}(D)=\{a, b, d\}=X\backslash D$.
Here $\text{Ext}(D)\subsetneq s^*\text{Ext}(D)\subsetneq s\text{Ext}(D)=X\backslash D$.
Let $E=\{a, b, c\}$.Then $\text{Ext}(E)=s^*\text{Ext}(E)=s\text{Ext}(E)=\phi$. Here $\text{Ext}(E)=s^*\text{Ext}(E)=s\text{Ext}(E)\subsetneq\{d\}=X\backslash E$.

Example 3.29. Consider the space (X, τ) in Example 3.5.
Let A={a, d, e, g}. Then Ext(A)=s*Ext(A)={b}; sExt(A)={b,f}.
Here, Ext(A)=s*Ext(A)⊊sExt(A)⊊{b, c, f}=X\A.
Let B={a, d, f, g}. Then Ext(B)={b}; s*Ext(B)=sExt(B)={b, e}.
Here, Ext(B)⊊s*Ext(B)=sExt(B)⊊X\B={b, c, e}.
Let C={a, c, g}.Then Ext(C)={b}; s*Ext(C)={b, e}; sExt(C)={b, e, f}.
Here, Ext(C)⊊s*Ext(C)⊊sExt(C)⊊X\C={b, d, e, f}.

Remark 3.30. *In* (x) *of Theorem 3.26, the inclusions may be strict and equality may hold as well. This can be seen from the following example.*

Example 3.31. Consider the space (X, τ) given in Example 3.5.
Let A={b, d, f, g}.Then *s*Int*(A)={b, f, g}; s*Ext(A)={a, e}; s*Ext(s*Ext(A))={b, f, g}.
Here, *s*Int*(A)=s*Ext(s*Ext(A)).
Let B={b, c, f}.Then *s*Int*(B)={b}; s*Ext(B)={a, e}; s*Ext(s*Ext(B))={b, f, g}.
Here, *s*Int*(B)⊊s*Ext(s*Ext(B)).

Remark 3.32. *In* (xii) *and* (xiii) *of Theorem 3.26, the inclusions may be strict and equality may hold as well. This can be seen from the following example.*

Example 3.33. Consider the space (X, τ) given in Example 3.5.
Let A={a, e}; B={c, e, f} Then A∪B={a, c, e, f}; A∩B={e} and s*Ext(A)={b, f, g}; s*Ext(B)={a, b}; s*Ext(A∪B)={b}; s*Ext(A∩B)={a, b, c, d, f, g}.
Here, s*Ext(A∪B)=s*Ext(A) ∩s*Ext(B) and s*Ext(A)∪s*Ext(B)⊊s*Ext(A∩B).
Let C={b, f} and D={a, c}. Then C∪D={a, b, c, f} and C∩D=ϕ. s*Ext(C)={a, c, d, e}; s*Ext(D)={b, e, f, g}; s*Ext(C)∪s*Ext(D)=X; s*Ext(C∩D)=X; s*Ext(C∪D)=ϕ; s*Ext(C)∩s*Ext(D)={e}.
Here, s*Ext(C)∪s*Ext(D)=s*Ext(C∩D) and s*Ext(C∪D)⊊s*Ext(C) ∩s*Ext(D).

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