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Some bounds on chromatic number of NI graphs

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Abstract

A connected graph is called neighbourly irregular (NI) if it contains no edge between the vertices of the same degree. In this paper we determine the upper bound for chromatic number of a neighbourly irregular graph. We also prove some results on neighbourly irregular graphs and its chromatic number.

Keywords: Regular graph, irregular graph, neighbourly irregular graph.

AMS Subject Classification(2010): 05C15.

1 Introduction

Throughout this paper we consider only finite, simple and connected graphs. Notations and terminology that we do not define here can be found in [9]. Let G be a connected graph of order n. For $1 \leq i \leq n$-1, the subset $V_i(G)$ (simply V_i) is defined as the set of all vertices of degree i in G, that is, $V_i(G) = \{ v \in V(G) \mid d(v) = i \}$. Note that $|V_i| \leq n$ for every i, $1 \leq i \leq n$-1. The degree set $D(G)$ of a graph G is the set of degrees of the vertices of G. A vertex of degree one is called a pendent vertex of G. The vertex which is adjacent to a pendent vertex is called a support vertex. For any two graphs G and H, the join $G \vee H$ is the graph obtained from $G \cup H$ by joining every vertex in G to each vertex in H. We denote the complement of G by G^c. For any real number x, $\lfloor x \rfloor$ denotes the greatest integer which does not exceed x and $\lceil x \rceil$ denotes the least integer which exceed x.

It is well known that all non-trivial graphs must contain at least two vertices of the same degree. A graph is said to be regular if each vertex has the same degree. Indeed, a graph is r-regular if $|V_r(G)| = n$. A graph that is not regular is called irregular [2]. Let I_n denote a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = \{v_{n+1-i}v_j, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, i \leq j \leq n-i\}$, which has precisely two vertices with the same degree [7]. I_n is a special type of irregular graph for which $D(I_n) = \{1, 2, \ldots, n-1\}$. In [11], the graph I_n is referred to as the pairlone graph and is denoted by PL_n. Also it has been proved that, for any $n \geq 2$, there exists a unique pairlone graph of order n. It is clear that and $\chi(PL_n) = n-\lfloor \frac{n-1}{2} \rfloor$. When n is odd, $\chi(PL_n) = (n+1)/2$.

S. Gnaana Bhragsam and Ayyaswamy [12] have introduced the concept of neighbourly irregular graph. A connected graph G is said to be a *neighbourly irregular graph* (NI graph) if no two adjacent vertices of G have the same degree. For example, the graphs shown in Figure 1 are NI. It has been proved that I_n is NI if and only if n is odd [7]. For more results in irregular graphs, one can refer to [1], [3], [4, [5], [8], [10], [13] and [14].

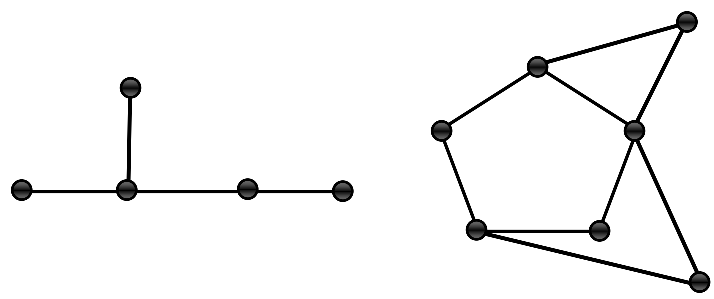


Figure 1.

It is clear that a connected graph G is NI if and only if for each i, $1 \leq i \leq n$-1, $V_i(G)$ is either empty or independent in G. In other words, a graph G is not NI if and only if for some i, V_i is neither empty nor independent in G.

The following facts can be verified easily:

Fact 1 [12] *If v is a vertex of maximum degree in a NI graph, then at least two of the adjacent vertices of v have the same degree.*

Fact 2 [12] $K_{m,n}$ is NI if $m \neq n$.

Fact 3 [12] *Let G be a NI graph of order n. Then for any positive integer m < n, there exists at most m vertices of degree n-m.*

Fact 4 [12] *If a graph G is NI, then* G^c *is not NI.*

The converse of the Fact 4 is not true. For example, neither P_4 nor P_4^c are NI.

Fact 5 [4] *Any NI graph G with clique number k has at least* $2k$-1 *vertices. For, the k vertices in the clique must have distinct degrees in G and therefore* $\Delta(G) \geq 2k$-2.

Fact 6 *If G is an NI graph of order n then for any* $m \geq 1$, $G \vee K_m^c$ *is NI if and only if* $V_{n-m}(G)$ *is empty.*

In this paper we prove that for any NI graph G, $\chi(G) \leq \left\lfloor \frac{\Delta}{2} \right\rfloor + 1$. Moreover we prove the existence of NI graph G of order n with $\chi(G) = k$ and $\Delta(G) = d$ for any k, d and n such that $4 \leq 2k$-$2 \leq d < n$. We also prove that, if $|d(u)$-$d(v)| \geq k > 2$ for any edge uv in an NI graph G, then $\chi(G) \leq \left\lceil \frac{\Delta}{k} \right\rceil$. In addition, we determine $\chi(G)$ of an NI graph G for which $|d(u)$-$d(v)| = k > 0$ for any edge uv in G. Some results on NI graphs in which $|d(u)$-$d(v)| = k > 0$ for any edge uv have been obtained in [6].

2 Main Results

First we prove the existence of an NI tree of order n with $\Delta(T) = d$ for any two positive integers n and d such that $3 \leq d < n$. P_3 is the only NI tree when $\Delta(T) = 2$.

Theorem 2.1. *For any two positive integers n and d such that* $n \geq d+1 \geq 4$, *there is an NI tree T of order n with* $\Delta(T) = d$.

Proof. Let $d \geq 3$ be a given positive integer. We prove the existence of an NI tree of order n with $\Delta(T) = d$ for any given $n > d$ by induction on n. If $n = d+1$, then $K_{1,d}$ is the required one for any $d \geq 3$.

Assume that the result is true for all n where $n \le d+k$, $k \ge 1$ and let T be an NI tree of order n with $\Delta(T) = d$. We claim that the result is true for $n+1$, by constructing an NI tree T_1 of order $n+1$ with $\Delta(T) = d$.

If T has a pendent vertex v whose support vertex is u such that $d(u) \ge 3$, then construct T_1 from T by adding a new vertex w and a new edge wv. Clearly T_1 is an NI tree of order $n+1$ with $\Delta(T_1) = d$. Otherwise in T, the degree of any support vertex is 2. Since $d \ge 3$, T must have at least 3 pendent vertices. Let v_1 and v_2 be any two pendent vertices in T. $T \setminus v_1$ is also an NI tree of order n-1 with $\Delta(T \setminus v_1) = d$. Now construct T_1 from $T \setminus v_1$ by adding two new vertices w_1 and w_2 and the edges w_1v_2 and w_2v_2. Clearly T_1 is an NI tree of order $n+1$ and $\Delta(T_1) = d$. ■

It is a well known fact that for any graph G, $\chi(G) \le \Delta+1$. Brook's [9] proved that the upper bound of $\chi(G)$ is Δ if the graph G is neither complete nor an odd cycle. But this bound can be reduced further in case of NI graphs, which is established in the following theorem.

Theorem 2.2. *For any NI graph G,* $\chi(G) \le \left\lfloor \frac{\Delta}{2} \right\rfloor + 1$.

Proof. We prove the result for the case when Δ is odd and in a similar way one can prove the same when Δ is even. Let $t = (\Delta+1)/2$ and let $S = \{ c_1, c_2, \ldots, c_t \}$ be a set with t distinct colours. For each i, $1 \le i \le t$ assign the colour c_i to the vertices in $V_{\Delta+1-i}$. Let S_i be the set of vertices which receive the colour c_i. Now it remains to assign colours to the vertices in V_i for all i, $1 \le i \le \Delta$-t. Choose V_k for some k, $1 \le k \le \Delta$-t. Clearly $k < t$. Let $v \in V_k$. Since $d(v) = k < t$, $N(v) \cap S_i$ must be empty for some i, $1 \le i \le t$. Now assign the colour c_i to the vertex v. Replace S_i by $S_i \cup \{ v \}$. Repeating the same process, we can colour all the vertices in V_k using colours $c_1, c_2, \ldots, c_t$. Since k is arbitrary, G is t-colourable. This completes the proof. ■

Since for the NI graph I_{2n+1}, $\Delta = 2n$ and $\chi = n+1$, the bound attained in the above theorem is sharp.

Corollary 2.31. *Any NI graph G with* $\chi(G) \ge k$ *has at least* 2k-1 *vertices.*

Proof. Let G be an NI graph with $\chi(G) \ge$ k. Suppose $n < 2k$-1. Then $\Delta(G) < 2k$-2. Now by Theorem 2.2, $k \le \chi(G) \le \left\lfloor \frac{\Delta}{2} \right\rfloor + 1 < \left\lfloor \frac{2k-2}{2} \right\rfloor + 1 = k$, a contradiction. Therefore, G must have at least 2k-1 vertices. ■

Theorem 2.4. *For any three positive integers k, d and n such that* $n > d \ge 2k$-2 ≥ 4, *there is an NI graph G of order n with* $\chi(G) = k$ *and* $\Delta(G) = d$.

Proof. Let $k \ge 3$ be given. We consider the following two cases.

Case 1: Suppose $n = d+1$.

If $d = 2k$-2, then $G \cong I_{2k-1}$ of order $d+1$ is the required graph. If $d > 2k$-2, then construct a new graph G from I_{2k-1} by joining d-(2k-2) new vertices to the vertex of degree 2k-2. Clearly G is an NI graph of order $d+1$ with $\Delta(G) = d$. Thus for any $k \ge 3$, there is an NI graph G of order $n = d+1$ with $\Delta(G) = d$, for any $d \ge 2k$-2.

Case 2: Suppose $n > d+1$.

Construct an NI graph G of order $d+1$ with $\Delta(G) = d$ as in Case 1. Let $t = n$-(d+1). If t = 1 or 2, then construct a new graph H from G by introducing t new vertices and joining them with a pendent vertex of G. If t = 3, then construct a new graph H from G by identifying a vertex of degree 2 of P_4 with a pendent vertex of G.

Suppose $t > 3$. Let T be an NI tree of order $t+1$ with $\Delta(T) = 3$ (T exists by Theorem 2.1). Let v be the vertex of degree d in G and let u be a pendent vertex which is adjacent to v. Now construct a new graph H from $G \setminus u$ by identifying a vertex of T with v. Clearly H is an NI graph of order n with $\chi(H) = k$ and $\Delta(H) = d$. This completes the proof. ■

In any graph G, $|d(u)-d(v)| \geq 0$ for any edge uv. G is regular if $|d(u)-d(v)| = 0$ for any edge uv in G. Clearly in an NI graph $|d(u)-d(v)| \geq 1$ for any edge uv. That is, a graph is NI if and only if the lowest bound for the difference between the degrees of the adjacent vertices is at least one. The following theorem gives the bound for the chromatic number if the lowest bound for the difference between the degrees of the adjacent vertices is known.

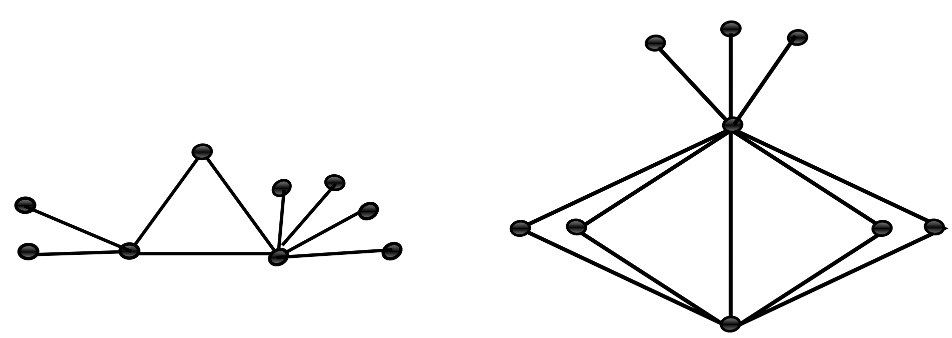
Theorem 2.5. *Let G be a graph in which* $|d(u)-d(v)| \geq k > 2$ *for any edge uv in G. Then* $\chi(G) \leq \left\lceil \frac{\Delta}{k} \right\rceil$.

Proof. Take $\Delta(G) = tk+r$ where $0 \leq r < k$. Suppose $r = 0$. Consider $S_i = \bigcup_{j=\Delta-ik+1}^{\Delta-(i-1)k} V_j(G)$, $i = 1, 2, \ldots, t$.

Clearly each S_i is independent in G. For each i, $1 \leq i \leq t$ assign the colour c_i to the vertices in S_i. Obviously, if S_i is empty for some i, $1 \leq i \leq t$, then the number of colours used to colour the vertices of G is fewer than t. Therefore, $\chi(G) \leq \left\lceil \frac{\Delta}{k} \right\rceil$.

If $r \neq 0$, then consider $S_{t+1} = \bigcup_{j=1}^{\Delta-tk} V_j(G)$. Clearly S_{t+1} is independent in G. For each i, $1 \leq i \leq t+1$ assign the colour c_i to the vertices in S_i. Clearly as discussed earlier, $\chi(G) \leq t+1 = \left\lceil \frac{\Delta}{k} \right\rceil$. ■

The above limit for $\chi(G)$ is tight. For example, consider the graph G_1 and G_2 in Figure 2, clearly $|d(u)-d(v)| \geq 2$ for any edge uv in G_1 and $|d(u)-d(v)| \geq 3$ for any edge uv in G_2. It can be verified that $\chi(G_1) = \frac{\omega}{2} = 3$, whereas $\chi(G_2) = \left\lceil \frac{\Delta}{3} \right\rceil = 3$.



G_1 G_2

Figure 2.

Fact 7 Let G be an NI graph such that $|d(u)-d(v)| \geq k > 2$ for any edge uv in G. If the clique number of G is ω, then G has at least $\omega+(\omega-1)k$ vertices.

For, any two vertices u and v in the clique, $|d(u)-d(v)| \geq k$ in G, and thus there must be at least one vertex of degree $(\omega-1)+(\omega-1)k$.

Let G be an NI graph of order n. The chromatic number becomes 2 if $|d(u)-d(v)| = k > 0$, for any two adjacent vertices u and v which is shown in the following theorem.

Theorem 2.6. *Let G be an NI graph of order n such that $|d(u)\text{-}d(v)| = k > 0$ for any edge uv in G. Then,* $\chi(G) = 2$.

Proof. Suppose G has an odd cycle $C = v, v_1, v_2, \ldots, v_m, u_m, u_{m-1}, \ldots, u_1, v$ of length $2m+1$. Let $d(v) = d$ in G. Since v_1 and u_1 are adjacent to v, the degree of v_1 and u_1 must be $d \pm k$. $|d(u_1)\text{-}d(v_1)|$ is either 0 or $2k$. Again degree of v_2 and u_2 must be d or $d \pm 2k$. Thus $|d(u_2)\text{-}d(v_2)|$ is either 0 or $2k$ or $4k$. Proceeding like this we get for each i, $1 \leq i \leq m$, $|d(v_i)\text{-}d(u_i)|$ is either 0 or greater than k. Indeed, $|d(v_m)\text{-}d(u_m)|$ is either 0 or greater than k. But v_m and u_m are adjacent in G. This is a contradiction to our assumption that $|d(v_m)\text{-}d(u_m)| = k$. Thus G contains no odd cycle and hence it is bipartite and hence $\chi(G) = 2$. ■

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