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Further results on product cordial graphs

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Abstract

A binary vertex labeling of graph G with induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ defined by $f^*(e = uv) = f(u)f(v)$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called *product cordial* if it admits product cordial labeling. We prove that the shell admits a product cordial labeling. Sundaram et al.[5] proved that if a graph with p vertices and q edges with $p \geq 4$ is product cordial then $q \leq \dfrac{p^2-1}{4} + 1$. We present here some families of graphs which satisfy this condition but not product cordial.

Keywords: Graph labeling, cordial labeling, product cordial labeling.
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1 Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with order p and size q. For all standard terminology and notations we follow Harary [3]. We give brief summary of definitions which are useful for the present study.

Definition 1.1. *A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).*

Graph labeling is one of the potential areas of research due to its diversified applications in computer network. An extensive survey on graph labeling can be found in Gallian [2].

Definition 1.2. *A mapping $f : V(G) \rightarrow \{0,1\}$ is called a binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f. The induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$.*

Let us denote

$$\left.\begin{array}{ll} v_f(i) = & \text{number of vertices of } G \text{ having label } i \text{ under } f \\ e_f(i) = & \text{number of edges of } G \text{ having label } i \text{ under } f^* \end{array}\right\} \text{where } i = 0 \text{ or } 1.$$

Definition 1.3. *A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called cordial if it admits cordial labeling.*

The concept of cordial labeling was introduced by Cahit [1]. Some variations in cordial labelings have been introduced such as Prime cordial labeling, E-cordial labeling, Product cordial labeling and Total product cordial labeling.

Definition 1.4. *A binary vertex labeling of graph G with induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ defined by $f^*(e = uv) = f(u)f(v)$ is called a product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called product cordial if it admits a product cordial labeling.*

The product cordial labeling was introduced by Sundaram et al. [4], [5]. The graphs obtained by joining apex vertices of k copies of stars, shells and wheels to a new vertex are proved to be product cordial by Vaidya and Dani [6] while some results on product cordial labeling for cycle related graphs are studied in Vaidya and Kanani [7]. In the same paper they studied product cordial labeling for shadow graph of cycle C_n. Vaidya and Barasara [8] have proved that the cycle with one chord, the cycle with twin chords, the friendship graph and the middle graph of path admit product cordial labeling. In [9] they proved that the graphs obtained by duplication of one edge, mutual vertex duplication and mutual edge duplication in cycle are product cordial graphs. Product cordial labeling in the context of tensor product of some graphs is studied by Vaidya and Vyas [10]. Vaidya and Barasara in [11] have obtained some results on product cordial labeling in the context of some graph operations. In the present work we investigate some families of graphs which admit product cordial labeling. We also present some families of graphs which satisfy the condition $q \leq \dfrac{p^2 - 1}{4} + 1$ but not product cordial.

Definition 1.5. *A* chord of cycle C_n *is an edge joining two non-adjacent vertices of cycle C_n.*

Definition 1.6. *The* shell S_n *is the graph obtained by taking $n - 3$ concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex. The shell S_n is also called the* fan f_{n-1}.

Definition 1.7. *The* shadow graph $D_2(G)$ *of a connected graph G is constructed by taking two copies* *of G say G' and G'' and joining each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.*

Definition 1.8. *The* square of a graph G *denoted by G^2 has the same vertex set as that of G and two vertices are adjacent in G^2 if they are at a distance of 1 or 2 apart in G.*

Definition 1.9. The splitting graph *of a graph G is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is denoted by $S'(G)$.*

Definition 1.10. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The* Cartesian product *of G_1 and G_2 which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that u and v are adjacent in $G_1 \times G_2$ whenever ($u_1 = v_1$ and u_2 adjacent to v_2) or ($u_2 = v_2$ and u_1 adjacent to v_1).*

Definition 1.11. *The* circular ladder graph CL_n *is defined as $C_n \times P_2$.*

Definition 1.12. *The* Möbius ladder M_n *is a graph obtained from the ladder $P_n \times P_2$ by joining the opposite end vertices of two copies of P_n.*

Definition 1.13. *The* middle graph $M(G)$ *of a graph G is the graph whose vertex set is $V(G) \bigcup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.*

Definition 1.14. *Let P_n be a path on n vertices denoted by $(1, 1), (1, 2), \ldots, (1, n)$ and with $n-1$ edges denoted by $e_1, e_2, \ldots, e_{n-1}$ where e_i is the edge joining the vertices $(1, i)$ and $(1, i+1)$. On each edge e_i, $i = 1, 2, \ldots, n-1$ we erect a ladder with $n-(i-1)$ steps including the edge e_i. The graph so obtained is called a* step ladder graph *which is denoted by $S(T_n)$, where n denotes the number of vertices in the base.*

Definition 1.15. *$H_{n,n}$ is the graph with vertex set $V(H_{n,n}) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and the edge set $E(H_{n,n}) = \{v_i u_j : 1 \le i \le n, n-i+1 \le j \le n\}$.*

2 Main Results

Theorem 2.1. *S_n is a product cordial graph for odd n and not product cordial for even n.*

Proof. Let $v_1, v_2, \ldots, v_n$ are the vertices of shell S_n with v_1 as an apex vertex. Then $|V(S_n)| = n$ and

$|E(S_n)| = 2n - 3$.
We consider the following two cases.
Case 1: When n is odd.

$f(v_i) = 1 \quad$ for $1 \leq i \leq \left\lceil \dfrac{n}{2} \right\rceil$

$f(v_i) = 0 \quad$ otherwise

Then we have

$1 + v_f(0) = v_f(1) = \left\lceil \dfrac{n}{2} \right\rceil$

$e_f(0) = e_f(1) + 1 = \left\lceil \dfrac{2n-3}{2} \right\rceil$.

Thus, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.
Therefore, S_n is a product cordial graph for odd n.
Case 2: When n is even.

Assign label 0 to $\dfrac{n}{2}$ vertices out of the n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least n edges to have label 0 and at most $n-3$ edges out of $2n-3$ edges to have label 1 .Therefore, $|e_f(0) - e_f(1)| = 3$.

Thus, the edge condition for product cordial graph is violated. Therefore, S_n is not a product cordial graph for even n. ∎

Illustration 2.2. *Shell S_7 and its product cordial labeling is shown in Figure 1.*

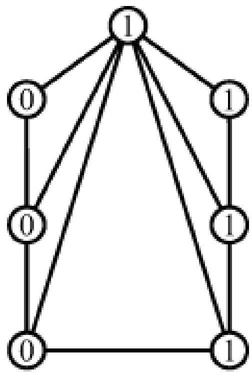

Figure 1.

Theorem 2.3. *$D_2(C_n)$ is not a product cordial graph.*

Proof. The shadow graph of cycle C_n has $2n$ vertices and $4n$ edges. We consider the following two cases.
Case 1: When $n = 3$.
Assign label 0 to any 3 vertices so that the vertex condition for the product cordial graph is satisfied. The

vertices with label 0 cause at least 6 edges to have label 0 and at most 3 edges to have label 1. Therefore $|e_f(0) - e_f(1)| = 6$. Thus, the edge condition for product cordial graph is violated.

Case 2: When $n \neq 3$.

Assign label 0 to n vertices out of total $2n$ vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $2n+4$ edges to have label 0 and at most $2n-4$ edges to have label 1. Therefore, $|e_f(0)-e_f(1)| = 8$. Thus the edge condition for product cordial graph is violated.

Hence, $D_2(C_n)$ is not a product cordial graph. ■

Theorem 2.4. *$D_2(P_n)$ is not a product cordial graph.*

Proof. The shadow graph of path P_n has $2n$ vertices and $4n-4$ edges. Assign label 0 to n vertices out of total $2n$ vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $2n$ edges to have label 0 and at most $2n-4$ edges to have label 1. Therefore $|e_f(0) - e_f(1)| = 4$. Thus the edge condition for product cordial graph is violated. Hence $D_2(P_n)$ is not a product cordial graph. ■

Theorem 2.5. *C_n^2 is not a product cordial graph.*

Proof. The square of the cycle C_n has n vertices and $2n$ edges. We consider the following two cases.

Case 1: When n is odd.

Assign label 0 to $\left\lfloor \frac{n}{2} \right\rfloor$ vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $n+2$ edges to have label 0 and at most $n-2$ edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = 4$. Thus the edge condition for product cordial graph is violated.

Case 2: When n is even.

Assign label 0 to $\frac{n}{2}$ vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $n+3$ edges to have label 0 and at most $n-3$ edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = 6$. Thus the edge condition for product cordial graph is violated.

Hence, C_n^2 is not a product cordial graph. ■

Theorem 2.6. *$M(C_n)$ is not a product cordial graph.*

Proof. The middle graph of cycle C_n has $2n$ vertices and $3n$ edges. Assign label 0 to n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $2n$ edges to have label 0 and at most n edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = n$. Thus the edge condition for product cordial graph is violated. Hence, $M(C_n)$ is not a product cordial graph. ■

Theorem 2.7. *$S'(C_n)$ is not a product cordial graph.*

Proof. The splitting graph of cycle C_n has $2n$ vertices and $3n$ edges. We consider the following three cases.
Case 1: When $n \equiv 0$ (mod 4).
Assign label 0 to n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $\frac{3n}{2}+2$ edges to have label 0 and at most $\frac{3n}{2}-2$ edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = 4$. Thus the edge condition for product cordial graph is violated.
Case 2: When $n \equiv 1$ (mod 4) or $n \equiv 3$ (mod 4).
Assign label 0 to n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $\frac{3n-1}{2}+3$ edges to have label 0 and at most $\frac{3n+1}{2}-3$ edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = 5$. Thus the edge condition for product cordial graph is violated.
Case 3: When $n \equiv 2$ (mod 4).
Assign label 0 to n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $\frac{3n}{2}+3$ edges to have label 0 and at most $\frac{3n}{2}-3$ edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = 6$. Thus the edge condition for product cordial graph is violated.
Hence $S'(C_n)$ is not a product cordial graph. ∎

Theorem 2.8. *CL_n is not a product cordial graph.*

Proof. Circular ladder CL_n has $2n$ vertices and $3n$ edges. Assign label 0 to n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $2n$ edges to have label 0 and at most n edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = n$. Thus the edge condition for product cordial graph is violated. Hence CL_n is not a product cordial graph. ∎

Theorem 2.9. *M_n is not a product cordial graph.*

Proof. Möbius ladder M_n has $2n$ vertices and $3n$ edges. We consider the following two cases.
Case 1: When $n = 3$.
Assign label 0 to 3 vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least 7 edges to have label 0 and at most 2 edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = 5$. Thus the edge condition for product cordial graph is violated.
Case 2: When $n \neq 3$.
Assign label 0 to n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $2n$ edges to have label 0 and at most n edges to have label 1.

Therefore, $|e_f(0) - e_f(1)| = n$. Thus the edge condition for product cordial graph is violated. Hence M_n is not a product cordial graph. ■

Theorem 2.10. $S(T_n)$ *is not a product cordial graph.*

Proof. Step ladder $S(T_n)$ has $\frac{n^2+5n+2}{2}$ vertices and n^2+3n edges. We consider the following two cases.
Case 1: When $\frac{n^2+5n+2}{2}$ is odd.
Assign label 0 to $\left\lfloor \frac{n^2+5n+2}{4} \right\rfloor$ vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $\frac{n^2+3n+4}{2}$ edges to have label 0 and at most $\frac{n^2+3n-4}{2}$ edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = 4$. Thus the edge condition for product cordial graph is violated.
Case 2: When $\frac{n^2+5n+2}{2}$ is even.
Assign label 0 to $\frac{n^2+5n+2}{4}$ vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $\frac{n^2+3n+4}{2}$ edges to have label 0 and at most $\frac{n^2+3n-4}{2}$ edges to have label 1. Therefore, $|e_f(0)-e_f(1)| = 4$. Thus the edge condition for product cordial graph is violated.

Hence in all cases, $S(T_n)$ is not a product cordial graph. ■

Theorem 2.11. $H_{n,n}$ *is a product cordial graph for* $n = 2$ *and not a product cordial graph for* $n > 2$.

Proof. The graph $H_{n,n}$ has $2n$ vertices and $\frac{n(n+1)}{2}$ edges. We consider the following three cases.
Case 1: When $n = 2$.
Since P_n is product cordial for all n, P_2 is a product cordial graph and $H_{2,2}$ is isomorphic to P_2. Thus $H_{2,2}$ is a product cordial graph.
Case 2: When n is odd.
Assign label 0 to n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $\frac{n^2+3}{4}$ edges to have label 0 and at most $\frac{n^2-1}{4}$ edges to have label 1. Therefore, $|e_f(0) - e_f(1)| = \left\lceil \frac{n}{2} \right\rceil$. Thus the edge condition for product cordial graph is violated.
Case 3: When n is even and $n > 2$.
Assign label 0 to n vertices so that the vertex condition for the product cordial graph is satisfied. The vertices with label 0 cause at least $\frac{n^2+4}{4}$ edges to have label 0 and at most $\frac{n^2}{4}$ edges to have label 1.

Therefore, $|e_f(0) - e_f(1)| = \dfrac{n}{2}$. Thus the edge condition for product cordial graph is violated.
Hence $H_{2,2}$ is a product cordial graph for $n = 2$ and not a product cordial graph for $n > 2$. ■

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