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On path connector sets

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Abstract

In this paper, we identify a few proper subsets of the vertex set of a non-empty connected graph and study their properties. Further, we obtain some results when the graph is semi-complete.

Keywords: Split path connector set,non-split path connector set, complementary nil path connector set,point set path connector set.
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1 Introduction

In defence , we come across situations where the complete strategy should not be known to a single individual but can be shared by two officials. Further, in some situations there should not be a direct contact between the two individuals, but can be linked by a superior (third) person.

In such situations, the notion of a complete graph does not serve the purpose. I.H. Naga Raja Rao and S.V. Siva Rama Raju introduced a new type of graph called "Semi-Complete Graph" and studied its various properties in [2], [3] and [4].

For standard terminology and results we refer [1].

2 Preliminaries

Definition 2.1. *[2] (i) A graph G is said to be semi-complete(SC) if it is simple and for any two vertices u,v of G there is a vertex w of G such that $\{u, w, v\}$ is a path in G.*
(ii) A graph G is said to be purely semi-complete if and only if G is semi-complete but not complete.

Theorem 2.2. *[2] If G is a semi-complete graph then there exists a unique path of length 2 between any two vertices of G if and only if the edge set of G can be partitioned into edge disjoint triangles.*

Theorem 2.3. *[2] If G is a semi complete graph in which no two triangles have a common edge, then all the triangles have a common vertex.*

Definition 2.4. *[4] (i) A Path Connector set(PC-Set) in a graph G is a subset V' of the vertex set V of G such that for any distinct pair of non-adjacent vertices in G there is a shortest path whose internal vertices are from V'.*
(ii) A Path Connector Set in G is said to be a minimum Path Connector Set (mPC-Set) in G if and only if it has the minimum cardinality among all the PC-Sets in G.

Theorem 2.5. *[4] Let G be a purely semi-complete graph with vertex set V. Then*
(a) Any PC-Set S in G is a dominating set in G.
(b) Further, if $\mid S \mid \geq 2$ then S is a connected dominating set in G.

Definition 2.6. *[5] Let G be a connected graph. A set $D \subseteq V(G)$ is a point-set dominating set (psd-Set) of G if for every set $S \subseteq V - D$, there exists a vertex $v \in D$ such that the subgraph $< S \bigcup \{v\} >$ induced by $S \bigcup \{v\}$ is connected. The point-Set domination number $\gamma_p(G)$ is the minimum cardinality of a psd-Set.*

3 Split/Non-split Path Connector Sets

Definition 3.1. *Let G be a connected graph with vertex set V.*

1. *A proper Path Connector set (PC-Set) S of G is said to be a Split Path Connector Set (SPC-Set) in G or a Non-Split Path Connector Set (NSPC-Set) in G according as the subgraph induced by $V - S$ is disconnected or connected.*

2. *An SPC-Set in G is said to be a minimum SPC-Set (mSPC-Set) if it has minimum cardinality among all the SPC-Sets in G.*

3. *An NSPC-Set in G is said to be a minimum NSPC-Set (mNSPC-Set) if it has minimum cardinality among all the NSPC-Sets in G.*

Example 3.2. $S = \{v_3, v_5, v_6\ (or\ v_7)\}$ *is an SPC-Set and also an mSPC-Set.* $S' = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ *is an NSPC-Set.*

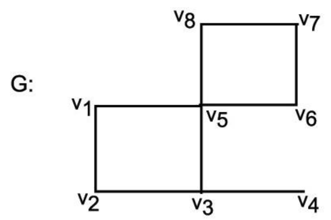

Figure 1.

Example 3.3. *For the graph given in Figure 2a, $\{v_1\}, \{v_4\}$ are PC-Sets(infact mPC-Sets) and they are also NSPC-Sets since the graph $< V - \{v_i\} >$(i=1 or 4) is connected.*

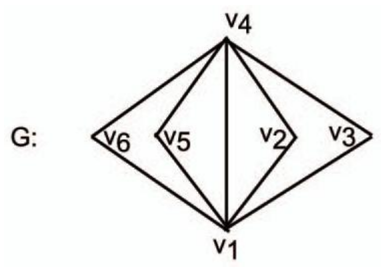

Figure 2a.

For the graph given in Figure 2b, $S = \{v_1, v_4\}$ is an SPC-Set in G, since $< V - S >$ is disconnected. A graphical representation of it is the null graph induced by $\{v_6, v_5, v_2, v_3\}$.
Observe that S is not a mPC-Set in G, but it is an mSPC-Set in G.

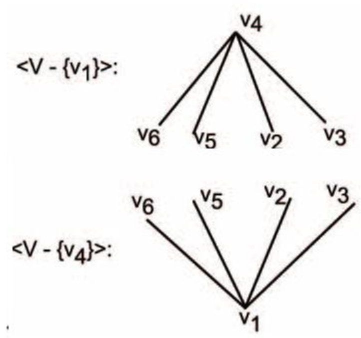

Figure 2b.

Example 3.4. *For the graph G given in Figure 3, $\{v_0\}$ is a mPC-Set and also an mSPC-Set. Clearly $V - \{v_i\}(i = 1, 2, ..., 6)$ are NSPC-Sets.*

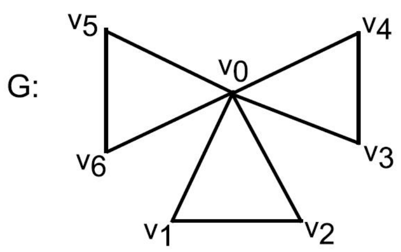

Figure 3.

Theorem 3.5. *If G is a connected graph having a cut-vertex, then any PC-Set in G is an SPC-Set.*

Proof. Let v_0 be a cut-vertex in G and S be a PC-Set in G. Then $v_0 \in S$ and there exists a pair of non-adjacent vertices in G, say v_1, v_2 such that v_0 is the internal vertex in every $v_1 - v_2$ path in $G \Rightarrow < V - S >$ is disconnected. Thus, S is an SPC-Set in G. ∎

The converse of the above theorem is false. For example, consider the graph G in Figure 4. Any PC-Set in G is an SPC-Set in G, but G does not have any cut-vertex.

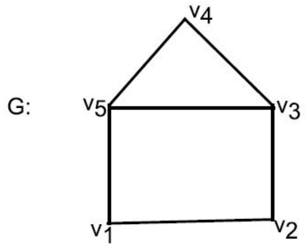

Figure 4.

We prove a necessary and sufficient condition for a proper subset of the vertex set of a (non empty) connected graph to be an SPC-Set.

Theorem 3.6. *Let G be a (non empty) connected graph with vertex set V and S be a proper subset of V. Then S is an SPC-Set in G if and only if there is a pair of non-adjacent vertices in $< V - S >$ such that every path between them in G has atleast one internal vertex from S.*

Proof. Since S is an SPC-Set, $< V - S >$ is disconnected. Hence, there is a pair of non adjacent vertices in $< V - S >$ such that there is no path between them in $< V - S >$. Since G is connected, there is a path between them in G. Hence, every path between them in G has atleast one internal vertex from S.

Conversely, by the hypothesis there is a pair of vertices in $< V - S >$ such that there is no path between them in $< V - S >$. Hence $< V - S >$ is disconnected. Since S is a PC-Set, S is an SPC-Set in G. ■

Corollary 3.7. *G is a non empty connected graph with vertex set V and S is a proper PC-Set in G. Then S is an NSPC-Set in G if and only if for every pair of non adjacent vertices in $< V - S >$ there is a path with no internal vertex from S.*

Observation 3.8. *If S and $V - S$ are PC-Sets in G, then S is an NSPC-Set in G.*

4 Complementary nil path connector set

Definition 4.1. *Let G be a connected graph with vertex set V. A proper path connector set S of G is said to be a complementary nil path connector set(CNPC-Set) in G if $V - S$ is not a path connector set in G.*

Example 4.2. *Consider the graph G given in Figure 5. $S = \{v_1, v_3, v_6\}$ is a PC-Set but $V - S = \{v_2, v_4, v_5, v_7\}$ is not a PC-Set, since there is no shortest $v_1 - v_4$ path(path of length 2) whose internal vertices are from $V - S$. Thus S is a CNPC-Set for G. Whereas, for the graph G given in Example 3.3 Figure 2a, $\{v_i\}$ and $V - \{v_i\}$ are PC-Sets in G for $i = 1, 4$. Hence, $\{v_1\}$, $V - \{v_1\}$ and $\{v_4\}$, $V - \{v_4\}$ are not CNPC-Sets in G.*

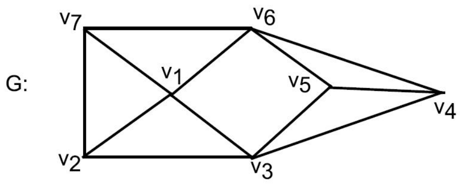

Figure 5.

Observation 4.3.

(i) An SPC-Set S in a connected graph G is a CNPC-Set in G, since $< V - S >$ is disconnected ($\Rightarrow$ $V - S$ is not a PC-Set in G). But the converse is not true. For the graph G in Figure 6, $S = \{v_3, v_5, v_7\}$ is a CNPC-Set in G, but not an SPC-Set in G.

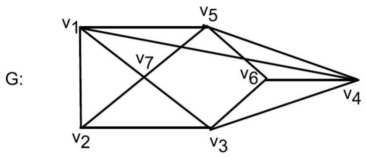

Figure 6.

(ii) An NSPC-Set S in a connected graph G needs not be a CNPC-Set in G. For example, for the graph G in Figure 7, $S = \{v_3, v_7\}$ is an NSPC-Set in G, but not a CNPC-Set in G.

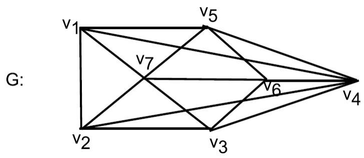

Figure 7.

(iii) An NSPC-Set S in a connected graph G can be a CNPC-Set in G. For the graph G in Figure 8, $S = \{v_1, v_4\}$ is an NSPC-Set in G, which is also a CNPC-Set in G.

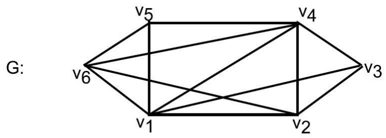

Figure 8.

Theorem 4.4. *Let G be a connected graph with vertex set V and S be a proper PC-Set in G. Then S is a CNPC-Set in G if and only if there is a pair of non-adjacent vertices in G such that every shortest path between them in G has atleast one internal vertex from S.*

Proof. Let S be a CNPC-Set in G. Hence by definition, $V - S$ is not a PC-Set in G. So there is a pair of non-adjacent vertices in G such that no shortest path between them (in G) have all its internal vertices from $V - S$. Hence every shortest path between them in G has atleast one internal vertex which is not in $V - S$ and hence is in S.

Conversely, suppose that there is a pair of non-adjacent vertices in G such that every shortest path between them in G has atleast one internal vertex from S. Then there is no shortest path between a pair of non-adjacent vertices in G whose internal vertices are from $V-S \Rightarrow V-S$ is not a PC-Set in G. Thus, S is a CNPC-Set in G. ∎

Corollary 4.5. *If G is a connected graph in which there exists a unique shortest path between a pair of non-adjacent vertices in G, then every PC-Set in G is a CNPC-Set.*

Proof. If S is any PC-Set in G then there is a pair of non-adjacent vertices in G such that every shortest path between them in G(infact there is only one such path) has all its internal vertices from S. Hence, by Theorem 4.4, S is a CNPC-Set in G. ∎

The converse of the above corollary is false in view of the graph given in Figure 8. Here, $S = \{v_1, v_2, v_4\}$ is a CNPC-Set, but in this graph there exists two shortest paths, each of length 2 between any pair of non-adjacent vertices.

Theorem 4.6. *Let G be a connected graph having a cut-vertex . Then any PC-Set in G is a CNPC-Set in G.*

Proof. Let v_0 be a cut-vertex in G. Therefore, $v_0 \in S$. Hence there is a pair of non-adjacent vertices in G, say v_1, v_2 such that v_0 is the internal vertex in every $v_1 - v_2$ path in $G \Rightarrow < V-S >$ is not a PC-Set in G. Thus, S is a CNPC-Set in G. ∎

The converse of the above theorem is false in view of the graph given in Figure 4. Any PC-Set in G is a CNPC-Set in G, but G does not have cut-vertices.

Corollary 4.7. *Let G be a purely semi-complete graph whose edge set is a union of edge disjoint triangles. Then any PC-Set for G is a CNPC-Set in G.*

Proof. The given hypothesis in virtue of Theorem 2.3 implies that all the triangles have a common vertex, v_0. Also there is a unique shortest path between any pair of non-adjacent vertices in G(each is of length 2 and the internal vertex being v_0). Hence any proper subset S of the vertex set of G such that $v_0 \in S$ is a PC-Set in G. Now by the above theorem, any PC-Set for G is a CNPC-Set in G. ∎

Theorem 4.8. *If G is a purely semi-complete graph in which no two triangles have a common edge, then the intersection of all CNPC-Sets is a singleton.*

Proof. By Theorem 2.3 all the triangles of G have a common vertex, say v_0. If S is any PC-Set in G then $v_0 \in S$. Clearly $\{v_0\}$ is also a PC-Set in G. Futher between any two non-adjacent vertices in G there is a unique shortest path (of length 2). By Corollary 4.5, each PC-Set is a CNPC-Set. Hence, the intersection of all CNPC-Sets is the singleton $\{v_0\}$. ∎

Remark 4.9. *The converse of the above theorem is false in view of the following example. Consider the following graph G given in Figure 9. We observe that any CNPC-Set in G contains the vertex v_0. Hence the intersection of all CNPC-Sets is $\{v_0\}$. But the edge set of G is not a union of edge disjoint triangles.*

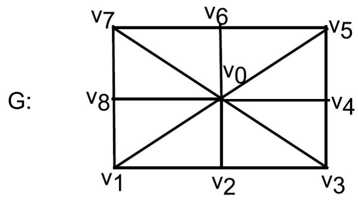

Figure 9.

5 Point set path connector set

Definition 5.1. *(i) Let G be a connected graph with vertex set V. A path connector set S in G is said to be a point set path connector set (PSPC-Set) if for every $P \subseteq V - S$ there is a $u \in S$ such that $< P \bigcup \{u\} >$ is connected.*
(ii) A PSPC-Set in G is said to be a minimum point set path connector set (mPSPC-Set) in G if it has minimum cardinality among all the PSPC-Sets in G.

Example 5.2. *Consider the graph given in Figure 4. Here, $S = \{v_3, v_5\}$ is a PSPC-Set, since for any $P \subseteq V - S = \{v_1, v_2, v_4\}$ there is a $u \in S$ such that $< P \bigcup \{u\} >$ is connected.*

Example 5.3. *For the following graph G given in Figure 10, $S = \{v_3\}$ is a PSPC-Set(infact mPSPC-Set), since for any $P \subseteq V - S = \{v_1, v_2, v_4, v_5\}, < P \bigcup \{v_3\} >$ is connected.*

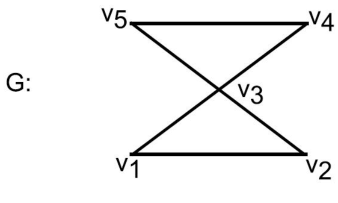

Figure 10.

Remark 5.4. *A PC-Set in a(connected) graph need not be PSPC-Set in view of the following example. Consider the following graph G in Figure 11. For this graph, $S = \{v_1, v_5\}$ is a PC-Set. It is not a PSPC-Set , for $P = \{v_4, v_6\}$ both $< P \bigcup \{v_1\} >, < P \bigcup \{v_5\} >$ are disconnected.*

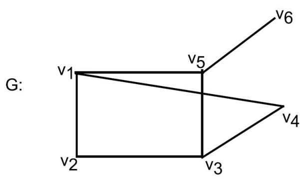

Figure 11.

Theorem 5.5. *G is a connected graph with vertex set V and S is a PC-Set in G. If for every $P \subseteq V - S, < P >$ has a PC-Set in itself, then S is a PSPC-Set in G.*

Proof. By the given hypothesis, $< P >$ is connected for every $P \subseteq V - S$. Since S is a PC-Set in G by Theorem 2.5, S is a dominating set for G. So there is a $u \in S$ such that u is adjacent with some vertex of P. Hence $< P \bigcup \{u\} >$ is connected. Thus, S is a PSPC-Set in G. ■

Observation 5.6. *The converse of the above theorem is false. For example, consider the graph G given in Figure 12. $S = \{v_1, v_2, v_4, v_6\}$ is a PSPC-Set in G. But for $P = V - S = \{v_3, v_5\}, < P >$ has no PC-Set in itself. Infact $< P >$ is not connected.*

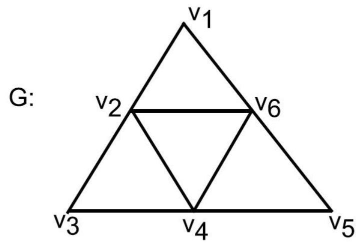

Figure 12.

Theorem 5.7. *Let G be a connected graph with vertex set V and S be a PSPC-Set in G. If for every $P \subseteq V - S, < P >$ has a PC-Set in P itself, then S is an NSPC-Set in G.*

Proof. Let S be a PSPC-Set in G. Then $< V - S >$ is connected and by definition S is an NSPC-Set in G. ■

Observation 5.8. *The converse of the above theorem is false. Consider the graph G in Figure 13. $S = \{v_0\}$ is an NSPC-Set in G. Since $< V - S > = < \{v_1, v_2, v_3, v_4, v_5, v_6\} >$ is connected. But $P = \{v_1, v_3\} \subseteq V - S$ is such that $< P >$ is not connected (and has no PC-Set in itself).*

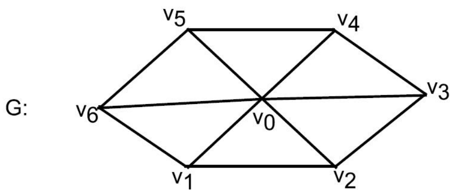

Figure 13.

Theorem 5.9. *Let G be a connected graph with vertex set V and S be a PSPC-Set in G. If for every $P \subseteq V - S$ with $|P| \geq 2, < P >$ has atleast two components then S is an SPC-Set in G.*

Proof. By the given hypothesis, $< V - S >$ has atleast two components and hence $< V - S >$ is disconnected $\Rightarrow S$ is an SPC-Set in G. ■

Observation 5.10. *The converse of the above theorem is false in view of the following. Consider the graph given in Figure 3.* $\{v_0\}$ *is a PSPC-Set which is an SPC-Set in* G*. Now* $P = \{v_1, v_2\} \subseteq V - \{v_0\}$ *with* $|P| = 2$ *and* $< P >$ *is connected and hence has no two components.*

Theorem 5.11. *If* G *is a connected graph and* S *is a PC-Set in* G *which is not a PSPC-Set in* G*, then* $|mpcs(G)| \geq 2$*.*

Proof. S is not a PSPC-Set in G. Hence, there is a $P \subseteq V - S$ such that $< P \bigcup \{u\} >$ is not connected for any $u \in S$. Then $< P >$ is not connected and hence has atleast two components. So atleast two vertices in P are not dominated by a single vertex from $S \Rightarrow |mpcs(G)| \geq 2$. ∎

Observation 5.12. *The converse of the above theorem is not true. For example consider the graph* G *given in Figure 14. Here,* $|mpcs(G)| = 2$*. But clearly every pc-Set in* G *is a PSPC-Set in* G*.*

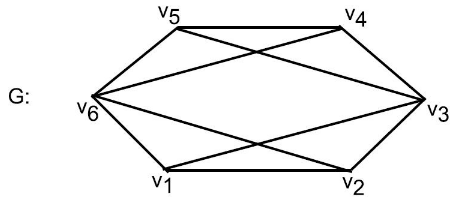

Figure 14.

Theorem 5.13. *If* G *is a purely semi-complete graph and* S *is a PC-Set in* G*, then* S *is a PSPC-Set in* G *if and only if for any independent set* $B \subseteq V - S$*, there is a* $v \in S$ *such that every vertex of* B *is adjacent to* v *in* G*.*

Proof. Let S be a PSPC-Set in G and B be any independent set in $V - S$. Then there is a $v \in S$ such that $< B \bigcup \{v\} >$ is connected. Since no two elements in B are adjacent we have for any $v_1, v_2 \in B$, $\{v_1, v, v_2\}$ is a path in G, of length 2. Thus the necessary part holds.

Conversely, suppose that S is a PC-Set in G and the stated condition holds. Let $P \subseteq V - S$. If $< P >$ is connected then the result is trivial. Otherwise, let $P_1, P_2, ..., P_n$ be the components of P, where $n \geq 2$. Select $v_i \in P_i (i = 1, 2, ..., n)$. So $B = \{v_1, v_2, ..., v_n\}$ is an independent set in G. Now, by the hypothesis there exists a $v \in S$ such that $< B \bigcup \{v\} >$ is connected (since each v_i is adjacent to v in G). Therefore, $< P \bigcup \{v\} >$ is connected. Hence, S is a PSPC-Set in G. ∎

Theorem 5.14. *If* G *is a purely semi-complete graph, then every PSPC-Set in* G *is a point set dominating set in* G*.*

Proof. Let S be a PSPC-Set in the purely semi-complete graph G and $P \subseteq V - S$. Let $B \subseteq P$ be any independent set. By the above theorem, there is a $v \in S$ such that $< B \bigcup \{v\} >$ is connected $\Rightarrow < P \bigcup \{v\} >$ is connected $\Rightarrow S$ is a point set dominating set in G. ∎

Observation 5.15. *The converse of the above theorem is false. For the graph G given in Figure 15, $\{v_2, v_4, v_6\}$ is a point set dominating set in G. But it is not even a pc-Set in G and hence not a PSPC-Set in G.*

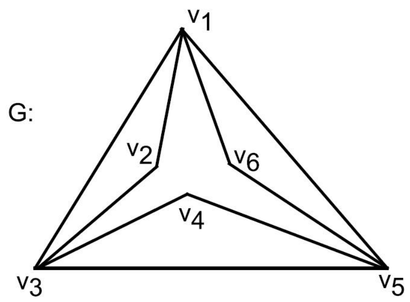

Figure 15.

Theorem 5.16. *Let G be a purely semi-complete graph with n vertices and S be a PC-Set in G such that $|S| \geq n-2$. Then S is a PSPC-Set in G.*

Proof. Let $P \subseteq V - S$. Then either P is singleton or two elements set. Let $P = \{v\}$. Since S is a pc-Set in G, by Theorem 2.5, S is a dominating set in G. So there is a $u \in S$ such that v is adjacent to $u \Rightarrow < P \bigcup \{u\} >$ is connected.

Let $P = \{v_1, v_2\}$. If v_1, v_2 are adjacent , then as in the previous case, there is a $u \in S$ such that $< P \bigcup \{u\} >$ is connected. If v_1 and v_2 are non-adjacent, then by the nature of S, there is a $u \in S$ such that $< P \bigcup \{u\} >$ is connected.Thus S is a PSPC-Set in G. ∎

Remark 5.17. *The converse of the above theorem is false in view of the graph given in Figure 14. $S = \{v_2, v_5\}$ is a PSPC-Set in G. But, $P = V - S = \{v_1, v_3, v_4, v_6\}$ is such that $|P| = 4 > 2$.*

Theorem 5.18. *If G is a purely semi-complete graph that has a unique path between any pair of vertices in G, then the intersection of all PSPC-Sets in G is a singleton.*

Proof. By Theorem 2.2, G is a union of edge disjoint triangles having a common vertex, say v_0. It follows that any PSPC-Set in G contains vertex v_0. Infact $\{v_0\}$ itself is a PSPC-Set in G. Hence, their intersection is $\{v_0\}$. ∎

Observation 5.19. *The converse of the above theorem is false. Consider the graph G given in Figure 16. Clearly all PSPC-Sets in G include v_1. But there are two shortest paths between v_1 and v_5 namely $\{v_1, v_4, v_5\}$ and $\{v_1, v_6, v_5\}$.*

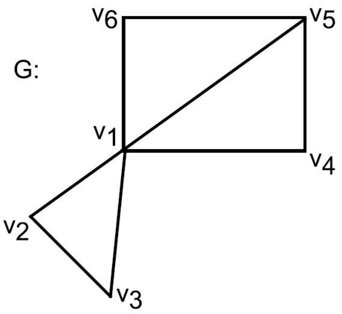

Figure 16.

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