International Journal of Mathematics and Soft Computing
Vol.2, No.2 (2012), 43 - 48.

ISSN 2249 - 3328

Ideals of a distributive lattice with respect to a closure operator

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Abstract

In this paper Quasi-complemented lattices are characterized in terms of dominator ideals. A closure operator is introduced on the ideal lattice and with the help of this operator the concept of closure ideals is introduced and then these ideals are characterized. Further, a one-to-one correspondence between the class of all prime closure ideals of a lattice and the ideal lattice of the lattice of all dominator ideals is established.

Keywords: Quasi-complemented distributive lattice, Boolean algebra, dense element, closure operator, closure ideal, isomorphism.
AMS Subject Classification(2010): 06D99.

1 Introduction

In 1968, the theory of relative annihilators was introduced in lattices by Mark Mandelker[3] and he characterized distributive lattices in terms of their relative annihilators. Several authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized many algebraic structures in terms of annihilators. In this paper we study the structure of closure ideals in distributive lattices with the support of annihilators.

Also the concept of closure ideals is introduced and characterized. The class of distributive $\star$-lattices[4] are also characterized in terms of dominator ideals. Finally, an isomorphism is obtained between the lattices of closure ideals and the ideal lattice of dominator ideals.

2 Definition and Preliminary results

For the elementary notions one can refer to [1] and [2]. We give some of the preliminary definitions and results which are useful for the present study.

For any $a \in L$ in a distributive lattice $(L, \vee, \wedge, 0)$ with zero, we define the annihilator of a by the set $(a]^* = \{ x \in L \mid x \wedge a = 0 \}$ where the principle ideal generated by a is written as $(a]$. An element $d \in L$ is called dense if $(d]^* = (0]$ and the set D of all dense elements is a filter whenever D is non-empty. An ideal I of L is called proper if $I \neq L$. A proper ideal P is called prime if for any $x, y \in L$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

A distributive lattice L is called quasi-complemented if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D$. Clearly every quasi-complemented lattice contains a dense element. Throughout this paper, L stands for a distributive lattice with 0, unless or otherwise specified.

3 Main Results

In this section we begin with the definition of a dominator ideal. Then we introduce the notion of closure ideals and obtain an isomorphism between the lattice of closure ideals and ideal lattice of dominator ideals.

Definition 3.1. *For any $a \in L$, the dominator of a is defied as*

$$(a)^e = \{x \in L \mid (a]^* \subseteq (x]^*\}.$$

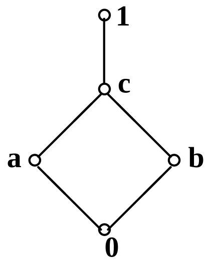
The following lemma is a direct consequence of the above definition.

Lemma 3.2. *For any $a, b \in L$, we have the following:*

(1) $(a)^e$ *is an ideal of* L.
(2) $(a] \subseteq (a)^e$.
(3) $a = 0$ *if and only if* $(a)^e = (0]$.
(4) $a \in (b)^e$ *implies* $(a)^e \subseteq (b)^e$.
(5) $a \leq b$ *implies* $(a)^e \subseteq (b)^e$.
(6) $a \in D$ *if and only if* $(a)^e = L$.
(7) $(a)^e \cap (b)^e = (a \wedge b)^e$.

For any $a \in L$, the ideal $(a)^e$ is called as a dominator ideal. Then the set $\mathcal{A}^e(L)$ of all dominator ideals of L is not a sublattice of the lattice $\mathcal{I}(L)$ of all ideals of L. It can be observed from the following example.

Consider the following distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given by:



For $a, b \in L$, it can be observed that $(a)^e = \{0, a\}$ and $(b)^e = \{0, b\}$. Also $(a \vee b)^e = (c)^e = L$. But $(a)^e \vee (b)^e = \{0, a, b, c\}$. Therefore, $\mathcal{A}^e(L)$ is not a sublattice of the ideal lattice $\mathcal{I}(L)$. However, we have the following:

Theorem 3.3. *For any distributive lattice L, $\mathcal{A}^e(L)$ forms a complete distributive lattice with smallest element $\{0\}$. Moreover, $\mathcal{A}^e(L)$ has the greatest element if and only if L has a dense element.*

Proof. For any $a, b \in L$, define $(a)^e \cap (b)^e = (a \wedge b)^e$ and $(a)^e \sqcup (b)^e = (a \vee b)^e$. Then clearly $\langle \mathcal{A}^e(L), \cap, \sqcup \rangle$ is a lattice. By the extension of the property (7) of Lemma 3.2, $\langle \mathcal{A}^e(L), \cap, \sqcup \rangle$ is a complete lattice with respect to the partial ordering defined as $(a)^e \leq (b)^e$ if and only if $(a)^e \subseteq (b)^e$.

Now, for any $a, b, c \in L$, we have $(a)^e \sqcup \{(b)^e \cap (c)^e\} = (a)^e \sqcup (b \wedge c)^e = \{a \vee (b \wedge c)\}^e = \{(a \vee b) \cap (a \vee c)\}^e = (a \vee b)^e \cap (a \vee c)^e = \{(a)^e \sqcup (b)^e\} \cap \{(a)^e \sqcup (c)^e\}$. Therefore, $\langle \mathcal{A}^e(L), \cap, \sqcup \rangle$ is a complete distributive lattice with the smallest element $\{0\}$.

Suppose that L has greatest element, say $(d)^e$. Let $x \in (d]^*$. Then $x \wedge d = 0$. Since $(d)^e$ is the greatest element of $\mathcal{A}^e(L)$, we get $(x)^e = (x)^e \cap (d)^e = (x \wedge d)^e = (0)^e = (0]$. Hence $x = 0$. Hence d is a dense element. Conversely, assume that L has a dense element, say d. Then by Lemma 3.2, $(d)^e = L$. Therefore, $(d)^e$ is the greatest element of $\mathcal{A}^e(L)$. ∎

The quasi-complemented distributive lattice is now characterized in terms of the lattice $\mathcal{A}^e(L)$ of all dominator ideals.

Theorem 3.4. *Let L be a distributive lattice with a dense element. Then L is quasi-complemented if and only if $\langle \mathcal{A}^e(L), \cap, \sqcup \rangle$ is a Boolean algebra.*

Proof. Suppose L has a dense element, say d. Then $(d)^e$ is the greatest element of $\mathcal{A}^e(L)$ and hence $\langle \mathcal{A}^e(L), \cap, \sqcup \rangle$ is a bounded distributive lattice. Now, assume that L is quasi-complemented. Let $(a)^e \in \mathcal{A}^e(L)$, where $a \in L$. Then there exists $a' \in L$ such that $a \wedge a' = 0$ and $a \vee a'$ is dense. Now $(a)^e \cap (a')^e = (a \wedge a')^e = (0)^e = \{0\}$. Also $(a)^e \sqcup (a')^e = (a \vee a')^e = L$(since $a \vee a'$ is dense). Thus $(a')^e$ is the complement of $(a)^e$ in $\langle \mathcal{A}^e(L), \cap, \sqcup \rangle$. Hence $\langle \mathcal{A}^e(L), \cap, \sqcup \rangle$ is a Boolean algebra.

Conversely assume that $\langle \mathcal{A}^e(L), \cap, \sqcup \rangle$ is a Boolean algebra. Let $a \in L$. Then $(a)^e \in \mathcal{A}^e(L)$. Then there exists $(b)^e \in \mathcal{A}^e(L)$ such that $(a)^e \cap (b)^e = \{0\}$ and $(a)^e \sqcup (b)^e = L$. Hence $(a \wedge b)^e = \{0\}$ and $(a \vee b)^e = L$. Thus we get $a \wedge b = 0$ and $a \vee b$ is dense. Therefore, L is quasi-complemented. ∎

We now introduce a closure operation on $\mathcal{I}(L)$.

Definition 3.5. *For any ideal I of L, define $\alpha(I) = \{ (a)^e \mid a \in I \}$.*

Definition 3.6. *For any ideal $\widetilde{I}$ of $\mathcal{A}^e(L)$, define $\beta(\widetilde{I}) = \{ a \in L \mid (a)^e \in \widetilde{I} \}$.*

Lemma 3.7. *Let L be a distributive lattice. Then we have the following:*

(a) For any ideal I of L, $\alpha(I)$ is an ideal in $\mathcal{A}^e(L)$.
(b) For any ideal $\widetilde{I}$ of $\mathcal{A}^e(L)$, $\beta(\widetilde{I})$ is an ideal in L.
(c) α and β are isotones.

Proof. (a). Let I be an ideal of L. Clearly $(0)^e \in \alpha(I)$. Let $(a)^e, (b)^e \in \alpha(I)$. Then $(a)^e = (a')^e$ and $(b)^e = (b')^e$ for some $a', b' \in I$. Hence $(a)^e \sqcup (b)^e = (a')^e \sqcup (b')^e = (a' \vee b')^e \in \alpha(I)$. Again, let $(x)^e \in \alpha(I)$ and $(r)^e \in \mathcal{A}^e(L)$. Then $(x)^e = (x')^e$ for some $x' \in I$. Now $(x)^e \cap (r)^e = (x')^e \cap (r)^e = (x' \wedge r)^e \in \alpha(I)$. Therefore, $\alpha(I)$ is an ideal in $\mathcal{A}^e(L)$.

(b). Let $\widetilde{I}$ be an ideal in $\mathcal{A}^e(L)$. Clearly $0 \in \beta(\widetilde{I})$. Let $a, b \in \beta(\widetilde{I})$. Then we get $(a)^e, (b)^e \in \widetilde{I}$. Since $\widetilde{I}$ is an ideal, we get $(a \vee b)^e = (a)^e \sqcup (b)^e \in \widetilde{I}$. Hence we get $a \vee b \in \beta(\widetilde{I})$. Again, let $x \in \beta(\widetilde{I})$ and $r \in L$. Then $(x)^e \in \widetilde{I}$. Since $\widetilde{I}$ is an ideal in $\mathcal{A}^e(L)$, we get $(x \wedge r)^e = (x)^e \wedge (r)^e \in \widetilde{I}$. Thus $x \wedge r \in \beta(\widetilde{I})$. Therefore, $\beta(\widetilde{I})$ is an ideal in L.

(c). Let I, J be two ideals of L such that $I \subseteq J$. Let $(x)^e \in \alpha(I)$. Then we get $(x)^e = (y)^e$ for some $y \in I \subseteq J$. Hence $(x)^e \in \alpha(J)$. Thus α is an isotone. Again, let $\widetilde{I}, \widetilde{J}$ be two ideals of $\mathcal{A}^e(L)$ such that $\widetilde{I} \subseteq \widetilde{J}$. Let $x \in \beta(\widetilde{I})$. Then $(x)^e \in \widetilde{I} \subseteq \widetilde{J}$. Thus $x \in \beta(\widetilde{J})$. Therefore, β is isotone. ∎

It can be easily observed that the set $\mathcal{I}[\mathcal{A}^e(L)]$ of all ideals of $\mathcal{A}^e(L)$ also forms a complete distributive lattice under set inclusion, in which infimum is $\bigcap J_i$, the set-theoretic intersection and the supremum of any two ideals $\widetilde{I}, \widetilde{J}$ of $\mathcal{A}^e(L)$ is given by $\{ (i)^e \sqcup (j)^e \mid (i)^e \in \widetilde{I} \text{ and } (j)^e \in \widetilde{J} \}$. Hence, a homomorphism is obtained between the lattice $\mathcal{I}(L)$ of all ideals of L and the lattice $\mathcal{I}[\mathcal{A}^e(L)]$ of all ideals of $\mathcal{A}^e(L)$.

Theorem 3.8. *The mapping α is a homomorphism of $\mathcal{I}(L)$ into $\mathcal{I}[\mathcal{A}^e(L)]$.*

Proof. Let I, J be two ideals in L. It is enough to prove that $\alpha(I \vee J) = \alpha(I) \sqcup \alpha(J)$ and $\alpha(I \cap J) = \alpha(I) \cap \alpha(J)$. Since α is isotone, we get $\alpha(I) \sqcup \alpha(J) \subseteq \alpha(I \vee J)$. Conversely, let $(x)^e \in \alpha(I \vee J)$. Then $(x)^e = (y)^e$ for some $y \in I \vee J$. Hence $(y)^e = (i \vee j)^e$ for some $i \in I$ and $j \in J$. Thus we get $(x)^e = (y)^e = (i)^e \sqcup (j)^e \in \alpha(I) \sqcup \alpha(J)$. Therefore, $\alpha(I \vee J) \subseteq \alpha(I) \sqcup \alpha(J)$.

Since α is an isotone, we get $\alpha(I \cap J) \subseteq \alpha(I) \cap \alpha(J)$. Conversely, let $(x)^e \in \alpha(I) \cap \alpha(J)$. Then $(x)^e = (i)^e$ and $(x)^e = (j)^e$ for some $i \in I$ and $j \in J$. Now $(x)^e = (x)^e \cap (x)^e = (i)^e \cap (j)^e = (i \wedge j)^e \in \alpha(I \cap J)$. Thus $\alpha(I) \cap \alpha(J) \subseteq \alpha(I \cap J)$. Therefore, α is a homomorphism. ∎

Lemma 3.9. *For any ideal I of L, the map $I \longrightarrow \beta \circ \alpha(I)$ is a closure operator on $\mathcal{I}(L)$. That is,*

(a) $I \subseteq \beta \circ \alpha(I)$
(b) $\beta \circ \alpha[\beta \circ \alpha(I)] = \beta \circ \alpha(I)$
(c) $I \subseteq J \Rightarrow \beta \circ \alpha(I) \subseteq \beta \circ \alpha(J)$ *for any two ideals I, J of L.*

Proof. (a). Let $x \in I$. Then we get $(x)^e \in \alpha(I)$. Since $\alpha(I)$ is an ideal in $\mathcal{A}^e(L)$, we get $x \in \beta \circ \alpha(I)$. Therefore, $I \subseteq \beta \circ \alpha(I)$.
(b). Since $\beta \circ \alpha(I)$ is an ideal in L, by the condition (a), we get $\beta \circ \alpha \subseteq \beta \circ \alpha[\beta \circ \alpha(I)]$. Conversely, let $x \in \beta \circ \alpha[\beta \circ \alpha(I)]$. Then we get $(x)^e \in \alpha[\beta \circ \alpha(I)]$. Hence it is obtained $(x)^e = (y)^e$ for some $y \in \beta \circ \alpha(I)$. Thus $(x)^e = (y)^e \in \alpha(I)$, which yields that $x \in \beta \circ \alpha(I)$. Therefore, $\beta \circ \alpha[\beta \circ \alpha(I)] \subseteq \beta \circ \alpha(I)$.
(c). Suppose I, J are the two ideals of L such that $I \subseteq J$. Let $x \in \beta \circ \alpha(I)$. Then we get $(x)^e \in \alpha(I)$. Hence $(x)^e = (y)^e$ for some $y \in I \subseteq J$. Thus $(x)^e = (y)^e \in \alpha(I)$. Therefore, $x \in \beta \circ \alpha(I)$. ∎

Definition 3.10. *An ideal I of L is called a closure ideal if $I = \beta \circ \alpha(I)$.*

Theorem 3.11. *Let I be an ideal of the lattice L. Then the following conditions are equivalent.*

(a) *I is a closure ideal.*
(b) *For $x, y \in L$, $(x)^e = (y)^e$ and $x \in I$ imply that $y \in I$.*
(c) $I = \bigcup\limits_{x \in I} (x)^e$.

Proof. $(a) \Rightarrow (b)$: Assume that I is a closure ideal of L. Then $I = \beta \circ \alpha(I)$. Let $x, y \in L$ be such that $(x)^e = (y)^e$. Suppose that $x \in I = \beta \circ \alpha(I)$. Then we get $(y)^e = (x)^e \in \alpha(I)$. Therefore, $y \in \beta \circ \alpha(I) = I$.

$(b) \Rightarrow (c)$: For any $x \in I$, we have $(x] \subseteq (x)^e$. Hence $I = \bigcup_{x \in I}(x] \subseteq \bigcup_{x \in I}(x)^e$. Again, let $x \in I$ and $y \in (x)^e$. Now,

$$\begin{aligned} y \in (x)^e &\Rightarrow (y)^e \subseteq (x)^e \\ &\Rightarrow (y)^e = (x)^e \cap (y)^e = (x \wedge y)^e \\ &\Rightarrow y \in I \qquad \text{by condition (2)} \end{aligned}$$

Hence $(x)^e \subseteq I$ for all $x \in I$. Therefore, $\bigcup_{x \in I}(x)^e \subseteq I$.

$(c) \Rightarrow (a)$: We have always $I \subseteq \beta \circ \alpha(I)$. Let $x \in \beta \circ \alpha(I)$. Then $(x)^e \in \alpha(I)$. Hence $(x)^e = (y)^e$ for some $y \in I$. Thus we get $x \in (x)^e = (y)^e \subseteq \bigcup_{y \in I}(y)^e = I$. Therefore, I is a closure ideal. ∎

From the above Lemma, it can be observed that the class of all closure ideals forms a complete distributive lattice in which the infimum of a set of closure ideals I_i is $\cap I_i$, their set-theoretic intersection. The supremum is $\beta \circ \alpha(\vee I_i)$ where $\vee I_i$ is their supremum in the lattice of all ideals of L.

In the following theorem, a one-to-one correspondence is obtained between the class of all closure ideals of L and the class of all ideals of $\mathcal{A}^e(L)$. For this we need the following Lemma.

Lemma 3.12. *The following conditions hold in a distributive lattice:*

(i) For any ideal I of L, $\beta \circ \alpha(I) = \bigcup_{a \in I}(a)^e$.
(ii) For any ideal $\widetilde{I}$ of $\mathcal{A}^e(L)$, $\alpha \circ \beta(\widetilde{I}) = \widetilde{I}$.

Proof. (i) Let $x \in \bigcup_{a \in I}(a)^e$. Then we get $x \in (a)^e$ for some $a \in I$. Hence $(x)^e \subseteq (a)^e$ for some $a \in I$. Now,

$$\begin{aligned} a \in I &\Rightarrow (a)^e \in \alpha(I) \\ &\Rightarrow (x)^e \in \alpha(I) \quad \text{(since } \alpha(I) \text{ is an ideal, } (x)^e \subseteq (a)^e) \\ &\Rightarrow x \in \beta \circ \alpha(I) \end{aligned}$$

Therefore, $\bigcup_{a \in I}(a)^e \subseteq \beta \circ \alpha(I)$.

Conversely, let $x \in \beta \circ \alpha(I)$. Then we get $(x)^e \in \alpha(I)$. Hence $(x)^e = (y)e$ for some $y \in I$. Thus $x \in (x)^e = (y)^e \subseteq \bigcup_{a \in I}(a)^e$, which yields that $\beta \circ \alpha(I) \subseteq \bigcup_{a \in I}(a)^e$. Hence $\beta \circ \alpha(I) = \bigcup_{a \in I}(a)^e$.

(ii). Let $\widetilde{I}$ be an ideal of $\mathcal{A}^e(L)$. Let $(x)^e \in \widetilde{I}$. Then we get $x \in \beta(\widetilde{I})$. Hence $(x)^e \in \alpha \circ \beta(\widetilde{I})$. Therefore, $\widetilde{I} \subseteq \alpha \circ \beta(\widetilde{I})$. Conversely, let $(x)^e \in \alpha \circ \beta(\widetilde{I})$. Then $(x)^e = (y)^e$ for some $y \in \beta(\widetilde{I})$. Since $y \in \beta(\widetilde{I})$, we get $(x)^e = (y)^e \in \widetilde{I}$. Hence $\alpha \circ \beta(\widetilde{I}) \subseteq \widetilde{I}$. Therefore, $\alpha \circ \beta(\widetilde{I}) = \widetilde{I}$. ∎

Theorem 3.13. *Let L be a distributive lattice. Then the class of all closure ideals of L are isomorphic to the class of all ideals of $\mathcal{A}^e(L)$. Under this isomorphism the prime closed ideals corresponds to the prime ideals of $\mathcal{A}^e(L)$.*

Proof. Let Θ be the restriction of α to the class $\mathcal{CI}(L)$ of all closure ideals of L. Let $I, J \in \mathcal{CI}(L)$. Then $\Theta(I) = \Theta(J) \Rightarrow \alpha(I) = \alpha(J) \Rightarrow \beta \circ \alpha(I) = \beta \circ \alpha(J) \Rightarrow I = J$(since $I, J \in \mathcal{CI}(L)$). Hence Θ is one-one. Again, let $\widetilde{I}$ be an ideal of $\mathcal{A}^e(L)$. Then $\beta(\widetilde{I})$ is an ideal in L. We now show that $\beta(\widetilde{I})$ is a closure ideal in L. We have always $\beta(\widetilde{I}) \subseteq \beta \circ \alpha[\beta(\widetilde{I})]$. Let $x \in \beta \circ \alpha[\beta(\widetilde{I})]$. Then we get $(x)^e \in \alpha \circ \beta(\widetilde{I}) = \widetilde{I}$(by Lemma 3.12). Hence $x \in \beta(\widetilde{I})$. Therefore, $\beta(\widetilde{I}) = \beta \circ \alpha[\beta(\widetilde{I})]$. Now for this $\beta(\widetilde{I}) \in L$, we get $\Theta[\beta(\widetilde{I})] = \alpha \circ \beta(\widetilde{I}) = \widetilde{I}$. Therefore, Θ is onto. Now, for any $I, J \in \mathcal{CI}(L)$,

$$\begin{aligned} \Theta(I \vee J) &= \Theta[\beta \circ \alpha(I \vee J)] \\ &= \alpha[\beta \circ \alpha(I \vee J)] \\ &= \alpha(I \vee J) \\ &= \alpha(I) \sqcup \alpha(J) \\ &= \Theta(I) \sqcup \Theta(J) \end{aligned}$$

Also $\Theta(I \cap J) = \alpha(I \cap J) = \alpha(I) \cap \alpha(J) = \Theta(I) \cap \Theta(J)$. Therefore, Θ is an isomorphism of $\mathcal{CI}(L)$ onto the lattice of ideals of $\mathcal{A}^e(L)$.

We have obtained a one-to-one correspondence between prime closure ideals of L and the prime ideals of $\mathcal{A}^e(L)$. Let I be a prime closure ideal of L. Suppose $(x \wedge y)^e = (x)^e \cap (y)^e = \in \alpha(I)$. Then $x \wedge y \in \beta \circ \alpha(I) = I$(since I is a closed ideal). Thus we get $x \in I$ or $y \in I$, which implies that $(x)^e \in \alpha(I)$ or $(y)^e \in \alpha(I)$. Therefore, $\Theta(I) = \alpha(I)$ is a prime ideal in $\mathcal{A}^e(L)$. Conversely, suppose that $\widetilde{I}$ is an ideal in $\mathcal{A}^e(L)$. Since Θ is onto, there exists a closure ideal I in $\mathcal{CI}(L)$ such that $\widetilde{I} = \Theta(I) = \alpha(I)$. Let $x, y \in L$ such that $x \wedge y \in I$. Then $(x)^e \cap (y)^e = (x \wedge y)^e \in \alpha(I)$. Hence $(x)^e \in \alpha(I)$ or $(y)^e \in \alpha(I)$. Since I is a closure ideal, we get $x \in \beta \circ \alpha(I) = I$ or $y \in \beta \circ \alpha(I) = I$. Therefore, I is a prime ideal in L. ∎

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